Linear Matrix Inequalities and SemiDefinite Programming Impacts on Control System Design

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Seminar Outline

- Linear Matrix Inequalities and SDP
- Tricks to reformulate into LMIs
- System concepts via LMIs
- Multi-channel/objective with LMIs
- Uncertain systems analysis
- Gain-scheduling and LPV synthesis
- Hard non-LMI problems
- Conclusions, perspectives.

Linear Matrix Inequalities and SDP

- Definitions, manipulations
- Schur's complements
- Classes of convex optimization problems
- Semi- Definite Programming
- Algorithms to solve SDP, duality, complexity
- Software, links.

linear matrix inequalities

an LMI is a constraint on a vector $x \in \mathbf{R}^n$:

$$F(x) := F_0 + x_1 F_1 + \ldots + x_n F_n \succeq 0,$$

where F_0, $F_1, \ldots, F_n$ are symmetric matrices

- $\succeq$ is inequality on symmetric matrix cone
- LMI equivalent to $\lambda_{\min}(F(x)) \geq 0$
- $F(x) \succeq 0$ iff $\eta' F(x) \eta \geq 0$, $\forall \eta$
- $F(x) \succeq 0$ iff $\det\{$ppal mat.$\} \geq 0$
- $F(x) \succ 0$ iff $\eta' F(x) \eta > 0$, $\forall \eta \neq 0$

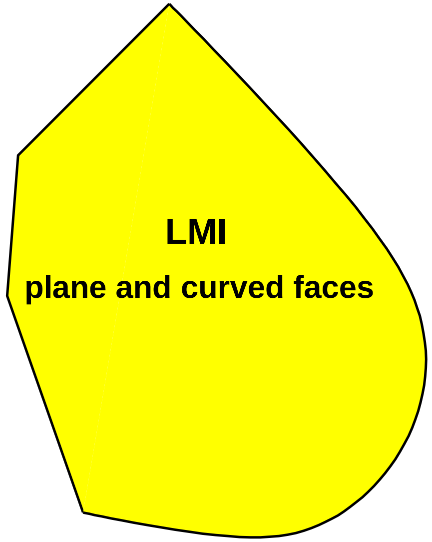
geometry of LMIs

- an LMI de fine a convex set

$$F(\lambda x+(1-\lambda)y) = \lambda F(x)+(1-\lambda)F(y) \succeq 0$$

whenever $F(x) \succeq 0$, $F(y) \succeq 0$

- set with non necessarily smooth boundary (corners)
- describe wide variety of constraints



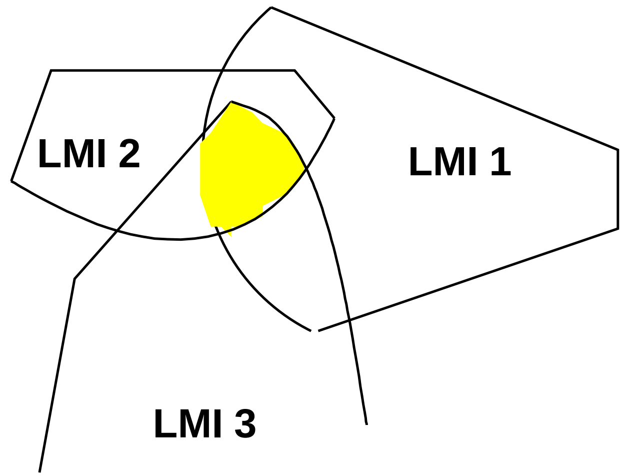
LMI - diagonal augmentation

LMI constraints

$$F_1(x) \succeq 0, \ldots, F_q(x) \succeq 0$$

are equivalent to single LMI constraint

$$\begin{bmatrix} F_1(x) & 0 & \ldots \\ 0 & \ddots & 0 \\ \vdots & 0 & F_q(x) \end{bmatrix} \succeq 0$$



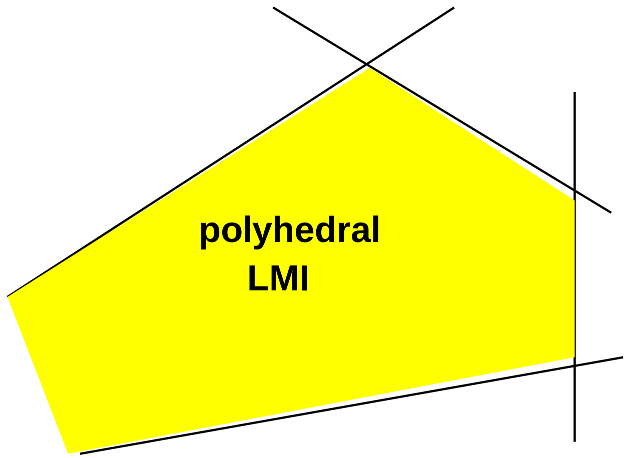
linear constraints

finite set of scalar linear (affine) constraints

$$a_i' x \leq b_i, \;\; i = 1, \ldots, m$$

can be represented as LMI $F(x) \preceq 0$, with

$$F(x) = \text{diag}(a_1' x - b_1, \ldots, a_m' x - b_m)$$



partitioned symmetric matrix

$$P := \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix}$$

$S = P_3 - P_2' P_1^{-1} P_2$ is the Schur complement of P_1 in P (provided P_1 invertible)

Schur complement lemmas

- $P \succ 0$ if and only if $P_1 \succ 0$ and $S \succ 0$
- if $P_1 \succ 0$, then $P \succeq 0$ if and only if $S \succeq 0$

Schur complement consequence

complicate constraint in variable x

$$P_3(x) - P_2(x)'P_1(x)^{-1}P_2(x) \succ 0$$

is turned into simpler one

$$\begin{bmatrix} P_1(x) & P_2(x) \\ P_2(x)' & P_3(x) \end{bmatrix} \succ 0\,.$$

provided that $P_1(x) \succ 0$.

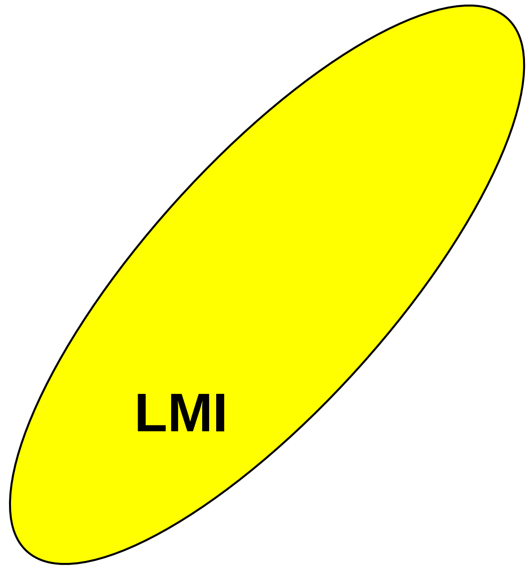
an ellipsoid can be described in different ways

- as $\|Ax+b\| \le 1$, iff

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)' & 1 \end{bmatrix} \succeq 0$$

- as $(x-x_0)'W(x-x_0) \le$ 1, with $W > 0$ iff

$$\begin{bmatrix} 1 & (x-x_0)' \\ (x-x_0) & W^{-1} \end{bmatrix} \succeq 0$$



fractional constraints

consider fractional constraints

$$\begin{aligned} \frac{(c'x)^2}{d'x} &\leq t \\ Ax + b &\geq 0 \end{aligned}$$

(assume $d'x > 0$, whenever $Ax + b \geq 0$)
can be represented as

$$\begin{aligned} \begin{bmatrix} t & c'x \\ c'x & d'x \end{bmatrix} &\succeq 0 \\ Ax + b &\geq 0 \end{aligned}$$

convex quadratic constraints

Convex quadratic constraints can be rewritten

$$(Ax+b)'(Ax+b) - c'x - d \leq 0$$

has the LMI representation

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)' & c'x+d \end{bmatrix} \succeq 0$$

- can be used to show that convex quadratic programming can be solved via SDP

- linear prog. (LP)

$$\text{minimize } c'x,\ Ax \preceq b$$

(componentwise)

- convex quadratic prog. (CQP) $Q_j \succeq 0$

$$\text{minimize } x'Q_0x + b_0'x + c_0$$
$$\text{s.t. } x'Q_ix + b_i'x + c_i \leq 0$$

All (and others) are generalized by SDP !:

with P variable

- Lyapunov inequality

$$A'P + PA \prec 0$$

can be represented in canonical form

$$F_0 + \sum_{i=1}^{n} x_i F_i \prec 0$$

pick a basis $(P_i)_i$ of the symmetric matrices,

$$P = \sum_i x_i P_i$$

hence recover the canonical form with

$$F_0 = 0, \quad F_i = A'P_i + P_i A$$

symmetric matrix expressions are LMIs

- Any (symmetric) linear constraints in the variables X, Y

$$AYB + (AYB)' + X + \ldots \preceq 0$$

can be represented in the canonical form

$$F(x) = F_0 + x_1 F_1 + \ldots + x_n F_n \preceq 0$$

by appropriate selection of the F_i's.

quadratic matrix inequality in P

$$A'P + PA + PBR^{-1}B'P + Q \preceq 0$$

where $R > 0$, is equivalent to LMI

$$\begin{bmatrix} A'P + PA + Q & PB \\ B'P & -R \end{bmatrix} \preceq 0$$

(proof by Schur complements)
Riccati-based control method can be solved via LMIs

classes of semidefinite programs

- I feasibility problem:

$$\text{find } x: \;\; F_0 + x_1 F_1 + \ldots + x_n F_n \preceq 0$$

- II linear objective minimization subject to LMIs

$$\text{minimize } c'x, \text{ s.t. } F_0 + x_1 F_1 + \ldots + x_n F_n \preceq 0$$

- III generalized eigenvalue minimization

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & A(x) - \lambda B(x) \succeq 0,\; B(x) \succeq 0,\; C(x) \succeq 0 \end{array}$$

(A, B, C affine symmetric expressions in x)

solving LMI - a rich set of algorithms

much work and progress since 1990 !

- primal interior-point method (method of centers)
- primal-dual interior-point method
- non-differentiable methods (bundle, ...)

Primal-dual methods very efficient.
other fast algorithms under development (aug. Lagrangian)

because of structure and convexity
algorithms are guaranteed to find global solutions !

primal-dual IPMs

ideas:

- instead of working in primal space, formulate problem in “primal-dual” space
- target objective is duality gap, and is zero at optimum
- try to solve (Lagrange) optimality conditions

SDP duality

- primal

$$\min c'x \text{ s.t. } F(x) \succeq 0$$

- dual

$$\max -\operatorname{Tr}(F_0 Z)$$
$$\text{s. t.} Z \succ 0, \ \operatorname{Tr} F_i Z = c_i$$

- optimality cond. if (x, Z) is primal-dual feasible

$$c'x = \sum_{i=1}^{n} x_i \operatorname{Tr} Z F_i = \overbrace{\operatorname{Tr} Z F(x)}^{\geq 0} - \operatorname{Tr} Z F_0 \geq -\operatorname{Tr} Z F_0$$

hence global optimality pairs (x, Z) such that

$\operatorname{Tr} Z F(x) = 0$ since primal and dual obj. coincide at solution

$$\begin{array}{ll} \text{solve} & \operatorname{Tr} ZF(x) = 0 \\ \text{subject to} & F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0 \\ & Z \succ 0, \ \operatorname{Tr} F_i Z = c_i, \ i = 1, \ldots, n \end{array}$$

- Actually, one tries to solve $\operatorname{Tr} ZF(x) = \mu I$ for decreasing value of μ ($\mu \longrightarrow 0$)
- Newton steps for the linearization of $\operatorname{Tr} ZF(x)$.
- superlinear convergence can be guaranteed

$$\|x_{k+1} - x_{\text{opt}}\| \leq \|x_k - x_{\text{opt}}\|^q, \ q > 1$$

very efficient in practice !

SDP software

- MATLAB LMI toolbox by Gahinet, Chilali, Laub, Nemirovski
- DSDP by Benson, Ye
- SDPpack by Alizadeh, Haeberly, Nayakkankuppam, Overton
- SeDuMi by Sturm
- lmitool-2.0 by Boyd et al.
- Cutting plane methods by Helmberg, Oustry, Kiwiel, etc.
- Many others ...

ftp addresses for SDP

- ftp addresses, codes, papers, courses on SDP

http://orion.math.uwaterloo.ca:80/ hwolkowi/henry/software/readme.html#

http://www.zib.de/helmberg/semidef.html

http://rutcor.rutgers.edu/ alizadeh/sdp.html

tricks to turn hard problems into LMIs

- Schur's complements (see previous)
- LMIs and quadratic forms
- multi-convexity, monotonicity, etc.
- Finsler's lemmas
- Projection lemmas
- changes of variables
- augmentation by slack

- **S-Procedure** transforms quadratic problems into LMIs(possibly conservative)

given Q_i's symmetric or hermitian matrices, define

$$F_0(x) = x'Q_0x, F_1(x) = x'Q_1x, \ldots, F_L(x) = x'Q_Lx,$$

$F_0(x) < 0$ **over the set** $F_1(x) \leq 0, \ldots, F_L(x) \leq 0$
whenever $\exists s_1 \geq 0, \ldots, s_L \geq 0$ **(slacks), such that**

$$F_0(x) - \sum_{i=1}^{L} s_i F_i(x) < 0 \text{ or LMI } Q_0 - \sum_{i=1}^{L} s_i Q_i \preceq 0$$

Finsler's Lemma

- converts checking the sign of a quadratic form over a subspace into solving an LMI problem

$$x'Qx < 0, \forall x \neq 0,\ Mx = 0$$

if and only there exists a scalar σ such that

$$Q - \sigma M'M \prec 0$$

$Mx = 0$ can also be formulated as $x'M'Mx = 0$

- proof via convexity of numerical ranges

generalized Finsler's Lemma

- convert family of constrained quadratic inequalities into an LMI feasibility problem
$Q = Q'$ and M given, and a compact subset of real matrices $\mathbf{U}$
we have the equivalence
- for all $U \in \mathbf{U}$,

$$x'Qx < 0, \ \forall x \neq 0 \text{ with } UMx = 0,$$

iff there exists Θ s.t.

$$\begin{aligned} Q + M'\Theta M &\prec 0 \\ \mathcal{N}_U'\Theta\mathcal{N}_U &\succeq 0, \qquad \forall U \in \mathbf{U} \end{aligned}$$

where $\mathcal{N}_U$ is basis of nullspace of U

multi-convexity

given a function $f(\delta_1, \dots, \delta_K)$

- it is multi-convex function if separately convex along each direction δ_i
- multi-convexity is weaker than convexity
- convexity iff
- multi-convexity iff

$$[\frac{\partial^2}{\partial\delta_i\delta_j} f(\delta)]_{1\le i,j\le K} \succeq 0 \qquad \frac{\partial^2}{\partial\delta_i^2} f(\delta) \ge 0, \quad i = 1, \dots, K$$

Turn parameter-dependent LMIs into finite set of LMIs.

projection Lemma - two-Sided

given $\Psi = \Psi' \in \mathbf{R}^{m \times m}$, P, Q of column dim. m
find X such that

$$\Psi + P'X'Q + Q'XP \prec 0$$

let columns of $\mathcal{N}_P, \mathcal{N}_Q$ form bases of the null spaces of P and Q
inequality is solvable for X if and only if

$$\mathcal{N}_P' \, \Psi \, \mathcal{N}_P \prec 0 \quad \mathcal{N}_Q' \, \Psi \, \mathcal{N}_Q \prec 0$$

(Gahinet & Apkarian 1993)

System concepts via LMIs

- Stability
- L_2 gain or H_∞ norm
- H_2 norm
- Pole clustering
- $\ldots$

stability & equilibria

- Equilibrium points

$$\dot{x} = f(x)$$

are defined as the solutions x^* of

$$0 = f(x^*).$$

system has trajectory $x(t) = x^*$, $\forall t \geq 0$ if initialized at x^*
From now on, we assume $x^* = 0$.

- stability (simple)

$$\forall R > 0, \exists r > 0, \|x(0)\| < r \Rightarrow \forall t \geq 0, \|x(t)\| < R$$

- asymptotic stability

if it is stable and

$$\exists r > 0, \ \|x(0)\| < r \Rightarrow x(t) \to 0, \ \text{as } t \to \infty$$

- exponentially stable

if $\exists\ \alpha > 0$ and $\lambda > 0$ s. t.

$$\forall t > 0, \ \|x(t)\| \leq \alpha \|x(0)\| e^{-\lambda t}$$

in some ball. λ rate of conv.

positive-definite functions of the state

Assume D is open region containing $x^* = 0$.

- A function $V(x)$ from $\mathbf{R}^n$ into $\mathbf{R}$ is positive semi-definite on a domain D if

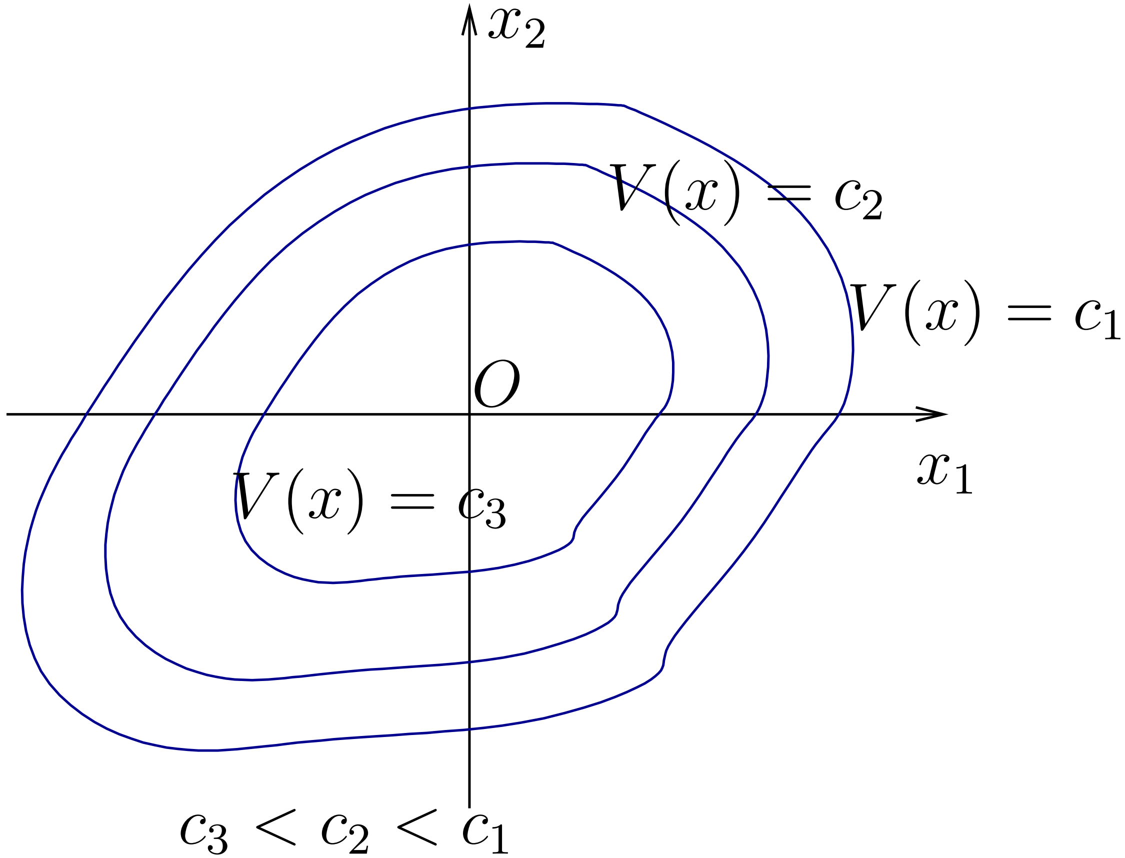
(1) $V(0) = 0$

(2) $V(x) \geq 0, \ \forall x \in D$

- A function $V(x)$ from $\mathbf{R}^n$ into $\mathbf{R}$ is positive definite on a domain D if

(1) $V(0) = 0$

(2) $V(x) > 0, \ \forall x \in D, \ x \neq 0$



Typical level curves of positive-definite functions

Lyapunov derivatives

- if x is state of system $\dot{x} = f(x)$, then $V(x)$ is implicitly a function of time. Its time derivative is

$$\dot{V}(x) = \frac{dV(x)}{dt} = \frac{\partial V}{\partial x}' \dot{x} = \frac{\partial V}{\partial x}' f(x)$$

since x is constrained to satisfy $\dot{x} = f(x)$.

- it is referred to as derivative of V along the system trajectories (also Lyapunov's derivative).

Lyapunov function: definition

- $V(x)$ is a Lyapunov function of the system $\dot{x} = f(x)$ if
  - it is $\mathcal{C}^1$ with respect to x on D
  - it is positive definite (see earlier) on D
  - its derivative on the system trajectories is negative semi-definite, that is,

$$\dot{V}(x) \leq 0, \text{ on } D$$

as a function of x.

Lyapunov theorem for local stability

- if in a ball around the origin ($= x^*$), there exists $V(x)$ in $\mathcal{C}^1$ such that
  - ➠ $V(x)$ is positive definite
  - ➠ $\dot{V}(x)$ is negative semi-definite

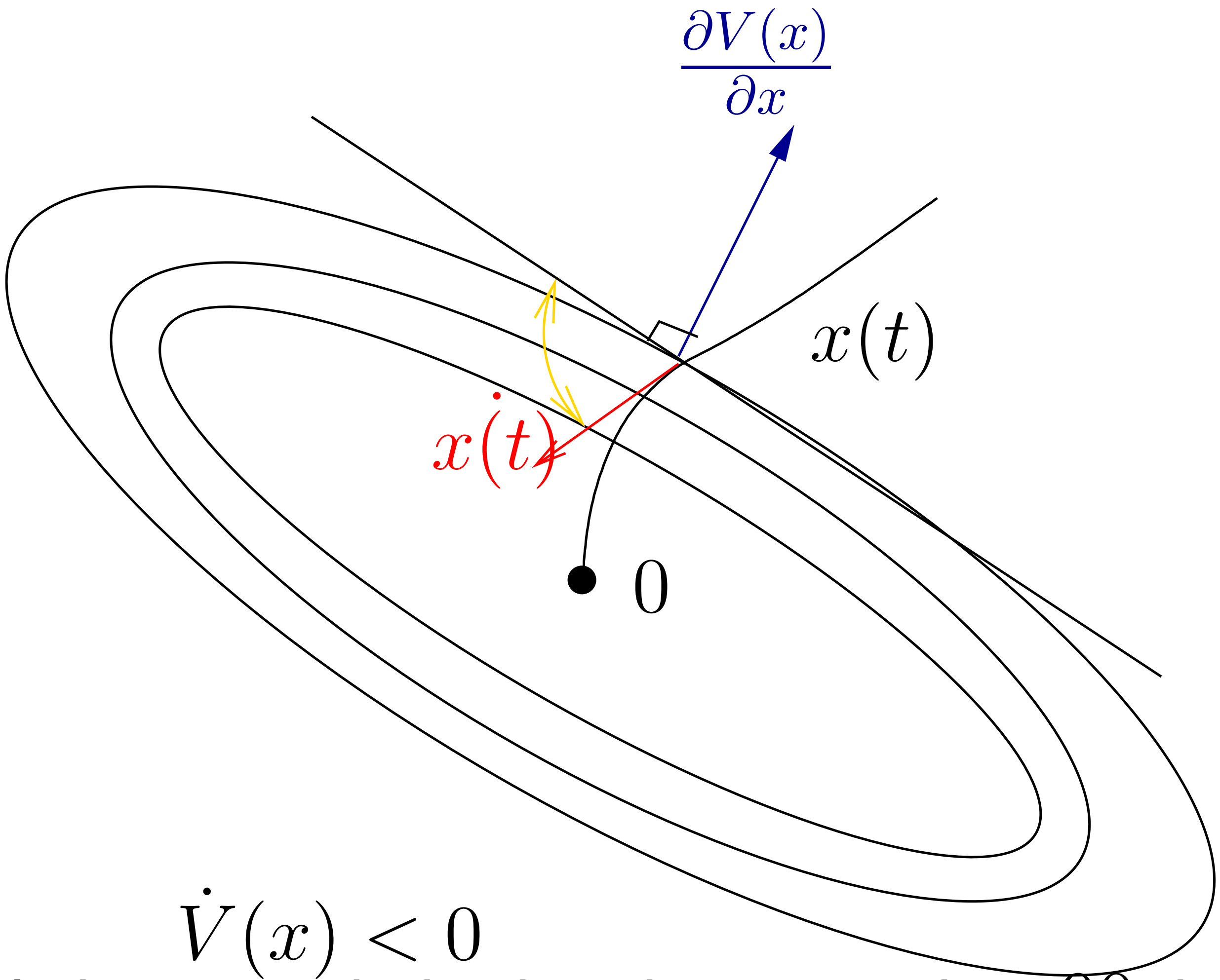
then the equilibrium point $x^* = 0$ is (loc.) stable. It is asymptotically stable if $V(x)$ is negative definite, i. e.,

$$\dot{V}(x) < 0,\ \forall x \neq 0,\ x \in \text{ball}$$

as a function of x.

- global stability if ball $= \mathbf{R}^n$.

Lyapunov geometry



$\dot{V}(x) < 0$

angle between derivatives is greater than 90 deg.

stability for linear systems

the system

$$\frac{d}{dt}x = Ax$$

is exponentially stable if and only if there exists X with

$$X \succ 0, \qquad A'X + XA \prec 0$$

why ?

$$V(x) = x'Xx$$

is a quadratic Lyapunov function

proof of exponential stability

perturb Lyapunov LMI to

$$A'X + XA + \varepsilon X \prec 0$$

for any state trajectory $x(t)$, we infer

$$x(t)'(A'X + XA)x(t) + \varepsilon x(t)'Xx(t) \leq 0$$

and thus

$$\frac{d}{dt}x(t)'Xx(t) + \varepsilon x(t)'Xx(t) \leq 0$$

- note that solution of

$$\frac{d}{dt}x(t)'Xx(t) + \varepsilon x(t)'Xx(t) = z(t) \text{ with } z(t) \leq 0$$

is

$$V(x(t)) = x(0)'Xx(0)e^{-\varepsilon t} + \int_0^t e^{-\varepsilon(t-\tau)}z(\tau)d\tau$$

hence

$$x(t)'Xx(t) \leq x(0)'Xx(0)e^{-\varepsilon t}, \ \forall t \geq 0 \, .$$

proof of exponential stability - continued

we have, with initial condition $x(0)$ yields

$$x(t)'Xx(t) \leq x(0)'Xx(0)e^{-\varepsilon t}$$

finally, using

$$\lambda_{\min}(X)\|x\|^2 \leq x'Xx \leq \lambda_{\max}(X)\|x\|^2$$

gives

$$\|x(t)\| \leq \|x(0)\|\sqrt{\frac{\lambda_{\max}(X)}{\lambda_{\min}(X)}}e^{-\varepsilon t/2} \quad \text{for } t \geq 0$$

system is exponentially stable !

necessity of Lyapunov inequalities

Assume A is stable $(\mathrm{Re}\,\lambda_i(A) < 0)$ and consider for $Q \succ 0$, the (well-defined) integral

$$
\begin{aligned}
-Q &= \int_0^\infty \frac{d}{dt}(e^{A't}Qe^{At})dt \\
&= \int_0^\infty (A'e^{A't}Qe^{At} + e^{A't}Qe^{At}A)dt \\
&= A'P + PA \text{ with } P := \int_0^\infty e^{A't}Qe^{At}dt \succ 0
\end{aligned}
$$

necessity continued

finally, we have

$$A'P + PA = -Q \prec 0, \quad P \succ 0\,.$$

LMI problem has a solution whenever A is stable.

- condition is iff
- for linear systems quadratic Lyapunov functions are rich enough

Energy gain not larger than γ: with $w \in L_2$ and $x(0) = 0$, every trajectory of

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bw \\ z &= Cx + Dw \end{aligned}$$

should satisfy

$$\|z\|_2 \leq \gamma \|w\|_2, \quad \forall w \in L_2$$

or

$$\int_0^\infty z(t)'z(t)\,dt \leq \gamma^2 \int_0^\infty w(t)'w(t)\,dt$$

H_∞ norm

- stable and the L_2 gain $w \longrightarrow z$ is smaller than γ if and only if there exists $X \succ 0$

$$\begin{bmatrix} A'X + XA & XB & C' \\ B'X & -\gamma I & D' \\ C & D & -\gamma I \end{bmatrix} \prec 0$$

- freq. domain $\|C(sI - A)^{-1} + D\|_\infty < \gamma$ via KYP.
- similarly, H_2 norm, LQ, LQG, many others ...

H_∞ norm

- necessity call for general LQ theory.
- we shall only prove sufficiency.

- Note first that the $(1,1)$ block of the LMI implies that A is stable

- By Schur complement, LMI is rewritten

$$\begin{bmatrix} A'X + XA & XB \\ B'X & -\gamma I \end{bmatrix} + \gamma^{-1} \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \prec 0$$

Left- and right-multiply with $\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$ yields ...

proof of sufficiency cont.

$$\overbrace{x'(A'X + XA)x + x'XBw + w'B'Xx}^{\frac{d}{dt}V}$$
$$-\gamma w'w + \gamma^{-1}z'z \leq 0$$

proof of sufficiency cont.

integrate over $[0,\ T]$ and exploit $x(0) = 0$:

$$x(T)'Xx(T) + \int_0^T \gamma^{-1}\|z(t)\|^2 - \gamma\|w(t)\|^2 dt \leq 0$$

Recall $X \succ 0$ and take $T \to \infty$ ($w \in L_2$):

$$\int_0^\infty \|z(t)\|^2\, dt \leq \gamma^2 \int_0^\infty \|w(t)\|^2\, dt \leq 0$$

Can perturb γ to $\gamma - \varepsilon$ to get strict inequality

For $\omega \in \mathbf{R}$, left- and right multiply with

$$\begin{bmatrix} (j\omega - A)^{-1}B \\ I \end{bmatrix}$$

to get

$$\gamma^{-1}T(j\omega)^*T(j\omega) - \gamma I \prec 0$$

hence

$$\|T(j\omega)\| < \gamma, \quad \forall \omega \in \mathbf{R}$$

From the right-lower block, we also get

$$\begin{bmatrix} -\gamma I & D' \\ D & -\gamma I \end{bmatrix} \prec 0 \quad \text{or } \|D\| < \gamma$$

relation to frequency domain

finally,

$$\|T(j\omega)\| < \gamma \text{ for } \omega \in \mathbf{R} \cup \{\infty\}$$

hence,

$$\|T\|_\infty := \sup_{\omega \in \mathbf{R} \cup \{\infty\}} \|T(j\omega)\| < \gamma.$$

H_2 performance

- H_2 norm of T defined as

$$\|T\|_2 := \sqrt{\frac{1}{2\pi}\mathrm{Tr}\int_{-\infty}^{\infty} T(j\omega)^* T(j\omega)\, d\omega}$$

- in the time domain (via Parseval)

$$\|T\|_2 := \sqrt{\int_0^{\infty} \mathrm{Tr}\,(Ce^{At}B)'(Ce^{At}B)\, dt}$$

H_2 performance computation

Easily computed by solving linear equation

$$AP_0 + P_0A' + BB' = 0 \quad \Rightarrow \quad \|T\|_2^2 = \mathrm{Tr}\,(CP_0C')$$
$$A'Q_0 + Q_0A + C'C = 0 \quad \Rightarrow \quad \|T\|_2^2 = \mathrm{Tr}\,(B'Q_0B)$$

Why ? see stability notes.

- note that $D = 0$ for H_2 norm to be well defined.

stochastic interpretation of H_2 norm

w white noise, $\dot{x} = Ax + Bw$, $x(0) = 0$, $z = Cx$.
Recall: with solution of

$$\dot{P}(t) = AP(t) + P(t)A' + BB', \quad P(0) = 0$$

we have $E(x(t)x(t)') = P(t)$.
Hence

$$\begin{aligned}
\lim_{t\to\infty} E(z(t)'z(t)) &= \lim_{t\to\infty} E(x(t)'C'Cx(t)) \\
&= \lim_{t\to\infty} \operatorname{Tr} E(Cx(t)x(t)'C') \\
&= \operatorname{Tr}(CP_0C') = \|T\|_2^2
\end{aligned}$$

- asymptotic variance of output of system.

let z_j be impulse response to $Be_j\delta(t)$ with standard unit vector e_j of

$$\dot{x} = Ax,\ x(0) = x_0,\ z = Cx$$

$$\int_0^\infty z_j(t)' z_j(t)\, dt = \int_0^\infty B_j' e^{A't} C' C e^{At} B_j\, dt$$

$v'v = \mathrm{Tr}\,(vv')$ and $\sum_j B_j B_j' = BB'$ implies

$$\sum_j \int_0^\infty \|z_j(t)\|^2\, dt = \|T\|_2^2\,.$$

How to get LMI characterization ?

With A stable, it is easy to see that

$$\mathrm{Tr}\,(CP_0C') < \gamma^2 \text{ for } AP_0 + P_0A' + BB' = 0$$

if and only if there exists X with

$$\mathrm{Tr}\,(CXC') < \gamma^2 \text{ and } AX + XA' + BB' \prec 0\,.$$

- for $\Leftarrow$ take difference of Lyapunov conditions

How to get LMI characterization ?

- for $\Rightarrow$ since trace inequality is strict and by continuity there exists $\varepsilon > 0$ and X such that

$$AX + XA' + BB' + \varepsilon I = 0, \quad \mathrm{Tr}\,(CXC') < \gamma^2\,.$$

Note that $AX + XA' + BB' \prec 0$ and

$$X = \int_0^\infty e^{At}(BB' + \varepsilon I)e^{A't}\,dt \succ P_0$$

Hence,

$$\|C(sI-A)^{-1}B\|_{H_2}^2 := \mathrm{Tr}\,(CP_0C') < \mathrm{Tr}\,(CXC') < \gamma^2\,.$$

LMI characterization for H_2 norm

♢ A is stable and $\|T\|_2^2 < \gamma$ if and only if $Y \succ 0$ with

$$\mathrm{Tr}\,(CYC') < \gamma, \qquad AY + YA' + BB' \prec 0$$

or if and only if $X \succ 0$ with

$$\mathrm{Tr}\,(B'XB) < \gamma, \qquad A'X + XA + C'C \prec 0$$

regional pole constraints

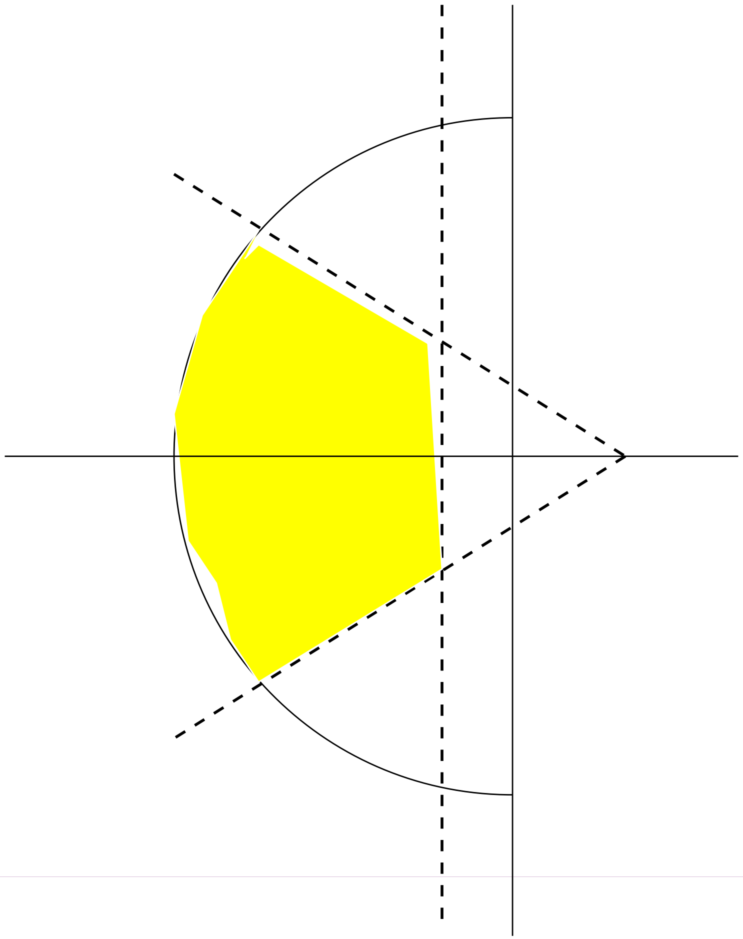
- to shape transient responses of closed-loop system
- damping, settling time, rise time related to location of poles
- useful regions: vertical strips, disks, conic sectors, etc

- An LMI region $\mathcal{R}$ is defined as

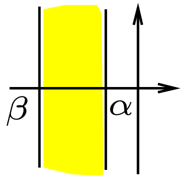
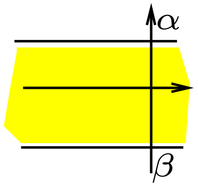
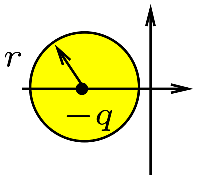
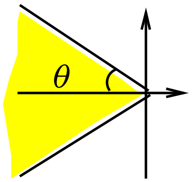
$$\mathcal{R} = \{z \in \mathbf{C} \, : U + zV + \bar{z}V' \prec 0\}\,.$$

LMI region intersections

- a large variety of regions can be represented this way
- intersections of LMI regions are LMI regions



a short catalog of useful LMI regions

LMI Regions	Characterization
	$f_{\mathcal{R}}(z) = \begin{bmatrix} -\alpha + \frac{1}{2}(z+\bar{z}) & 0 \\ 0 & \beta - \frac{1}{2}(z+\bar{z}) \end{bmatrix}$
	$f_{\mathcal{R}}(z) = \begin{bmatrix} -\alpha - \frac{i}{2}(z-\bar{z}) & 0 \\ 0 & \beta + \frac{i}{2}(z-\bar{z}) \end{bmatrix}$
	$f_{\mathcal{R}}(z) = \begin{bmatrix} -r & q+z \\ q+\bar{z} & -r \end{bmatrix}$
	$f_{\mathcal{R}}(z) = \begin{bmatrix} \sin\theta(z+\bar{z}) & \cos\theta(z-\bar{z}) \\ \cos\theta(\bar{z}-z) & \sin\theta(z+\bar{z}) \end{bmatrix}$

Lyapunov theorem for LMI regions

- System $\frac{d}{dt}x = Ax$ has all its poles in LMI region $\mathcal{R}$ iff there exists $X \succ 0$ s. t.

$$U \otimes X + V \otimes (A'X) + V' \otimes (XA) \prec 0\,.$$

is an LMI with respect to X.
($\otimes$ is Kronecker product $A \otimes B := ((A_{ij}B)))$

- classical Lyapunov theorem with $U = 0$, $V = 1$
- intersection by diagonal augmentation of U, V.

other specs. can be combined by just merging LMI constraints condition is

$$X \succ 0, \qquad U \otimes X + V \otimes (A'X) + V' \otimes (XA) \prec 0\,.$$

pick an eigenpair of A, (λ, v), $Av = \lambda v$, and pre- and post-multiply inequality by $I \otimes v^*$, $I \otimes v$, gives

$$\overbrace{(v^* X v)}^{>0} (U + \lambda^* V + \lambda V') < 0$$

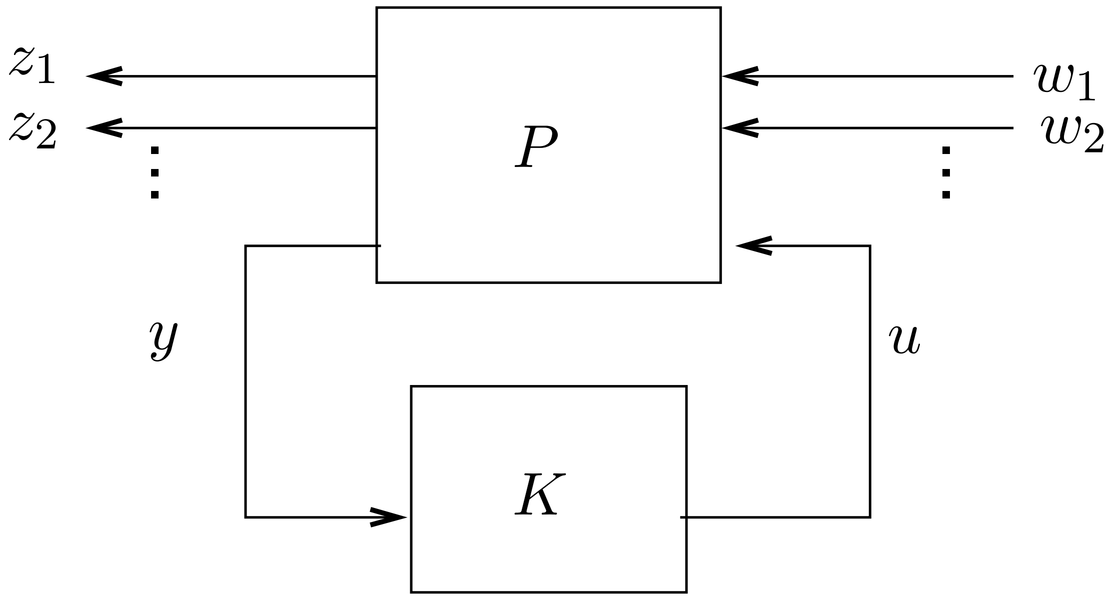
Hence,

$$U + \lambda^* V + \lambda V' < 0.$$

Implies λ^*, λ are in $\mathcal{R}$.

- formulation
- linearizing change of variables
- state-feedback synthesis
- output-feedback synthesis
- projected form.

- synthesis structure



- given $P(s)$, find $K(s)$ to achieve a set of specifications for channels $w_1 \longrightarrow z_1$, $w_2 \longrightarrow z_2$, ...

$$\min \|T_{w_2^1 \leftarrow z_2^1}\|_2$$

$$\|T_{w_\infty^1 \leftarrow z_\infty^1}\|_\infty < \gamma_1, \quad \|T_{w_2^2 \leftarrow z_2^2}\|_2 < \gamma_2$$

poles in LMI region $\mathcal{R}$.

controller synthesis - data

- synthesis interconnection

$$P(s)\quad \begin{cases} \frac{d}{dt}x &= Ax + B_1 w + B_2 u, \qquad A \in \mathbf{R}^{n\times n} \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w \end{cases}$$

- controller

$$K(s)\quad \begin{cases} \frac{d}{dt}x_K &= A_K x_K + B_K y, \qquad A_K \in \mathbf{R}^{n\times n} \\ u &= C_K x_K + D_K y \end{cases}$$

- Stability, Perfo.: H_∞, H_2, pole plac. on various channels

derivation

- compute closed-loop data
- write stability/performance (ineq.) conditions in closed loop
- apply congruence transformations
- use suitable linearizing transformations

- turns out to be very simple problem

$$P(s) \quad \begin{cases} \dot{x} &= Ax + B_1 w + B_2 u, \qquad A \in \mathbf{R}^{n \times n} \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= x \quad \longleftarrow \quad \text{measurable state vector} \end{cases}$$

and

$$u = Kx \quad \longleftarrow \quad \text{state-feedback}$$

closed-loop data are

$$\begin{aligned} \dot{x} &= (A + B_2 K)x + B_1 w \\ z &= (C_1 + D_{12} K)x + D_{11} w \end{aligned}$$

- characterization is $X \succ 0$ and

$$\begin{bmatrix} (A+B_2K)'X + * & * & * \\ B_1'X & -\gamma I & * \\ C_1 + D_{12}K & D_{11} & -\gamma I \end{bmatrix} \prec 0$$

perform congruence transformation
diag$(Y = X^{-1}, I, I)$ to get $Y \succ 0$ and

$$\begin{bmatrix} (A+B_2K)Y + Y(A+B_2K)' & * & * \\ (C_1 + D_{12}K)Y & -\gamma I & * \\ B_1' & D_{11}' & -\gamma I \end{bmatrix} \prec 0,$$

note Y is invertible perform change of variable $W = KY$ to get LMI !: $Y \succ 0$ and

$$\begin{bmatrix} AY + YA' + B_2W + (B_2W)' & * & * \\ C_1Y + D_{12}W & -\gamma I & * \\ B_1' & D_{11}' & -\gamma I \end{bmatrix} \prec 0\,.$$

- note change of variable is without loss (NSC)
- when solved, deduce (state-feedback) controller

using

$$K = WY^{-1}\,.$$

trick

- $\Leftarrow (Y, KY)$ solution $\rightarrow (Y, W)$ easy
- $\Rightarrow (Y, W)$ solution $\rightarrow (Y, KY)$

note that term $B_2 W$ is $B_2 W Y^{-1} Y$ hence

$$(Y, K = WY^{-1}) \text{ is a solution.}$$

state-feedback H_2 synthesis

- similar derivation
- characterization

$$(A + B_2K)'X + * + (C_1 + D_{12}K)'(C_1 + D_{12}K) \prec 0,$$
$$\mathrm{Tr}\,(B_1'XB_1) < \eta^2$$

become via Schur complements

$$\begin{bmatrix} (A + B_2K)'X + * & * \\ (C_1 + D_{12}K) & -I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} Z & B_1' \\ B_1 & X^{-1} \end{bmatrix} \preceq 0, \;\; \mathrm{Tr}\, Z < \eta^2$$

- perform congruence transformations

diag$(Y = X^{-1}, I)$ and diag(I, Y) to get

$$\begin{bmatrix} AY + B_2KY + * & * \\ C_1Y + D_{12}KY & -I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} Z & B_1'Y \\ YB_1 & Y \end{bmatrix} \succeq 0, \ \mathrm{Tr}\, Z < \eta^2$$

- change of variable $W = KY$ yields LMIs !

$$\begin{bmatrix} AY + B_2W + * & * \\ C_1Y + D_{12}W & -I \end{bmatrix} \prec 0,$$

with

$$\begin{bmatrix} Z & B_1'Y \\ YB_1 & Y \end{bmatrix} \succeq 0, \ \mathrm{Tr}\, Z < \eta^2$$

state-feedback pole clustering

- similarly $Y \succ 0$ and

$$U \otimes Y + V \otimes (A + B_2 K)Y + V' \otimes Y(A + B_2 K)' \prec 0\,.$$

change of variable $W = KY$ leads to LMI!:

$$Y \succ 0$$
$$U \otimes Y + V \otimes (AY + B_2 W) + V' \otimes (AY + B_2 W)' \prec 0\,.$$

multiple constraints

- the Y's are not the same for all perfs.
- hard problem is relaxed by taking a single Y for all perfs.
- technique is constantly refined to exploit different Y's by spec. (active area).

- $$\left[\begin{array}{c|c}\mathcal{A} & \mathcal{B}_1 \\ \hline \mathcal{C}_1 & \mathcal{D}_{11}\end{array}\right] := \left[\begin{array}{cc|c} A & 0 & B_1 \\ 0 & 0 & 0 \\ \hline C_1 & 0 & D_{11}\end{array}\right] + \left[\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & D_{12}\end{array}\right] \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \left[\begin{array}{cc|c} 0 & I & 0 \\ C_2 & 0 & D_{21}\end{array}\right],$$

- Above analysis condition must be satisfied in closed-loop. Synthesis conditions in 3 steps

1- introduce a single variable $\mathcal{P}$ common specification/channel (conservative step),

2- perform adequate congruence transformations,

3- use linearizing changes of variables to end up with LMI synthesis conditions.

linearizing change of variable

Introduce notation

$$\mathcal{P} = \begin{bmatrix} \boldsymbol{X} & N \\ N' & \star \end{bmatrix}, \qquad \mathcal{P}^{-1} = \begin{bmatrix} \boldsymbol{Y} & M \\ M' & \star \end{bmatrix}$$

From $\mathcal{P}\mathcal{P}^{-1} = I$ infer

$$\mathcal{P}\Pi_Y = \Pi_X \;\text{ with }\; \Pi_Y := \begin{bmatrix} \boldsymbol{Y} & I \\ M' & 0 \end{bmatrix}, \; \Pi_X := \begin{bmatrix} I & \boldsymbol{X} \\ 0 & N' \end{bmatrix}.$$

Define change of variable (wlog N, M are invertible)

$$(1)\begin{cases} \widehat{\boldsymbol{A}}_K & := \; NA_KM' + NB_KC_2\boldsymbol{Y} + \boldsymbol{X}B_2C_KM' + \boldsymbol{X}(A + B_2D_KC_2)\boldsymbol{Y}, \\ \widehat{\boldsymbol{B}}_K & := \; NB_K + \boldsymbol{X}B_2D_K, \; \widehat{\boldsymbol{C}}_K := C_KM' + D_KC_2\boldsymbol{Y}, \widehat{\boldsymbol{D}}_K := D_K. \end{cases}$$

and, perform congruence transformations to get

linear terms in the new variables $\boldsymbol{X}$, $\boldsymbol{Y}$, $\widehat{\boldsymbol{A}}_K$, $\widehat{\boldsymbol{B}}_K$, $\widehat{\boldsymbol{C}}_K$, $\widehat{\boldsymbol{D}}_K$!

$$\begin{bmatrix} L_{11} & \widehat{\boldsymbol{A}}_K' + (A + B_2 \widehat{\boldsymbol{D}}_K C_2) & * & * \\ \widehat{\boldsymbol{A}}_K + (A + B_2 \widehat{\boldsymbol{D}}_K C_2)' & L_{22} & * & * \\ (B_1 + B_2 \widehat{\boldsymbol{D}}_K D_{21})' & (\boldsymbol{X} B_1 + \widehat{\boldsymbol{B}}_K D_{21})' & -\gamma I & * \\ C_1 \boldsymbol{Y} + D_{12} \widehat{\boldsymbol{C}}_K & C_1 + D_{12} \widehat{\boldsymbol{D}}_K C_2 & D_{11} + D_{12} \widehat{\boldsymbol{D}}_K D_{21} & -\gamma I \end{bmatrix} \prec 0$$

where

$$L_{11} := A\boldsymbol{Y} + \boldsymbol{Y}A' + B_2 \widehat{\boldsymbol{C}}_K + (B_2 \widehat{\boldsymbol{C}}_K)', \; L_{22} := A'\boldsymbol{X} + \boldsymbol{X}A + \widehat{\boldsymbol{B}}_K C_2 + (\widehat{\boldsymbol{B}}_K C_2)' \,.$$

- similarly for H_2 and LMI region specs.
- for multi- channel/objective just stack together various LMI specs.

LMI for H_2 specification

$$\begin{bmatrix} A\boldsymbol{Y}+\boldsymbol{Y}A'+B_2\widehat{\boldsymbol{C}}_K+(B_2\widehat{\boldsymbol{C}}_K)' & * & * \\ \widehat{\boldsymbol{A}}_K+(A+B_2\widehat{\boldsymbol{D}}_KC_2)' & A'\boldsymbol{X}+\boldsymbol{X}A+\widehat{\boldsymbol{B}}_KC_2+(\widehat{\boldsymbol{B}}_KC_2)' & * \\ C_1\boldsymbol{Y}+D_{12}\widehat{\boldsymbol{C}}_K & C_1+D_{12}\widehat{\boldsymbol{D}}_KC_2 & -I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \boldsymbol{Y} & I & B_1+B_2\widehat{\boldsymbol{D}}_KD_{21} \\ I & \boldsymbol{X} & \boldsymbol{X}B_1+\widehat{\boldsymbol{B}}_KD_{21} \\ (B_1+B_2\widehat{\boldsymbol{D}}_KD_{21})' & (\boldsymbol{X}B_1+\widehat{\boldsymbol{B}}_KD_{21})' & Q \end{bmatrix} \succ 0,$$

$$\mathrm{Tr}\,(Q) < \nu,\; D_{11}+D_{12}\widehat{\boldsymbol{D}}_KD_{21}=0.$$

$$\begin{bmatrix} \boldsymbol{Y} & I \\ I & \boldsymbol{X} \end{bmatrix} \succ 0$$

- congruence $\mathrm{diag}(\Pi_Y, \dots, \Pi_Y)$ yields

$$\left(\lambda_{jk}\begin{bmatrix} \boldsymbol{Y} & I \\ I & \boldsymbol{X} \end{bmatrix} + \mu_{jk}\begin{bmatrix} A\boldsymbol{Y}+B_2\widehat{\boldsymbol{C}}_K & A+B_2\widehat{\boldsymbol{D}}_K C_2 \\ \widehat{\boldsymbol{A}}_K & \boldsymbol{X}A+\widehat{\boldsymbol{B}}_K C_2 \end{bmatrix} + *\right) \prec 0.$$

$$\begin{bmatrix} \boldsymbol{Y} & I \\ I & \boldsymbol{X} \end{bmatrix} \succ 0$$

multiple constraints

- again for multiple constraints take the same X, Y and $\widehat{\boldsymbol{A}}_K$ $\widehat{\boldsymbol{B}}_K, \ldots$ for all LMIs.
- controller construction: just reverse the change of variables

For a single objective, LMI can be simplified,
Projection Lemma yields

$$\left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array}\right]' \left[\begin{array}{cc|c} A\boldsymbol{Y}+\boldsymbol{Y}A' & \boldsymbol{Y}C_1' & B_1 \\ C_1\boldsymbol{Y} & -\gamma I & D_{11} \\ \hline B_1' & D_{11}' & -\gamma I \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array}\right] \prec 0$$

$$\left[\begin{array}{c|c} \mathcal{N}_X & 0 \\ \hline 0 & I \end{array}\right]' \left[\begin{array}{cc|c} A'\boldsymbol{X}+\boldsymbol{X}A & \boldsymbol{X}B_1 & C_1' \\ B_1'\boldsymbol{X} & -\gamma I & D_{11}' \\ \hline C_1 & D_{11} & -\gamma I \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_X & 0 \\ \hline 0 & I \end{array}\right] \prec 0$$

$$\begin{bmatrix} \boldsymbol{Y} & I \\ I & \boldsymbol{X} \end{bmatrix} \succ 0 .$$

$\mathcal{N}_Y$ and $\mathcal{N}_X$ null spaces of $\begin{bmatrix} B_2' & D_{12}' \end{bmatrix}$ and $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$,

avantages of LMI formulations

- very general wrt DGKF, no assumptions required
- singular problems
- admits similar discrete-time counterpart
- has educational value for students (shorter proofs)
- See http://www.cert.fr/dcsd/cdin/apkarian/ for details
- See MATLAB LMI Control Toolbox for codes.

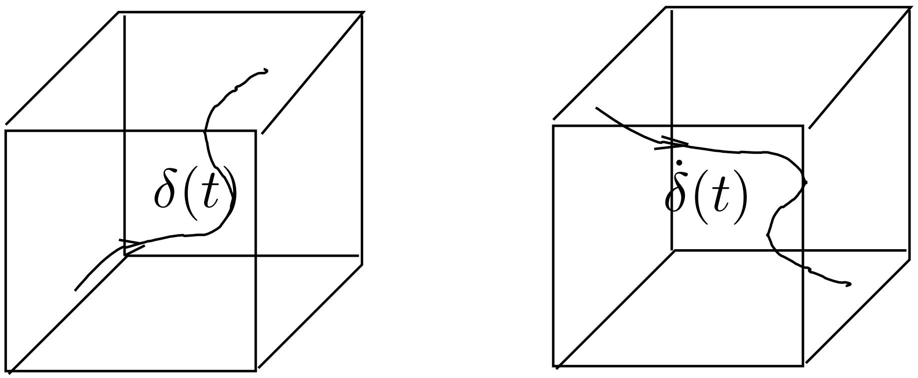
uncertain systems analysis

- Lyapunov technique
- Time-invariant and time-varying parameters
- Parameter-dependent Lyapunov functions.

Consider the uncertain system

$$\frac{d}{dt}x(t) = A(\delta)\, x(t); \qquad x(0) = x_0$$

- $\delta = [\delta_1, \ldots, \delta_L]' \in \mathbf{R}^L$ uncertain and possibly time-varying real parameters
- $A(\delta) = A_0 + \delta_1 A_1 + \ldots + \delta_L A_L$



is the system stable for all admissible $\delta(t)$?

Affine Quadratic Stability (AQS)

The system is Affinely Quadratically Stable, if $\exists$

$$V(x,\delta) \; := x' P(\delta) x, \qquad P(\delta) = P_0 + \delta_1 P_1 + \ldots + \delta_L P_L$$

s. t. $\quad V(x,\delta) > 0, \qquad dV/dt < 0$ along all admissible parameter trajectories.

- Lyapunov theory $\Rightarrow$ (exponential) stability.

$$\begin{aligned} P(\delta) \; &:= P_0 + \delta_1 P_1 + \ldots + \delta_L P_L && > \; 0 \\ L(\delta, \tfrac{d}{dt}\delta) \; &:= A(\delta)' P(\delta) + P(\delta) A(\delta) + \tfrac{dP(\delta)}{dt} && < \; 0 \end{aligned}$$

- turned into LMIs $\Rightarrow$ **multi-convexity, S-procedure ,...** !

analysis - continued

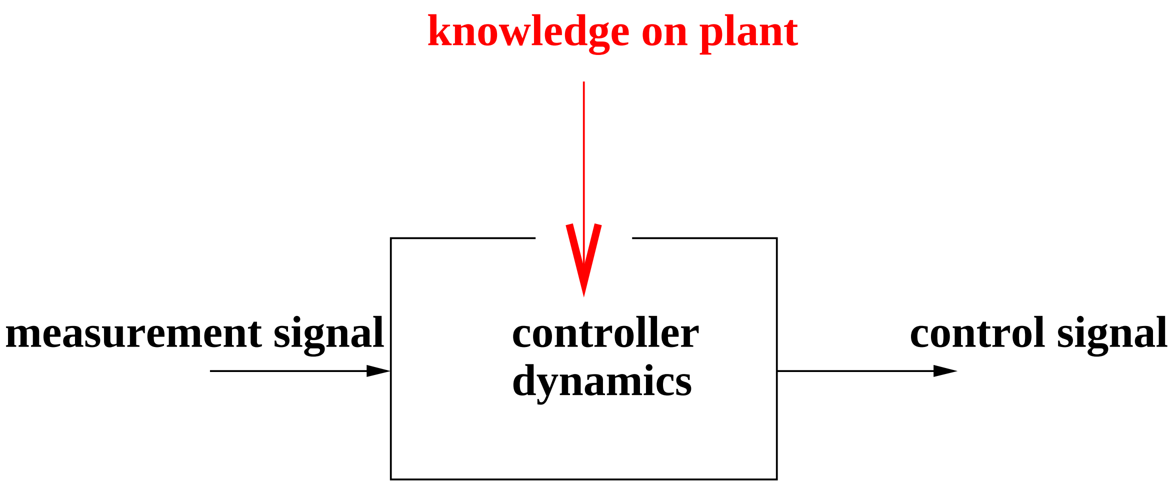
- **cases**
  Time-Invariant
  Parameters
  Arbitrary rate of variation
  (quad. stab.)

- **extensions**
  H_∞, H_2, LMI regions,...

- **components**
  LFT uncertainties
  nonlinear components
  (IQC theory, (Rantzer & Megretsky)
  μ analysis

- motivations and concepts
- classes of LPV system
- synthesis conditions for LFT systems

- handle full operating range
- gain-scheduled controllers exploit knowledge on the plant's dynamics in real time



controller mechanism is changed during operation

Gain-Scheduling techniques are applicable to

- Linear Parameter-Varying Systems (LPV):

$$\begin{aligned} \frac{d}{dt}x &= A(\theta)x + B(\theta)u \ , \\ y &= C(\theta)x + D(\theta)u. \end{aligned}$$

where $\theta := \theta(t)$ is an exogenous variable.

- " Quasi-Linear" Systems:

$$\begin{aligned} \frac{d}{dt}x &= A(y_{\text{sche}})x + B(y_{\text{sche}})u \ , \\ y &= C(y_{\text{sche}})x + D(y_{\text{sche}})u. \end{aligned}$$

where y_{sche} is a sub-vector of the plant's output y.

motivations $\#3$

- to get higher performance
- some LPV system are not stabilizable via a fixed LTI controller
- bypass critical phases of pointwise interpolation and switching
- engineering insight is preserved (freeze scheduled variable for analysis).
- nonlinear models can be handled by immersion into an LPV plant.

- **Aeronautics** (longitudinal motion of aircraft)

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -Z_\alpha & 0 \\ -m_\alpha & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ m_\delta \end{bmatrix} \delta, \qquad \begin{bmatrix} a_z \\ q \end{bmatrix} = \begin{bmatrix} -Z_\alpha V & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix},$$

  where Z_α, m_α and m_δ are functions of speed, altitude and angle of attack.

- **Robotics** (flexible two-link manipulator)

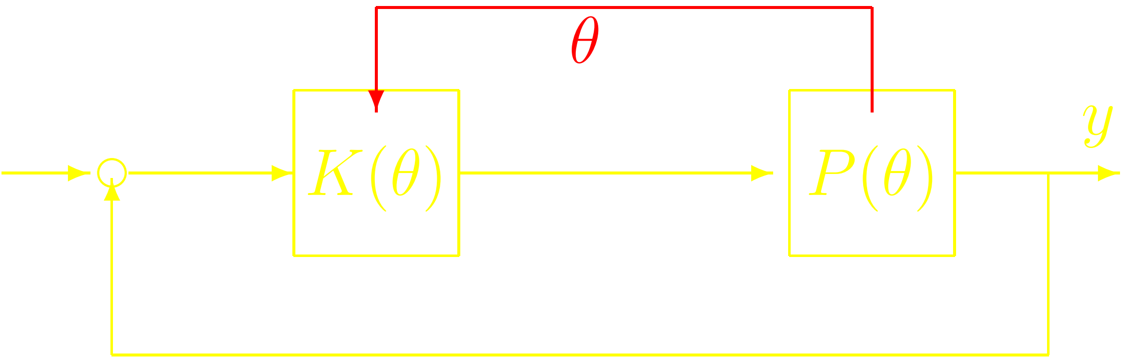
$$M(\theta_2)\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Fu(t),$$

  where θ_2 is the scheduled variable (conf. of 2nd beam).

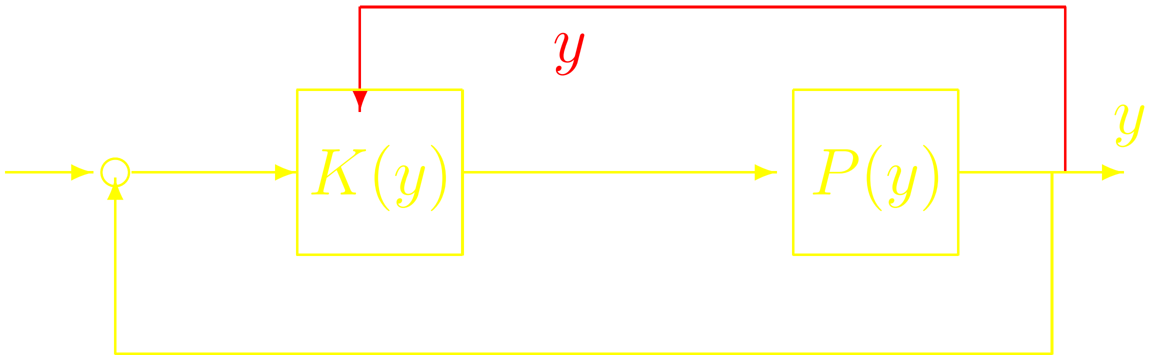
- and many others

example: different control principles

Robust control



LPV control



Output gain-scheduling

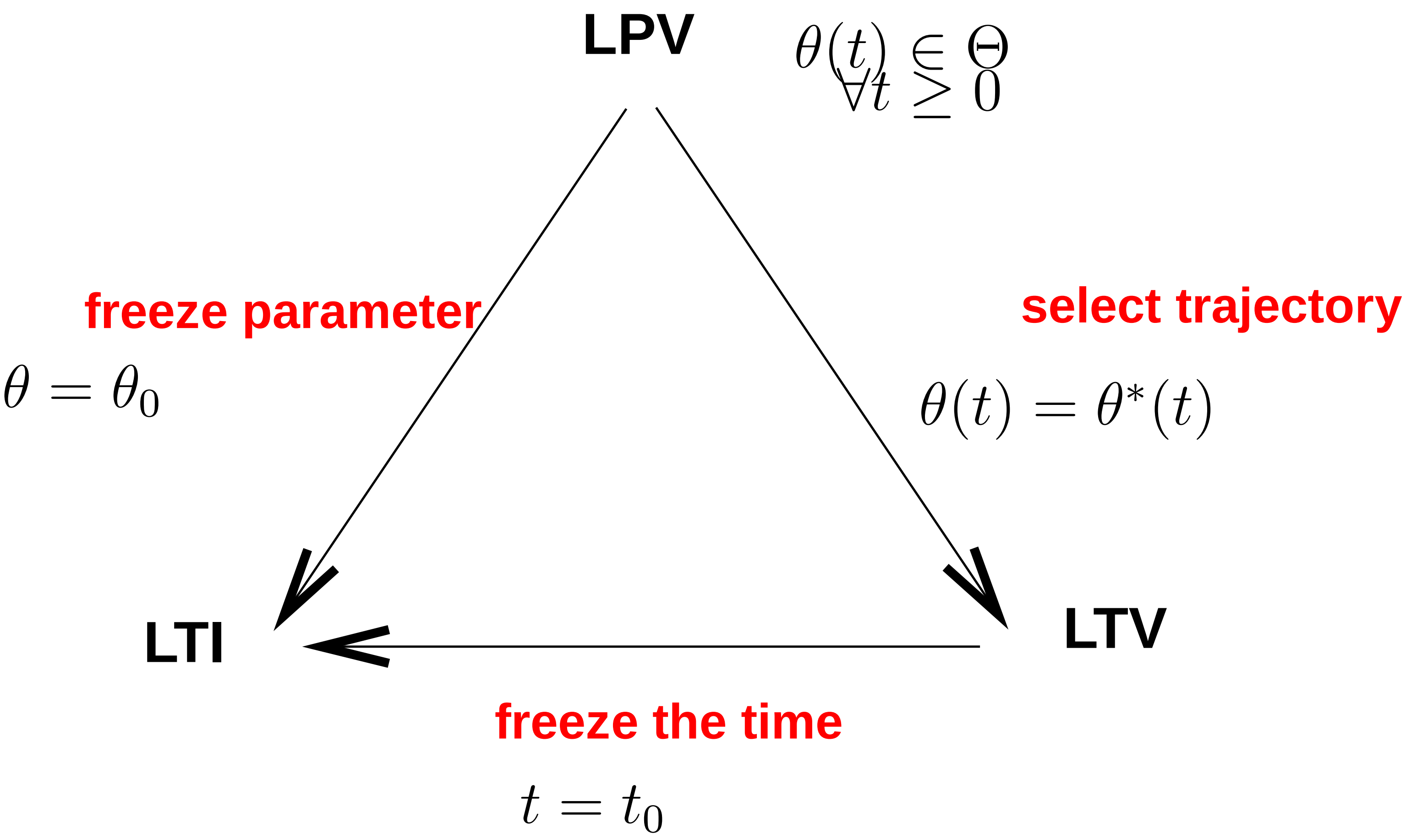
description of LPV systems

- LPV systems

$$\begin{aligned} \dot{x} &= A(\theta)x + B(\theta)u \ , \\ y &= C(\theta)x + D(\theta)u. \end{aligned}$$

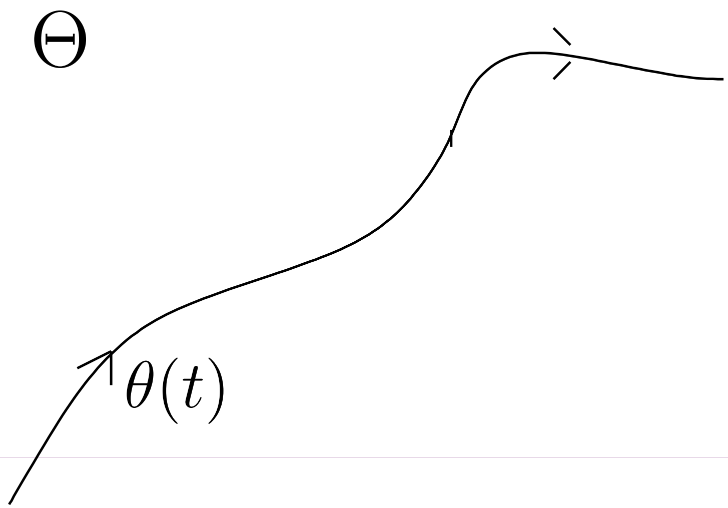
are characterized by

- the functional dependence of $\begin{bmatrix} A() & B() \\ C() & D() \end{bmatrix}$ on θ,
- the operating domain Θ of the system trajectories, $\theta(t) \in \Theta$,
- the rate of variations of $\theta(t)$ (if available) in the form of bounds $\dot{\theta}_i(t) \in [\underline{\theta}_i;\ \bar{\theta}_i]$.


LPV
$\theta(t) \in \Theta$
$\forall t \geq 0$
freeze parameter
$\theta = \theta_0$
select trajectory
$\theta(t) = \theta^*(t)$
LTI
LTV
freeze the time
$t = t_0$

LPV / LTV / LTI - off-line vs. on-line

- LTI and LTV systems are off-line systems, the state-space data A, B,... and $A(t)$, $B(t)$,... must be known in advance.
- LPV systems are on-line systems since the dynamics depend on the trajectory $\theta(t)$ experienced by the plant in Θ.



LPV systems interpretations

$$\begin{aligned}\dot{x} &= A(\theta)x + B(\theta)u, \qquad \theta(t) \in \Theta \\ y &= C(\theta)x + D(\theta)u.\end{aligned}$$

- θ may be subject to various assumptions:
  - $\theta(t)$ is uncertain $\longrightarrow$ robust control problem,
  - $\theta(t)$ is known in real-time $\longrightarrow$ Gain-scheduling problem,
  - $\theta(t) := \begin{bmatrix}\theta_1(t) \\ \theta_2(t)\end{bmatrix}$, where θ_1 is known and θ_2 is uncertain $\longrightarrow$ mixed problem

LPV pathologies - LPV/LTI stabilities

- stability over a domain
  - LTI Stability : $\text{Re}\lambda_i(A(\theta)) < 0, \ \forall\theta \in \Theta$,
  - LPV Stability : $\Phi_\theta(t) \longrightarrow 0$, for $t \longrightarrow \infty$, for all trajectory $\theta(t)$ in Θ.
- intuitive conjectures like
  - LTI stability $\Rightarrow$ LPV stability,
  - LPV stability $\Rightarrow$ LTI stability,

are FALSE !

LPV vs. LTI Stability

- Conjecture $\#1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 + a\theta_1^2 & 1 + a\theta_1\theta_2 \\ -1 + a\theta_1\theta_2 & -1 + a\theta_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

with trajectories $\theta_1 := \cos(t)$ and $\theta_2(t) := \sin(t)$ is LTI stable (for $a < 2$) but LPV unstable.

- Conjecture $\#2$

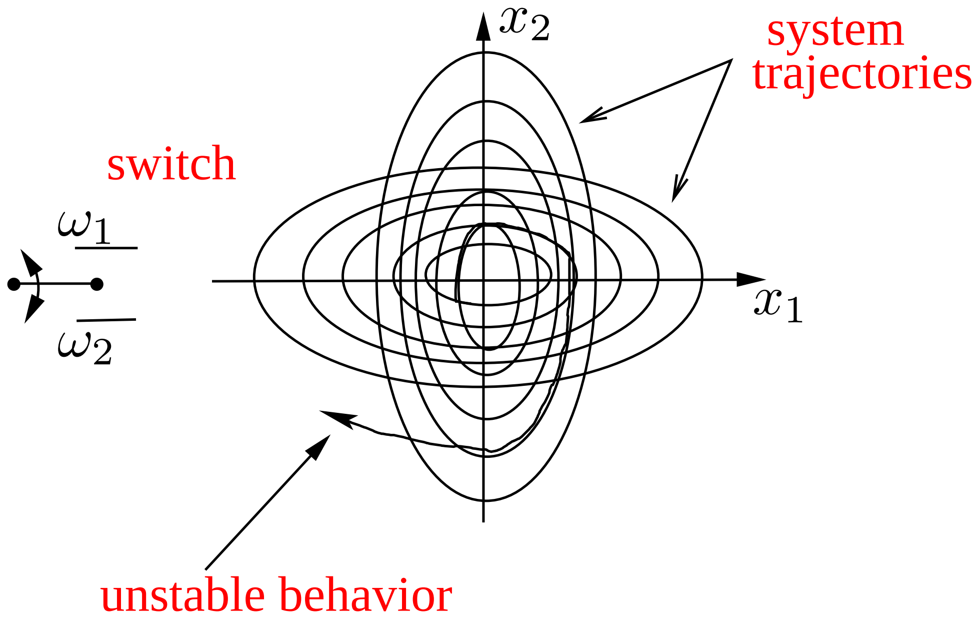
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 - 5\theta_1\theta_2 & 1 - 5\theta_1^2 \\ -1 + 5\theta_2^2 & -1 + 5\theta_1\theta_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

with trajectories $\theta_1 := \cos(t)$ and $\theta_2(t) := \sin(t)$ is LTI unstable (poles $+1$ and -3) but LPV stable.

consider the autonomous LPV system:

$$\ddot{x} + \omega^2(t)x = 0\,,$$

where we are allowed to switch between two values ω_1 and ω_2.



- **Sufficient stability cond.**

(1) $\text{Re}\lambda_i(A(\theta)) < 0$,
(2) $\|\dot{\theta}\| < \alpha$, with α sufficiently small,
$\Rightarrow$ LPV stability (Rosen. 63)

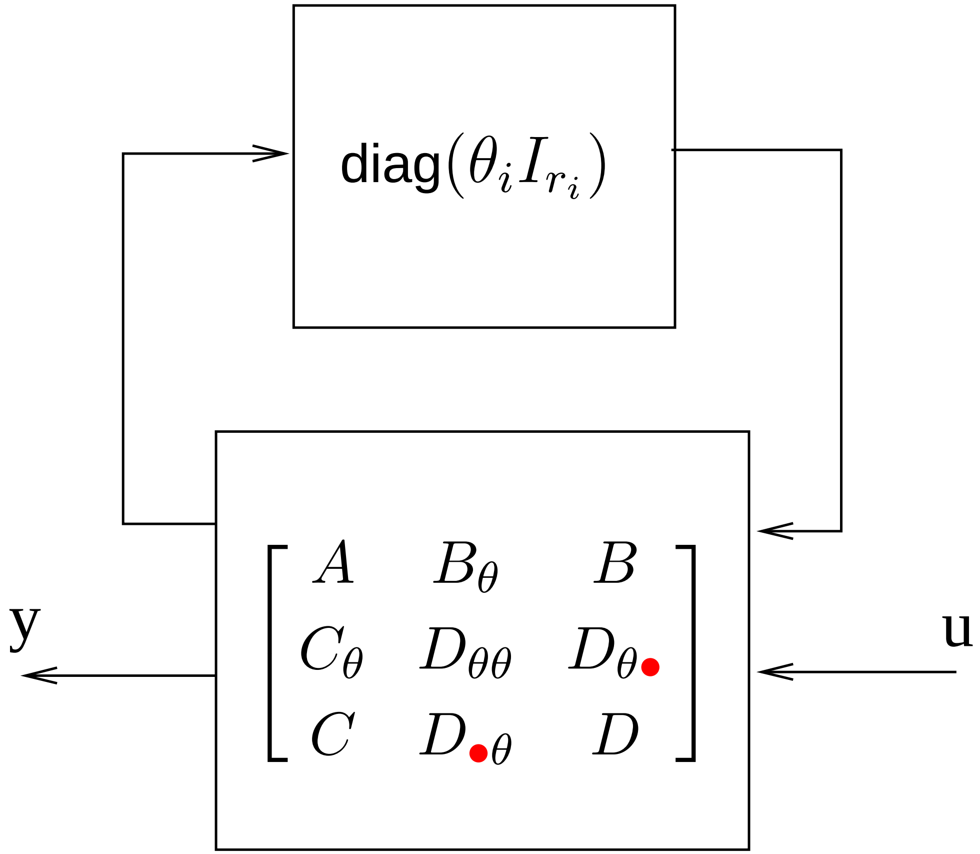
- **Sufficient instability cond.**

(1) $\text{Re}\lambda_i(A(\theta)) < 0$, $i = 1, \ldots, k$
(2) $\text{Re}\lambda_i(A(\theta)) > 0$, $i = k+1, \ldots, n$
(3) stable and unstable eigenvalues do not mix
(4) $\|\dot{\theta}\| < \alpha$, with α sufficiently small,

$\Rightarrow$ LPV instability (Skoog 72)

LPV stability can be inferred from LTI stability for slowly varying parameters (but not constructive conditions).

LPV systems in the LFT class

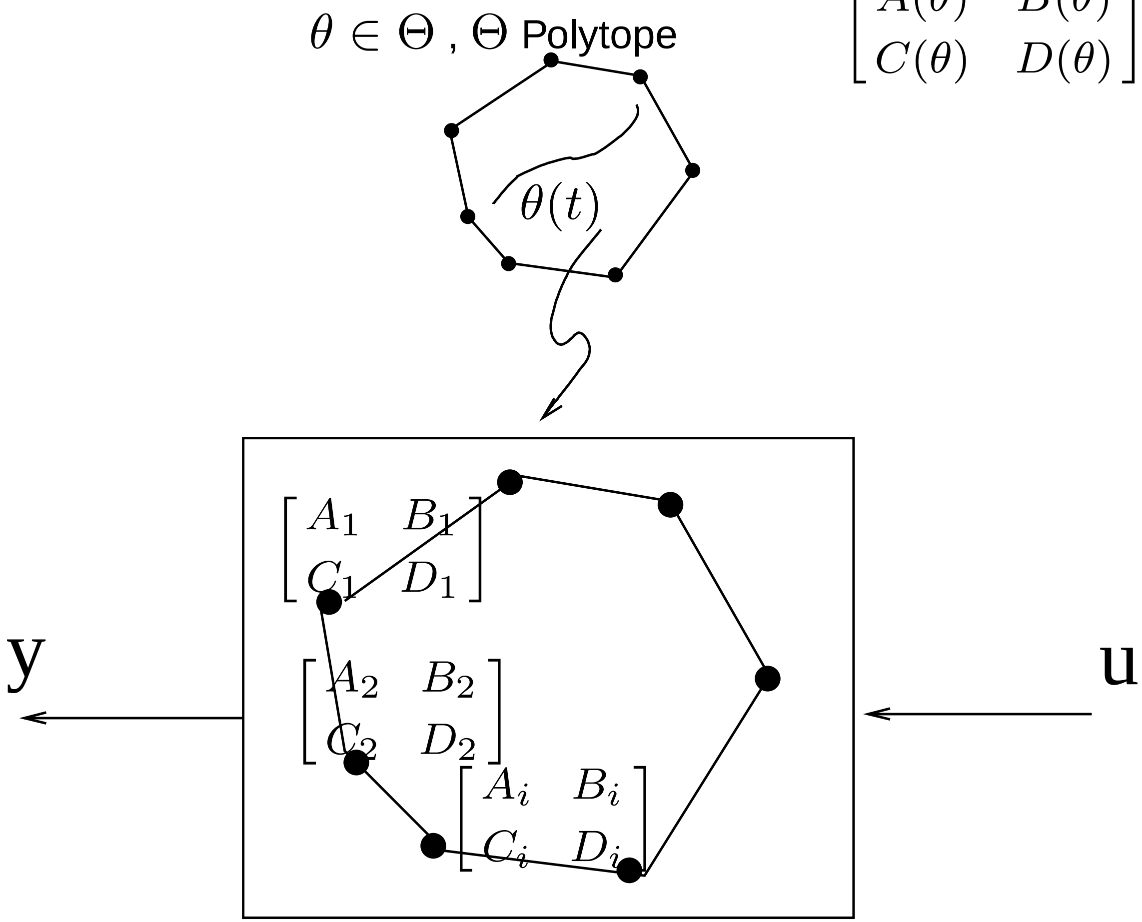


$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} B_\theta \\ D_{\bullet\theta} \end{bmatrix} \Theta(I - D_{\theta\theta}\Theta)^{-1} \begin{bmatrix} C_\theta & D_{\theta\bullet} \end{bmatrix},$$

where

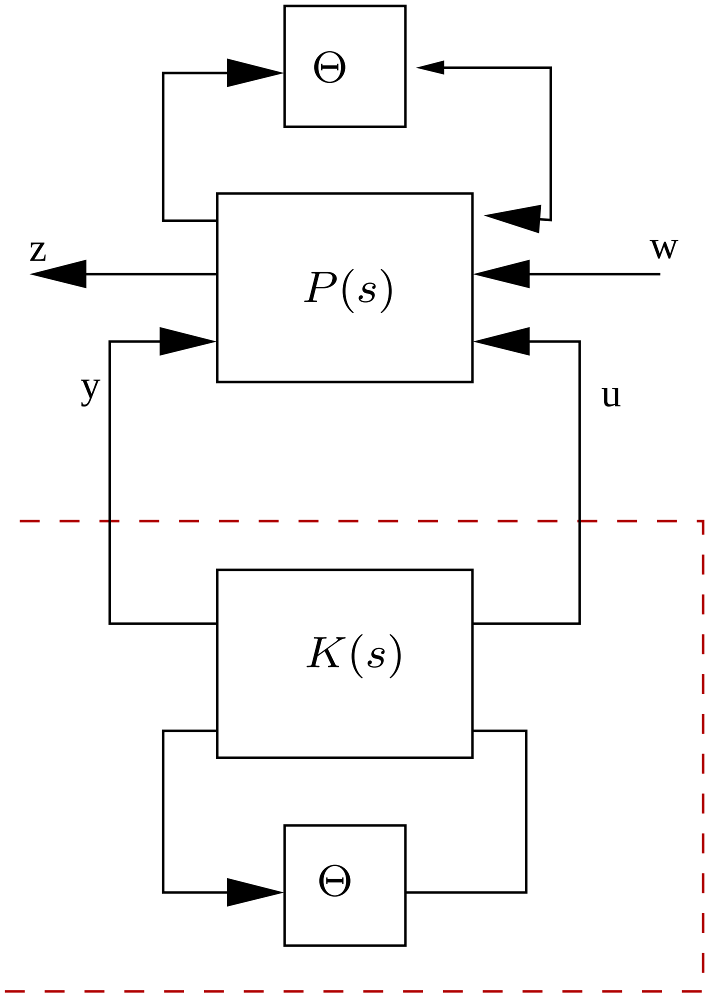
LPV systems in the polytopic class

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} \in \text{Cov} \left\{ \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, i = 1, 2, \ldots, r \right\}$$



$A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$ are arbitrary but continuous matrix-valued function of θ.

- far more difficult to handle but of great practical interest since they capture arbitrary nonlinearities



gain−scheduled controller

find LPV controller $F_l(K(s), \Theta(t))$ s.t.

- closed-loop stability,
- the L_2-induced norm of the operator $T_{w\to z}$ satisfies $\|T_{w\to z}(\Theta)\| < \gamma$

for all admissible trajectory $\theta(t)$.

-

$$P(s) = \begin{bmatrix} D_{\theta\theta} & D_{\theta 1} & D_{\theta 2} \\ D_{1\theta} & D_{11} & D_{12} \\ D_{2\theta} & D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_\theta \\ C_1 \\ C_2 \end{bmatrix} (sI-A)^{-1} \left[\, B_\theta \quad B_1 \quad B_2 \, \right] ,$$

Assumptions: (A, B_2, C_2) stabilizable and detectable, $D_{22} = 0$.

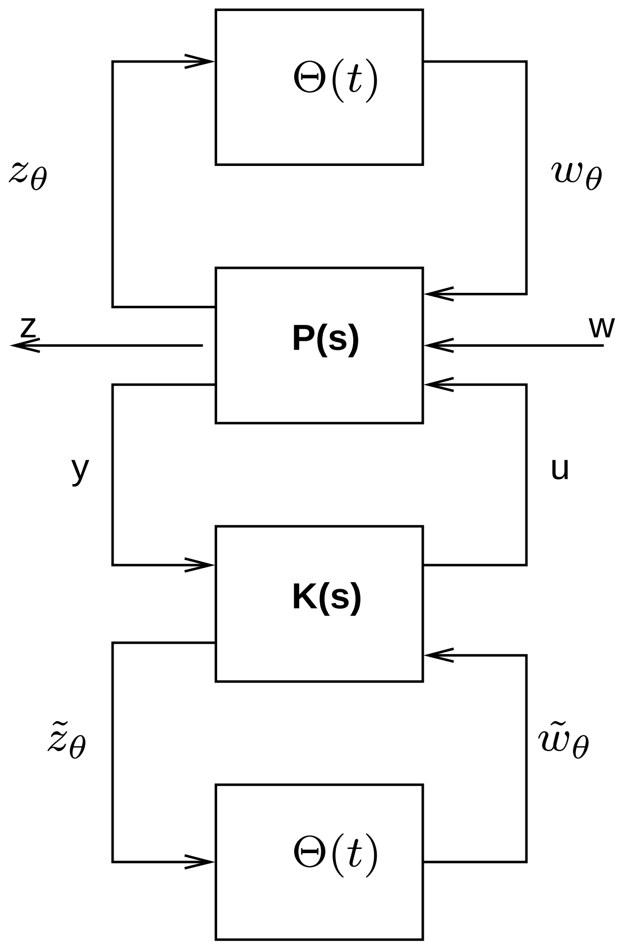
Notations: $\widehat{B}_1 = [\, B_\theta \quad B_1 \,]$, $\widehat{C}_1 = \begin{bmatrix} C_\theta \\ C_1 \end{bmatrix}$, $\widehat{D}_{11} = \begin{bmatrix} D_{\theta\theta} & D_{\theta 1} \\ D_{1\theta} & D_{11} \end{bmatrix}$,

$\mathcal{N}_Y := \mathrm{Ker}\,[\, B_2^T \quad D_{\theta 2}^T \quad D_{12}^T \quad 0 \,]$,

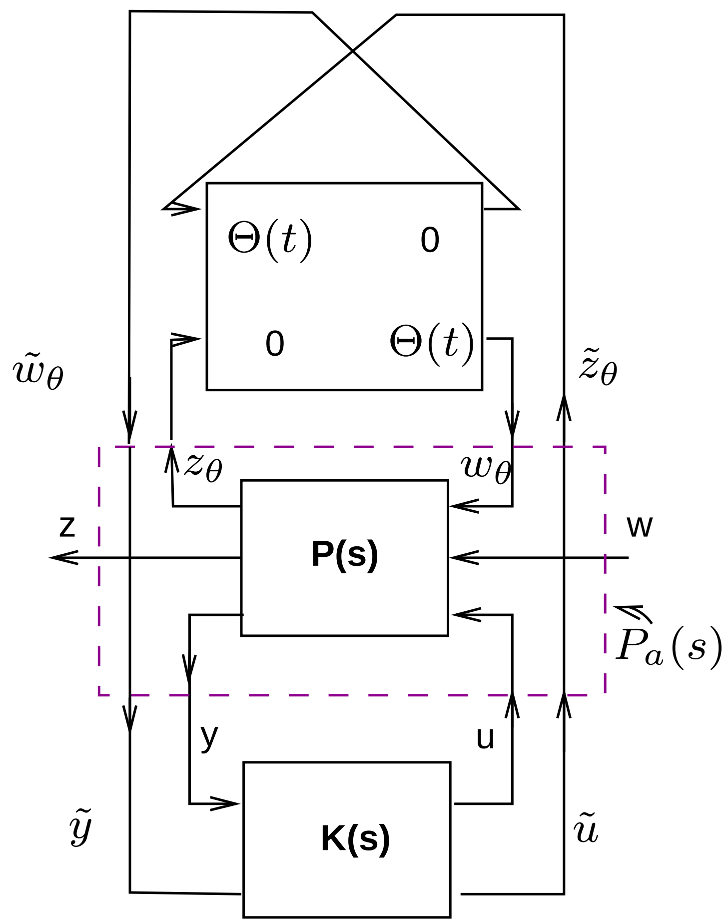
$\mathcal{N}_X := \mathrm{Ker}\,[\, C_2 \quad D_{2\theta} \quad D_{21} \quad 0 \,]$.

LPV-LFT systems - proof scheme

synthesis structure



synthesis structure
with parameter augmentation



LPV-LFT systems - proof scheme

- redraw the control configuration into a robust control problem with repeated uncertainty,
- formulate the Bounded Real Lemma with scalings for the closed-loop system,
- apply the Projection Lemma to derive the LMI characterization.

$$\mathcal{N}_Y^T \begin{bmatrix} AY + YA^T & \star & \star & \star & \star \\ C_\theta Y + \Gamma_3 B_\theta^T & -\Sigma_3 + \Gamma_3 D_{\theta\theta}^T - D_{\theta\theta}\Gamma_3 & \star & \star & \star \\ C_1 Y & -D_{1\theta}\Gamma_3 & -\gamma I & \star & \star \\ \Sigma_3 B_\theta^T & \Sigma_3 D_{\theta\theta}^T & \Sigma_3 D_{1\theta}^T & -\Sigma_3 & \star \\ B_1^T & D_{\theta 1}^T & D_{11}^T & 0 & -\gamma I \end{bmatrix} \mathcal{N}_Y \prec 0,$$

$$\mathcal{N}_X^T \begin{bmatrix} A^T X + XA & \star & \star & \star & \star \\ B_\theta^T X + T_3 C_\theta & -S_3 + T_3 D_{\theta\theta} - D_{\theta\theta}^T T_3 & \star & \star & \star \\ B_1^T X & -D_{\theta 1}^T T_3 & -\gamma I & \star & \star \\ S_3 C_\theta & S_3 D_{\theta\theta} & S_3 D_{\theta 1}^T & -S_3 & \star \\ C_1 & D_{1\theta} & D_{11} & 0 & -\gamma I \end{bmatrix} \mathcal{N}_X \prec 0,$$

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} \succeq 0$$

$$S_3 \succ 0,\ \Sigma_3 > 0;\ T_3,\ \Gamma_3 \ \text{skew} - \text{symmetric}\,.$$

- symmetric

$$S_\Theta := \{S : S > 0,\ S\Theta = \Theta S\}$$

- symmetric augmented

$$S_{\boldsymbol{\Theta}\oplus\boldsymbol{\Theta}} = \left\{ \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} : S_1,\ S_2 \in S_{\boldsymbol{\Theta}} \text{ and } S_2\Theta = \Theta S_2, \forall \Theta \in \boldsymbol{\Theta} \right\}.$$

- skew-symmetric

$$T_{\boldsymbol{\Theta}\oplus\boldsymbol{\Theta}} = \left\{ \begin{bmatrix} T_1 & T_2 \\ -T_2^T & T_3 \end{bmatrix} : T_1,\ T_2 \in T_{\boldsymbol{\Theta}} \text{ and } T_2\Theta = \Theta T_2, \forall \Theta \in \boldsymbol{\Theta} \right\}.$$

$$\begin{bmatrix} A_{c\ell}^T X_{c\ell} + X_{c\ell} A_{c\ell} & X_{c\ell} B_{c\ell} + C_{c\ell}^T T^T & C_{c\ell}^T \\ B_{c\ell}^T X_{c\ell} + T C_{c\ell} & -S + T D_{c\ell} + D_{c\ell}^T T^T & D_{c\ell}^T \\ C_{c\ell} & D_{c\ell} & -S^{-1} \end{bmatrix} \prec 0$$

where

- $A_{c\ell}, B_{c\ell}, \ldots$ closed-loop data
- S, T scalings for $\boldsymbol{\Theta} \otimes \boldsymbol{\Theta} \otimes \Delta$, and Δ fictitious performance block.

Can be rewritten

$$\Psi + Q_X^T \Omega P + P^T \Omega^T Q_X \prec 0,$$

where

$$\Psi = \begin{bmatrix} \mathcal{A}^T X_{c\ell} + X_{c\ell}\mathcal{A} & X_{c\ell}\mathcal{B}_1 + \mathcal{C}_1^T T^T & \mathcal{C}_1^T \\ \mathcal{B}_1^T X_{c\ell} + T\mathcal{C}_1 & -S + TD_{11} + D_{11}T^T & D_{11}^T \\ \mathcal{C}_1 & D_{11} & -S^{-1} \end{bmatrix},$$

$$P = \begin{bmatrix} \mathcal{C}_2 & \mathcal{D}_{21} & 0 \end{bmatrix}, \quad Q_X = \begin{bmatrix} \mathcal{B}_2^T X_{c\ell} & \mathcal{D}_{12}^T T^T & \mathcal{D}_{12}^T \end{bmatrix}.$$

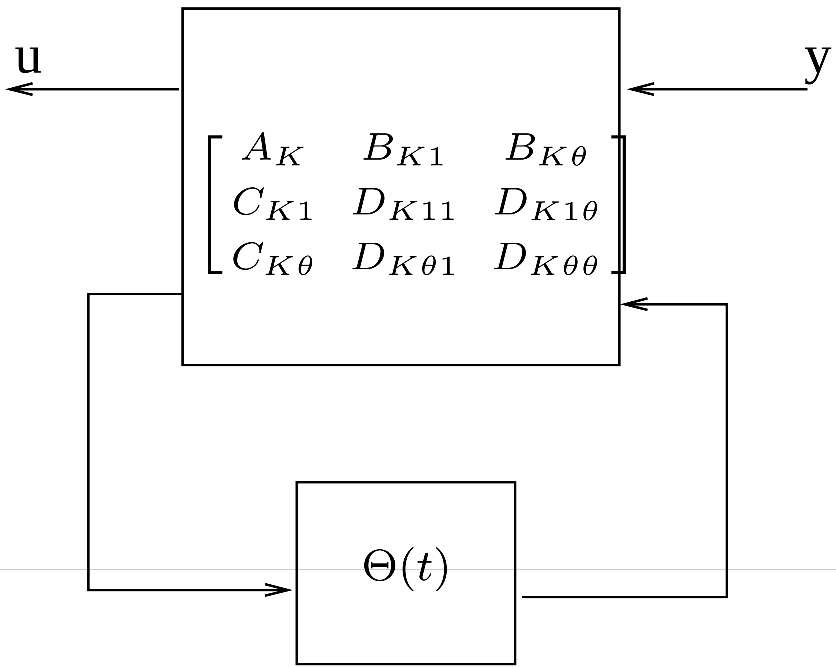
$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \Omega^T \end{bmatrix} = \left[\begin{array}{cc|ccc|ccc} A & 0 & 0 & B_\theta & B_1 & 0 & B_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ C_\theta & 0 & 0 & D_{\theta\theta} & D_{\theta 1} & 0 & D_{\theta 2} & 0 \\ C_1 & 0 & 0 & D_{1\theta} & D_{11} & 0 & D_{12} & 0 \\ \hline 0 & I & 0 & 0 & 0 & A_K^T & C_{K1}^T & C_{K\theta}^T \\ C_2 & 0 & 0 & D_{2\theta} & D_{21} & B_{K1}^T & D_{K11}^T & D_{K\theta 1}^T \\ 0 & 0 & I & 0 & 0 & B_{K\theta}^T & D_{K1\theta}^T & D_{K\theta\theta}^T \end{array}\right]$$

and

$$\widehat{B}_1 = [\, B_\theta \quad B_1 \,], \quad \widehat{C}_1 = \begin{bmatrix} C_\theta \\ C_1 \end{bmatrix}, \quad \widehat{D}_{11} = \begin{bmatrix} D_{\theta\theta} & D_{\theta 1} \\ D_{1\theta} & D_{11} \end{bmatrix}.$$

- LMI characterization follows from explicit computation of projections and using matrix completion Lemmas.

- Testing solvability falls within the scope of convex semi-definite programming
- A gain-scheduled controller is easily constructed from the quadruple (Y, X, L_3, J_3) by solving a scaled Bounded Real Lemma LMI condition.



other variants of this technique

- polytopic LPV systems
- general LPV systems (capture slow variations of parameters)
- LFT systems ang generalized scalings
- multi-objective/channel LPV synthesis

see webpage: **http://www.cert.fr/dcsd/cdin/apkarian/**

hard non-LMI problems

- most analysis problems reduce to LMIs
- some synthesis problems reduce to LMIs but
- many practical problems do not reduce to LMI/SDP (synthesis)

  - reduced- and fixed-order synthesis (PID H_∞, etc.)
  - structured and decentralized synthesis problems
  - general robust control with uncertain and/or nonlinear components
  - simultaneous model/controller design, multimodel control
  - unrelaxed LTI and LPV multi-objective
  - combinations of the above

new algorithms needed ! good research direction stabilize

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with $u = Ky$ (K static)

has characterization

$$\begin{aligned} \mathcal{N}_C'(A'X + XA)\mathcal{N}_C &< 0 \\ \mathcal{N}_{B'}'(YA' + AY)\mathcal{N}_{B'} &< 0 \\ \begin{bmatrix} X & I \\ I & Y \end{bmatrix} &> 0 \end{aligned}$$

$$XY - I = 0$$

constraints $XY - I = 0$ leads to hard problems

LMI + nonlinear equality constraints

with $g(x) = 0$ equ. constraints and $\mathcal{A}(x) \prec 0$ LMI, replace the difficult program by the more convenient

$$(P_{\lambda,\mu}) \qquad \begin{array}{ll} \text{minimize} & c'x + \lambda' g(x) + \frac{1}{\mu}\|g(x)\|^2 \\ \text{subject to} & \mathcal{A}(x) \preceq 0 \end{array}$$

- μ is penalty, $x_\mu \to x^*$ when $\mu \to 0$
- for *good* estimates λ (Lagrange multiplier), solution of $(P_{\lambda,\mu})$ is close to solution of original problem
- use first-order update rule to improve estimate λ
- solve $(P_{\lambda,\mu})$ by a succession of SDPs

- B. Fares and P. Apkarian and D. Noll, IJC, 2001
- B. Fares and D. Noll and P. Apkarian , SIAM Cont. Optim. 2002
- P. Apkarian and D. Noll and H. D. Tuan, 2002, IJRNC to appear.
- D. Noll and M. Torki and P. Apkarian, working paper, 2002

- A single framework for a great variety of methods
- LMI techniques extend the scope of classical techniques
- LPV control is a very successful example (industrial)
- Analysis meth. immediately applicable for validation
- Have educational merits
  see http://www.cert.fr/dcsd/cdin/apkarian/ for course plan
- not discussed: robust filtering and estimation, combinatorial optimization, graphs, etc.

recent concrete control applications

- Analysis robustness evaluation of controllers for:
  - ARIANE Launcher
  - satellites
  - long flexible civil aircraft (structural modes)
- Synthesis Preliminary tests show that LPV controllers are competitive for launcher control in atmospheric flight
- Synthesis control of the landing phase for civil aircraft under study with multiobjective LMI methods
- Synthesis Missiles ? still on paper

GRAZIE MILLE !