nonsmooth structured control design

P. APKARIAN (ONERA and UPS/Institut de Mathématiques de Toulouse)

V. BOMPART (ONERA UPS/Institut de Mathématiques de Toulouse)

D. NOLL (UPS/Institut de Mathématiques de Toulouse)

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outline

- motivation
- nonsmooth optimization
- controller design
- conclusion

context

- challenging practical design problems:
  - reduced- and fixed-order synthesis
  - structured and fixed-architecture synthesis problems
  - general robust control with uncertain and/or nonlinear components
  - simultaneous model/controller design, multimodel control
  - multi-objective frequency- and time-domain designs
  - combinations of the above, etc
- classical design methods (Riccati, LMI) fail on sizeable plants.
- D-K schemes and heuristic methods (homotopy,. . .) not satisfactory
- solutions based on LMI relaxations often very conservative

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an illustrative example

Boeing 767 at flutter condition

stabilization with static output feedback $u = Ky$:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

$A = 55 \times 55 \qquad B = 55 \times 2 \qquad C = 2 \times 55 \qquad K = 2 \times 2$

- BMI: $(A + BKC)^T P + P(A + BKC) \prec 0 \quad \text{et} \quad P = P^T \succ 0$
  $\Rightarrow$ 1544 variables most of them (1540) for Lyapunov matrix P
- $\min_K \alpha(A + BKC)$ where $\alpha \triangleq \max_i \operatorname{Re} \lambda_i$ is spectral abscissa
  $\Rightarrow$ 4 variables : controller parameters

$\Rightarrow$ recast controller design as a nonsmooth program

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controller design as nonsmooth (minimax) optimization

- stabilization

$$\min_K \max_i \operatorname{Re} \lambda_i(A + BKC)$$

- H_∞ synthesis

$$\min_K \max_{\omega \in [0,+\infty]} \max_i \sigma_i\big(T_{w \to z}(K, j\omega)\big)$$

- time-domain design

$$\min_K \max_{t \in [0,+\infty]} \max\left\{\big(z(K,t) - z_{max}(t)\big)^+, \big(z_{min}(t) - z(K,t)\big)^+\right\}$$

- generic BMI problem

$$\min_x \max_i \lambda_i\left(A_0 + \sum_k x_k A_k + \sum_k \sum_l x_k x_l B_{kl}\right)$$

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controller design as nonsmooth optimization

- these optimization problems are nonsmooth and/or nonconvex

- special composite structure

$$\max_{x \in X} \lambda_1 \circ T(K, x)$$

where $\max_{x \in X} \lambda_1$ is convex and $T(K, x)$ differentiable wrt. K, $\forall x$

Clarke subdifferential

Clarke regularity $\Rightarrow$ specialized and efficient techniques can be developed

- **directly** formulated in controller parameter space (no Lyapunov variables)
- **flexible**: to handle controller structural constraints
- **efficient**: to handle large control problems

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subdifferential and subgradients

nonsmooth minimax optimization problems of the form

$$\min_{K} \underbrace{\max_{x \in X} f(K,x)}_{\triangleq f_\infty(K)}$$

X stands for:

- a finite set of indices : $X = \{1, \cdots, q\}$

$$\min_{K} \max_{i} \operatorname{Re} \lambda_i(A + BKC)$$

- a time- or frequency-domain closed interval : $X = [0, +\infty]$, $X = [0, t_{max}]$, $X = [\omega_1, \omega_2]$, $\cdots$

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assume (for simplicity) that f is continuous and $K \mapsto f(K,x)$ is $\mathcal{C}^1$ $\forall x \in X$.
then f_∞ is loc. Lipschitz and admits a Clarke subdifferential $\forall K$:

$$\partial f_\infty(K) = \underset{x \in \hat{X}(K)}{\text{co}} \nabla_K f(K,x)$$

where $\hat{X}(K) \triangleq \{x \in X \; : \; f_\infty(K) = f(x,K)\}$ active set.
vectors $\phi \in \partial f_\infty(K)$ are Clarke subgradients of f_∞ at K.

nonsmooth optimality condition

Theorem ([Clarke, 1983])

$K^\star$ is a local minimum of $f_\infty \implies 0 \in \partial f_\infty(K^\star)$

- define optimality function θ to check whether $0 \in \partial f_\infty(K)$ satisfied at K.
- if not, find a descent step $H(K)$ for $f_\infty(K)$, i.e. $\exists t_K > 0$ such that, for all $t \in (0, t_K]$

$$f_\infty(K + tH(K)) < f_\infty(K)$$

- steepest descent:

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$$H(K) \triangleq -\arg \underbrace{\min_H \max_{\phi \in \partial f_\infty(K)} \langle \phi, H \rangle + \tfrac{1}{2}\|H\|^2}_{\triangleq \theta_1(K)}$$

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$\Rightarrow K \mapsto H(K)$ not continuous ! convergence problem.

nonsmooth descent direction

$$H(K) \triangleq \arg \min_{H} \max_{x \in \hat{X}_e(K)} \underbrace{f(K,x) + \langle \nabla_K f(K,x), H \rangle + \tfrac{1}{2}\delta \|H\|^2}_{\text{1st-order quadratic local model of } f(\cdot,x) \text{ at } K} - f_\infty(K)$$

$\delta > 0$, $\hat{X}_e(K) \supset \hat{X}(K)$, finite extension of active set

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Proposition ([Apkarian and Noll, 2006-I])

- $\theta(K) \leq 0$ and $\theta(K) = 0 \iff 0 \in \partial f_\infty(K)$
  $\theta(K)$ criticality measure of K for f_∞
- $\theta(K)$ **termination test and (local) optimality certificate**
- $K \mapsto \theta(K)$ continuous
- $f'_\infty(K; H(K)) \leq \theta(K) - \frac{1}{2\delta}\|H(K)\|^2 < 0$
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nonsmooth optimality condition

- primal QP problem

$$\underbrace{\min_{H} \max_{x \in \hat{X}_e(K)} f(K,x) + \langle \nabla_K f(K,x), H \rangle + \tfrac{1}{2}\delta \|H\|^2 - f_\infty(K)}_{\text{optimality function } \theta(K)}$$

- dual QP problem (equivalent)

$$\theta(K) = - \min_{\substack{\tau_x \geq 0 \\ \sum \tau_x = 1}} \sum_{x \in \hat{X}_e(K)} \tau_x \left(f_\infty(K) - f(K,x)\right) + \frac{1}{2\delta} \left\| \sum_{x \in \hat{X}_e(K)} \tau_x \nabla_K f(K,x) \right\|^2$$

with

$$H(K) = -\frac{1}{\delta} \sum_{x \in \hat{X}_e(K)} \tau_x^\star \nabla_K f(K,x)$$

- select primal or dual with smaller dimension

nonsmooth optimality condition

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nonsmooth descent algorithm

Algorithm ([Apkarian and Noll, 2006-I, Polak, 1997])

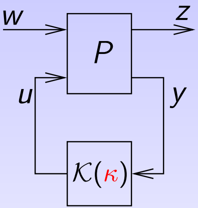
set $\beta \in (0,1)$, $\delta > 0$, $\varepsilon_\theta > 0$.

Initialize with controller K_0.

1. set counter $l \leftarrow 0$
2. find active set $\hat{X}(K_l)$.
3. build a finite extension $\hat{X}_e(K_l)$.
4. compute the optimality function value (convex QP) $\theta(K_l)$ and the search direction $H(K_l)$.
5. if $|\theta(K_l)| < \varepsilon_\theta$ (criticality test), stop.
   else compute the step-size t_l such that
   $$f_\infty(K_l + t_l H(K_l)) - f_\infty(K_l) \leq t_l \beta \theta(K_l)$$
6. set $K_{l+1} \leftarrow K_l + t_l H(K_l)$, $l \leftarrow l + 1$ and go to step 2.

synthesis formulation

$$P(s)\left\{\begin{bmatrix}\dot{x}\\ z\\ y\end{bmatrix} = \begin{bmatrix}A & B_1 & B_2\\ C_1 & D_{11} & D_{12}\\ C_2 & D_{21} & 0\end{bmatrix}\begin{bmatrix}x\\ w\\ u\end{bmatrix}\right.$$



- $\mathcal{K}(\kappa)$ defines controller structure
- find κ free parameters to achieve stability, frequency- or time-domain constraints

structured controller design

controller structural constraints

$$\mathcal{K}(\kappa) \quad \text{with } \kappa \in \mathbb{R}^q \text{ vector of free controller parameters}$$

structured controller design

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decentralized control (N subcontrollers)

$$\mathcal{K}(\kappa) = \left[\begin{array}{cccc|cccc} A_K^1 & & & & B_K^1 & & & \\ & A_K^2 & & & & B_K^2 & & \\ & & \ddots & & & & \ddots & \\ & & & A_K^N & & & & B_K^N \\ \hline C_K^1 & & & & D_K^1 & & & \\ & C_K^2 & & & & D_K^2 & & \\ & & \ddots & & & & \ddots & \\ & & & C_K^N & & & & D_K^N \end{array}\right]$$

with $\kappa = \left[(\mathrm{vec}\, A_K^1)^T \quad \dots \quad (\mathrm{vec}\, D_K^N)^T \right]^T$.

structured controller design

controller structural constraints

$\mathcal{K}(\kappa)$ with $\kappa \in \mathbb{R}^q$ vector of free controller parameters

PID control ($m_2 \times m_2$)

$$\mathcal{K}(\kappa) = \left[\begin{array}{cc|c} 0 & 0 & R_i \\ 0 & -\tau I_{m_2} & R_d \\ \hline I_{m_2} & I_{m_2} & D_K \end{array}\right]$$

such that $K(s) = D_K + \frac{R_i}{s} + \frac{R_d}{s+\tau}$.

here $\kappa = \left[\tau \quad (\text{vec}\, R_i)^T \quad (\text{vec}\, R_d)^T \quad (\text{vec}\, D_K)^T\right]^T$.

structured controller design

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$\mathcal{K}(\kappa)$ with $\kappa \in \mathbb{R}^q$ vector of free controller parameters

observer-based control (LQG)

$$\mathcal{K}(\kappa) = \left[\begin{array}{c|c} A - B_2 K_c - K_f C_2 & K_f \\ \hline -K_c & 0 \end{array}\right]$$

K_f state estimator gain,
K_c state feedback gain.
$\kappa = \left[(\mathrm{vec}\, K_f)^T \quad (\mathrm{vec}\, K_c)^T \right]^T$

structured controller design

controller structural constraints

$\mathcal{K}(\kappa)$ with $\kappa \in \mathbb{R}^q$ vector of free controller parameters

fractional representations

$$K(s,\kappa) = (N_m s^m + \ldots + N_0)(D_n s^n + \ldots + D_0)^{-1}$$

N_i numerator coefficients
D_j denominator coefficients
$\kappa = \left[\ldots(\mathrm{vec}\, N_i)^T \ldots (\mathrm{vec}\, D_j)^T \ldots\right]^T$

etc

any differentiable possibly nonlinear $\mathcal{K}(\kappa)$

structured controller design

controller structural constraints

$$\mathcal{K}(\kappa) \quad \text{with } \kappa \in \mathbb{R}^q \text{ vector of free controller parameters}$$

the nonsmooth problem becomes

$$\boxed{\min_{\kappa \in \mathbb{R}^q} \underbrace{\max_{x \in X} f(\mathcal{K}(\kappa), x)}_{f_\infty(\mathcal{K}(\kappa))}}$$

chain rule for subdifferentials with $\mathcal{K} \in \mathcal{C}^1(\mathbb{R}^q)$:

$$\partial(f_\infty \circ \mathcal{K})(\kappa) = \mathcal{K}'(\kappa)^\star \left[\partial f_\infty(\mathcal{K}(\kappa))\right]$$

subgradients $\psi = J_{\mathcal{K}}(\kappa)^T \phi$
where $J_{\mathcal{K}}$ Jacobian matrix of $\mathcal{K}$, and $\phi \in \partial f_\infty(\mathcal{K}(\kappa))$

stabilization

$$\min_K \underbrace{\max_i \text{Re } \lambda_i(A + B_2KC_2)}_{f_\infty(K)}$$

minimize spectral abscissa $\alpha \triangleq \max_i \text{Re } \lambda_i$.

- active set of closed-loop eigenvalues indices :

$$\hat{X}(K) = \{i = 1 \dots n \ : \ \text{Re } \lambda_i(A + B_2KC_2) = \alpha(A + B_2KC_2)\}$$

  extended set $\hat{X}_e(K) \supset \hat{X}(K)$ with neighboring eigenvalues.
- working assumption: all the closed-loop eigenvalues λ_i, $i \in \hat{X}_e(K)$, are simple.
- subgradients $\phi(K) \in \partial f_\infty(K)$:

$$\phi(K) = \sum_{i \in \hat{X}(K)} \left(\tau_i \text{Re } (C_2 v_i u_i^H B_2)^T \right)$$

  - v_i right eigenvector associated with λ_i
  - u_i^H left eigenvector associated with λ_i
  - $\tau_i \geq 0$ and $\sum_{i \in \hat{X}(K)} \tau_i = 1$
- stop before convergence if $\alpha(A + B_2KC_2) < 0$ (K stabilizing).

stabilization

$$\min_K \underbrace{\max_i \text{Re } \lambda_i(A + B_2 K C_2)}_{f_\infty(K)}$$

minimize spectral abscissa $\alpha \triangleq \max_i \text{Re } \lambda_i$.

- active set of closed-loop eigenvalues indices :

$$\hat{X}(K) = \{i = 1 \dots n \; : \; \text{Re } \lambda_i(A + B_2 K C_2) = \alpha(A + B_2 K C_2)\}$$

  extended set $\hat{X}_e(K) \supset \hat{X}(K)$ with neighboring eigenvalues.
- working assumption: all the closed-loop eigenvalues λ_i, $i \in \hat{X}_e(K)$, are simple.
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problem	(n, m_2, p_2)	iter. stab.	cpu (s)	iter. conv.	cpu (s)	α	θ
AC8 (SOF)	(9, 1, 5)	1	< 0.1	9	0.2	$-4.45 \cdot 10^{-1}$	$-5.60 \cdot 10^{-17}$
AC8 ($n_K = 1$)	(9, 1, 5)	5	0.2	17	0.4	$-4.45 \cdot 10^{-1}$	$-5.01 \cdot 10^{-27}$
HE2 (SOF)	(4, 2, 1)	1	< 0.1	216	2.3	$-2.39 \cdot 10^{-1}$	$-9.77 \cdot 10^{-6}$
HE2 ($n_K = 1$)	(4, 2, 1)	1	< 0.1	(28)	0.7	$-2.31 \cdot 10^{-1}$	-1.53
REA2 (SOF)	(4, 2, 2)	1	< 0.1	(49)	0.83	-2.46	$-1.1 \cdot 10^{-2}$
REA2 (PID)	(4, 2, 2)	1	< 0.1	(19)	0.6	-1.27	$-1.4 \cdot 10^{-1}$
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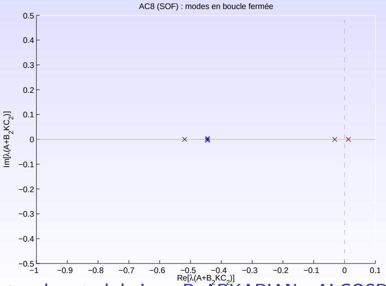

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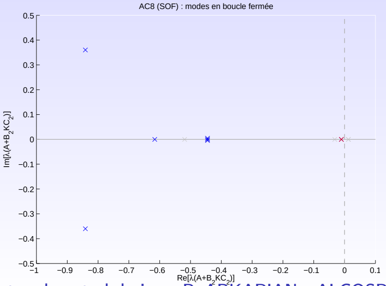

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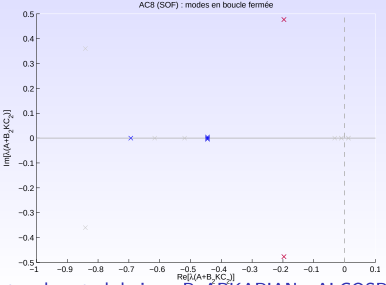

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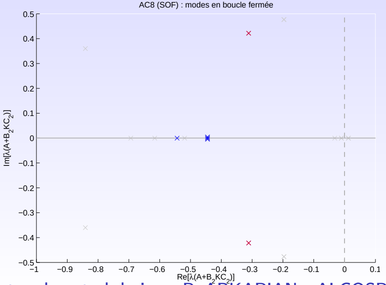

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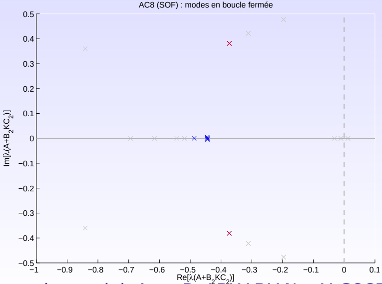

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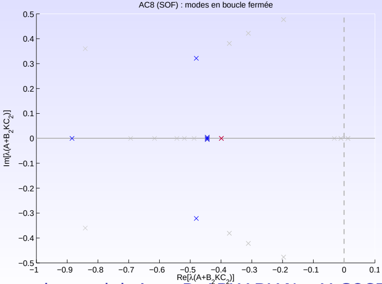

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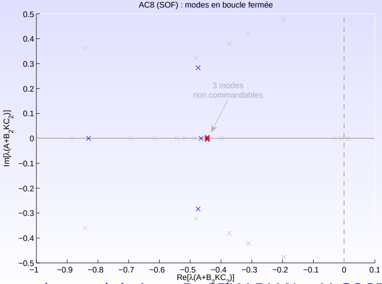
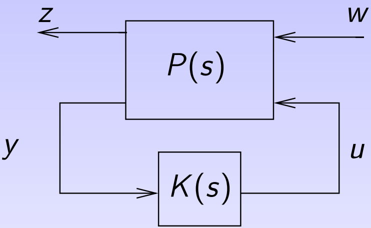


H_∞ synthesis

$$T_{w\to z}(K(s)) =$$



$$\boxed{\min_{K} \underbrace{\max_{\omega\in[0,+\infty]} \max_{i} \sigma_i\big(T_{w\to z}(K, j\omega)\big)}_{=\|T_{w\to z}(K,.)\|_\infty = f_\infty(K)}}$$

$$T_{w\to z}(K(s)) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

- composite $\|\cdot\|_\infty \circ T_{w\to z}$ is Clarke regular as a composite of convex and differentiable maps
- $\Rightarrow$ exhaustive description of subdifferential

$$\partial(\|\cdot\|_\infty \circ T_{w\to z})(K)$$

Clarke subdifferential of H_∞ norm

with standard notations define closed-loop data

$$\begin{array}{rclrcl} \mathcal{A}(K) & := & A + B_2 K C_2, & \mathcal{B}(K) & := & B_1 + B_2 K D_{21}, \\ \mathcal{C}(K) & := & C_1 + D_{12} K C_2, & \mathcal{D}(K) & := & D_{11} + D_{12} K D_{21}, \end{array}$$

introduce notation

$$\begin{bmatrix} T_{w \to z}(K, s) & G_{12}(K, s) \\ G_{21}(K, s) & \star \end{bmatrix} :=$$

$$\begin{bmatrix} \mathcal{C}(K) \\ C_2 \end{bmatrix} (sI - \mathcal{A}(K))^{-1} \begin{bmatrix} \mathcal{B}(K) & B_2 \end{bmatrix} + \begin{bmatrix} \mathcal{D}(K) & D_{12} \\ D_{21} & \star \end{bmatrix}.$$

- use appropriate plant augmentation if dynamic controller

Clarke subdifferential of H_∞ norm

Apkarian & Noll, 2006, [Apkarian and Noll, 2006-I]

subdif. is convex-compact set of subgradients Φ_Y's

$$\frac{\sum_{\nu=1}^{p} \mathrm{Re}\left\{ G_{21}(K, j\omega_\nu)\, T_{w\to z}(K, j\omega_\nu)^H Q_\nu Y_\nu Q_\nu^H G_{12}(K, j\omega_\nu) \right\}^T}{\| T_{w\to z}(K) \|_\infty}$$

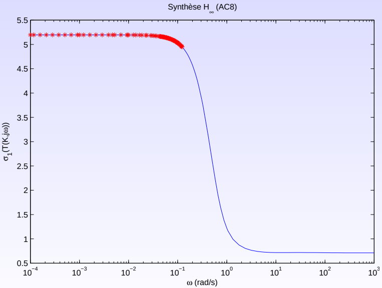
- ω_ν are active (peak) frequencies at K
- $Y := (Y_1, \ldots, Y_p)$ ranges over spectraplex (convex)

$$\{Y = (Y_1, \ldots, Y_p) : \; Y_i = Y_i^H, \succeq 0, \sum_{\nu=1}^{p} \mathrm{Tr}\,(Y_\nu) = 1\}$$

- use chain rule $\mathcal{K}(\kappa)'^* \partial(\| \cdot \|_\infty \circ T_{w\to z})(\mathcal{K}(\kappa))$ if structured controllers

examples from Leibfritz's collection

problem	(n, m, p)	order	iter	cpu (sec.)	nonsmooth H_∞	H_∞ AL	FW	H_∞ full
AC8	(9, 1, 5)	0	20	45	2.005	2.02	2.612	1.62
HE1	(4, 2, 1)	0	4	7	0.154	0.157	0.215	0.073
REA2	(4, 2, 2)	0	31	51	1.192	1.155	1.263	1.141
AC10	(55, 2, 2)	0	15	294	13.11	*	*	3.23
AC10	(55, 2, 2)	1	46	408	10.21	*	*	3.23
BDT2	(82, 4, 4)	0	44	1501	0.8364	*	*	0.2340
HF1	(130, 1, 2)	0	11	1112	0.447	*	*	0.447
CM4	(240, 1, 2)	0	2	3052	0.816	*	*	*
CM5	(480, 1, 2)	0	2	4785	0.816	*	*	*



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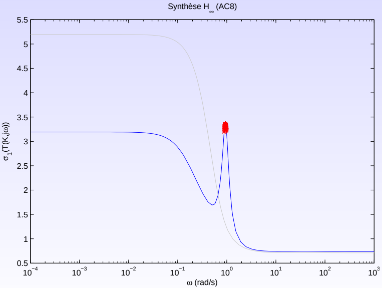

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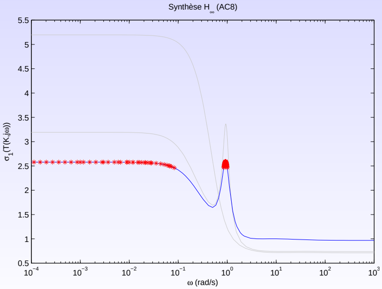

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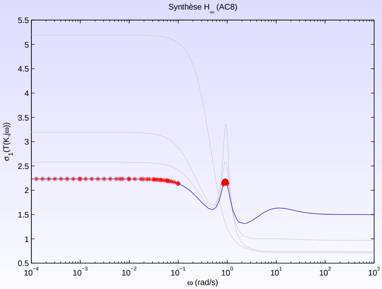

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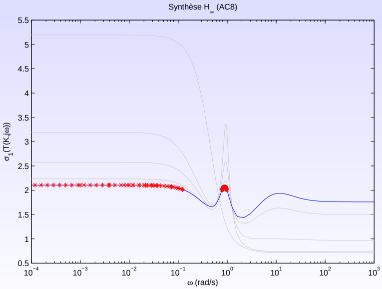

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HF1	$(130, 1, 2)$	0	11	1112	0.447	$*$	$*$	0.447
CM4	$(240, 1, 2)$	0	2	3052	0.816	$*$	$*$	$*$
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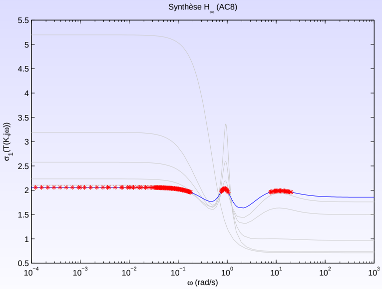

examples from Leibfritz's collection

problem	(n, m, p)	order	iter	cpu (sec.)	nonsmooth H_∞	H_∞ AL	FW	H_∞ full
AC8	(9, 1, 5)	0	20	45	2.005	2.02	2.612	1.62
HE1	(4, 2, 1)	0	4	7	0.154	0.157	0.215	0.073
REA2	(4, 2, 2)	0	31	51	1.192	1.155	1.263	1.141
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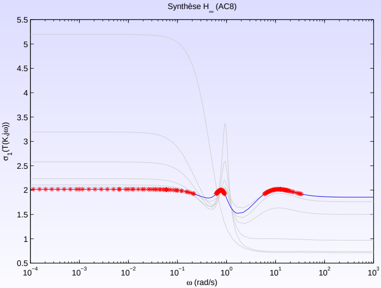

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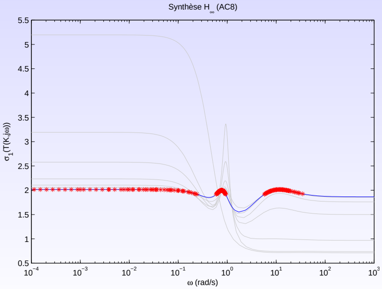

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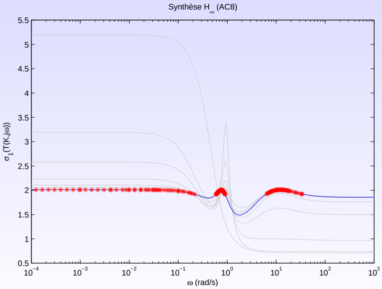

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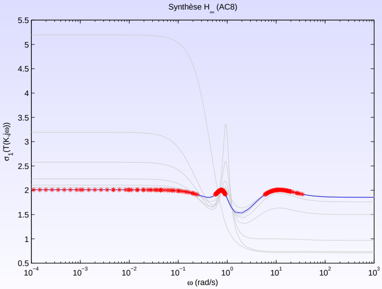

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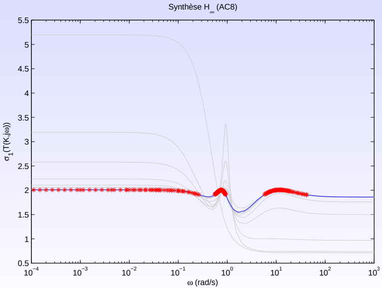

time-domain design

$$\min_{K\in} \underbrace{\max_{t\in[0,+\infty]} \left\{ \left(z(K,t) - z_{max}(t)\right)^+, \left(z_{min}(t) - z(K,t)\right)^+ \right\}}_{=f_\infty(K)}$$

minimize maximum violation of SISO step response envelope specification

$$z_{min}(t) \le z(K,t) \le z_{max}(t)$$

- stabilizing initial controller K_0.
- simulate step response for $t_l \in [0, T]$.
- active times (finite set) :

$$\hat{X}(K) = \left\{ t_l \;:\; z(K,t_l) - z_{max}(t) = f_\infty(K) \text{ or } z_{min}(t) - z(K,t_l) = f_\infty(K) \right\}$$

  extended set $\hat{X}_e(K) \supset \hat{X}(K)$ with neighboring times.
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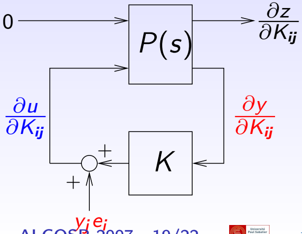
simulated subgradients

$$\phi(K) = \sum_{t_l \in \hat{X}(K)} \pm \tau_l \left[\frac{\partial z}{\partial K_{ij}}(K, t_l) \right]_{i,j}$$

with $\tau_l \geq 0$ and $\sum_{t_l \in \hat{X}(K)} \tau_l = 1$



numerical results

2-DOF PID control of a SISO plant

(4th-order)

t_s settling time at $\pm 2\%$

z_{os} overshoot

plant	contr.	t_s (s)	z_{os} (%)
G_1	ZN*	*46.22*	*46.92*
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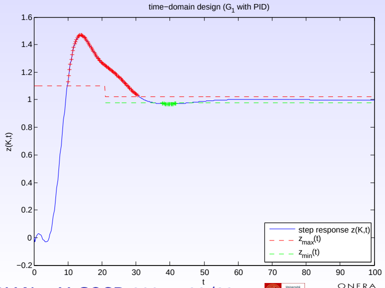

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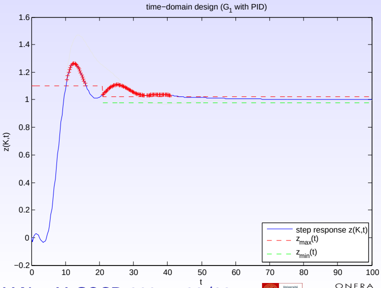

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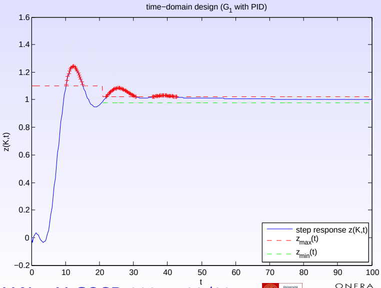

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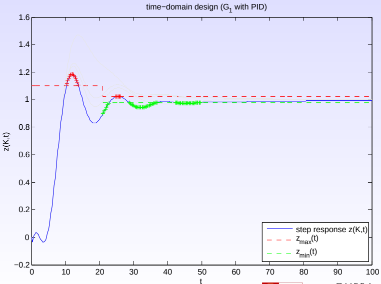

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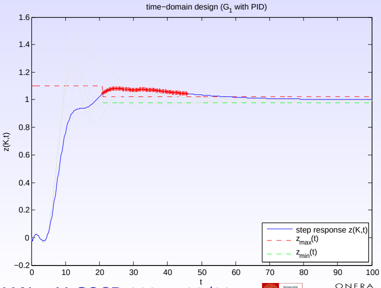

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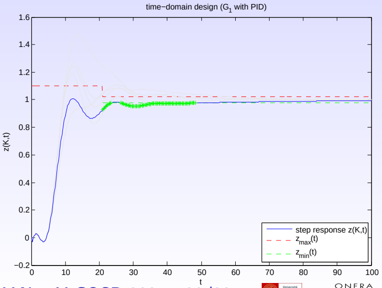

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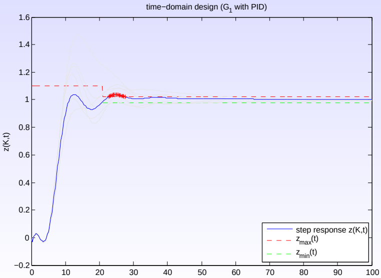

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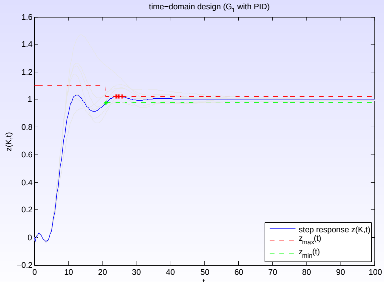

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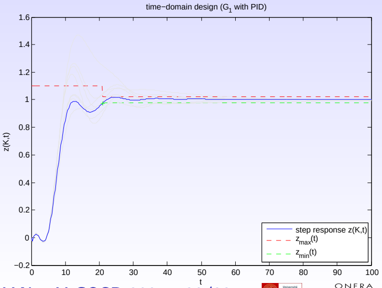

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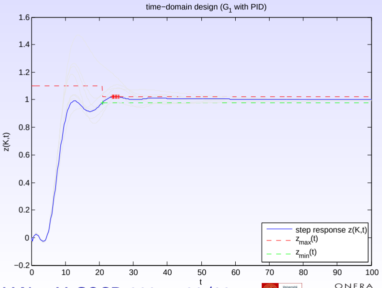

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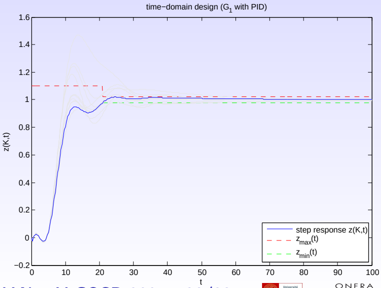

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  - numerically efficient for complex plants
  - flexible for structured practical controllers
  - provide local termination test and optimality certificates
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key references

F.H. Clarke.
Optimization and Nonsmooth Analysis.
John Wiley & Sons, 1983.

E. Polak.
On the mathematical foundations of nondifferentiable optimization in engineering design.
SIAM Review, 29(1):21-89, 1987.

E. Polak.
Optimization: Algorithms and Consistent Approximations.
Springer, Applied Mathematical Sciences, 1997.

J.V. Burke, D. Henrion, A.S. Lewis and M.L. Overton.
Stabilization via Nonsmooth, Nonconvex Optimization,
IEEE Trans. on Automatic Control, 51(11):1760-1769, 2006.

key references

http://www.cert.fr/dcsd/cdin/apkarian/

P. Apkarian and D. Noll.
Nonsmooth H_∞ synthesis.
IEEE Trans. on Automatic Control, 51(1):71-86, 2006.

P. Apkarian and D. Noll.
Nonsmooth Optimization for Multidisk H_∞ Synthesis.
Eur. J. of Control, 3(12):229-244, 2006.

P. Apkarian and D. Noll.
IQC analysis and synthesis via nonsmooth optimization.
Systems and Control Letters, 55(12):971-981, 2006.

V. Bompart, D. Noll and P. Apkarian.
Second-order nonsmooth optimization for H_∞ synthesis.
Numerische Mathematik, in press, 2007.

P. Apkarian, V. Bompart and D. Noll.
Nonsmooth Structured Control Design with Application to PID Loop-Shaping of a Process.
Int. J. of Robust and Nonlinear Control, in press, 2007.