PARAMETERIZED LMIS IN CONTROL THEORY

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Abstract. A wide variety of problems in control system theory fall within the class of parameterized Linear Matrix Inequalities (LMIs), that is, LMIs whose coefficients are functions of a parameter confined to a compact set. Such problems, though convex, involve an infinite set of LMI constraints, hence are inherently difficult to solve numerically.

This paper investigates relaxations of parameterized LMI problems into standard LMI problems using techniques relying on directional convexity concepts. An in-depth discussion of the impacts of the proposed techniques in quadratic programming, Lyapunov-based stability and performance analysis, μ analysis and Linear Parameter Varying control is provided. Illustrative examples are given to demonstrate the usefulness and practicality of the approach.

Key words. Linear Matrix Inequalities, robust semidefinite programming, directional convexity, robustness analysis, parametric uncertainty, Linear Parameter-Varying control.

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1. Introduction. LMI techniques are now well-rooted as a unifying framework for formulating and solving problems in control theory with a remarkable degree of simplicity. The main thrust of these techniques is that certain complicated control problems can be solved very efficiently. Specifically, the interior-point methods for semi-definite programming have worst-case polynomial complexity with respect to the problem size. From a practical viewpoint, extensive experience shows that interior-point methods solve problems in roughly less than a hundred iterations, independently of the problem size. Each elementary iteration reduces to solving a least-square problem which incurs the main computational overhead. Recent and thorough studies of interior-point techniques for semi-definite programming are, among others, Jarre [23], Vandenberghe and Boyd [43], Rendl, Vanderbei and Wolkowicz [34] and the master book by Nesterov and Nemirovski [28].

Basically, the simple feasibility problem of semidefinite programming consists in seeking a solution to the LMI

$$F_0 + z_1 F_1 + \ldots + z_r F_r < 0\,, \tag{1}$$

where the F_i's are given real symmetric matrices and the z_i's are the sought decision variables. A significantly more complicated generalization of problem (1) is the feasibility problem

$$F_0(\theta) + z_1(\theta) F_1(\theta) + \ldots + z_r(\theta) F_r(\theta) < 0, \tag{2}$$

where $\theta := [\theta_1, \ldots, \theta_N]^T$ is an additional parameter allowed to take any value in a compact set H of $\mathbf{R}^N$, typically a polytope. In contrast to problem (1) the problem data, $F_i(\theta)$, are now symmetric matrix-valued functions of θ, and we are seeking (arbitrary) functions of θ, $z_i(\theta)$ such that the LMI constraints (2) hold for any admissible value of θ. The complexity of problem (2) is twofold:

1. It is infinite-dimensional since the $z_i(.)$'s are sought in the infinite-dimensional space of functions of θ.

2. This is an infinitely constrained LMI problem for which each constraint corresponds to a given point in the range of θ.

A common and practical approach to overcome the difficulties arising from dimensionality, is to select a finite basis of functions for the z_i's and reconsider the problem over the resulting spanned finite-dimensional space. In such case, problem (2) simplifies to an LMI problem of the form

$$F_0(\theta) + z_1 F_1(\theta) + \ldots + z_r F_r(\theta) < 0, \qquad \forall \theta \in H \tag{3}$$

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where $z_1, \dots, z_r$ are conventional scalar decision variables as in (1) and H is a compact set. Such problems are referred to as *robust semidefinite programming* problems in [4] and are designated here as *Parameterized LMI* (PLMI) problems to stress the connections with the LMI control theory literature.

For reasons raised above, PLMI feasibility problems have still high complexity and are even known to be NP-hard [4]. The aim of this paper is to develop systematic relaxation techniques to turn, potentially conservatively, this problem into a standard LMI problem. A fruitful technique for turning PLMI problems into conventional LMI problems is the well-known **S**-procedure [45, 14]. With this approach, scaling or multipliers are utilized to eliminate the LMI parameter-dependence. The price to pay is the insuperable conservatism of the resulting conditions and also the extra computational effort, often prohibiting, introduced by the multiplier variables. This paper exploits competitive techniques invoking directional convexity concepts to derive a finite set of LMI conditions. Generally speaking, the approach requires significantly less variables than **S**-procedure techniques whilst producing more LMI constraints. Since the flop cost of interior-point techniques is roughly linear with respect to the size of the LMI constraint but polynomial with respect to the number of decision variables, the proposed techniques offer a valuable alternative to **S**-procedure techniques. It is however difficult to draw definitive conclusions at this stage since the respective performance of each technique is probably problem-dependent. As demonstrated in the body of the paper, the techniques therein also offer possibilities for handling polytopic representations, that is when the parameter θ designates polytopic coordinates, $\sum_{i=1}^{N} \theta_i = 1,\ \theta_i \geq 0$. We also briefly discuss relaxations of linear objective minimization problems subject to PLMI constraints and PLMI problems subject to algebraic constraints on the parameter θ.

The scope of applications of PLMIs is quite large and goes far beyond the area of robust control theory. In [4, 5], Ben-Tal and Nemirovski lay the foundations of *robust convex programming* and investigate its theoretical *tractability* in conjunction with the analysis of some generic uncertain convex programs. The same stream of ideas are applied to a truss topology design problem in [6]. In [29], the authors provide a thorough study of the regularity properties of solutions to PLMIs using the **S**-procedure and discuss its implications for a variety of topics: linear programming, polynomial interpolation, integer programming, ... Our contribution is in line with that of [29] or what is called “Approximate robust counterpart” of an uncertain semidefinite programming problem in [4]. The general instance of the problems is essentially intractable and we are constructing relaxed forms, generally conservative, that are directly amenable to the use of interior-point methods. Note also that alternative techniques to those considered here are developed in [40] using either convex approximations or d.c (difference convex) representations.

This work is mostly control theory oriented, and special attention is paid to the following topics:

(i) *Quadratic programming.* It is shown that some neither convex nor concave quadratic programming problems can be converted into boolean programming problems. The results so introduced constitute the core of the subsequent derivations and have a direct impact for relaxing PLMI problems.

(ii) *Lyapunov-based stability and performance analysis.* A rich catalog of Lyapunov-based stability and performance criteria for uncertain systems can be handled via PLMIs, thus providing generalizations of the single quadratic Lyapunov function approach.

(iii) *μ-analysis.* PLMIs have direct applications in the μ-analysis context or robust non-singularity analysis and can be utilized to refine the computation of upperbounds.

(iv) *Linear Parameter-Varying (LPV) control synthesis.* PLMIs and the concepts developed here are also central in LPV control synthesis to overcome the difficulties arising from gridding phases and reduce the computational efforts.

The paper is structured as follows. Section 3 discusses a variety of directional convexity concepts and its implications in functional optimization. These results are then extended to PLMI problems in Section 4. Important robust and LPV control issues mentioned above are investigated in Section 5. Numerical examples illustrating the techniques and tools are given in Section 6.

2. Preliminaries. The following definitions and notations are used throughout the paper.

R and **C** denotes the sets of real and complex numbers, respectively. M^T is the transpose of the matrix M, and M^* denotes its complex-conjugate transpose. The notation $\operatorname{Tr} M$ stands for the trace of M. For

Hermitian or symmetric matrices, $M > N$ means that $M - N$ is positive definite and $M \geq N$ means that $M - N$ is positive semi-definite.

Let S be a convex subset of $\mathbf{R}^n$. A function $f : S \to \mathbf{R}$ is quasi-convex if and only if for all u, v in S and α in $[0, 1]$,

$$f(\alpha u + (1-\alpha)v) \leq \max\{f(u), f(v)\} .$$

Strict quasi-convexity is obtained when the inequality is strict for all $0 < \alpha < 1$. This notion is weaker than convexity which requires

$$f(\alpha u + (1-\alpha)v) \leq \alpha f(u) + (1-\alpha) f(v) .$$

The relative interior, the closure and the relative boundary of S are denoted as $\operatorname{ri} S$, $\operatorname{cl} S$ and $\operatorname{rbd} S$, respectively. We then have $\operatorname{rbd} S = \operatorname{cl} S \setminus \operatorname{ri} S$.

A polytope Π in $\mathbf{R}^n$ is defined as the compact set

$$\Pi := \left\{ \sum_{i=1}^{L} \alpha_i v_i : \sum_{i=1}^{L} \alpha_i = 1,\ \alpha_i \geq 0,\ v_i \in \mathbf{R}^n \right\}$$

Equivalently, it is also the convex hull of the set $V = \{v_1, \ldots, v_L\}$, denoted $\operatorname{co} V$. The notation $\operatorname{vert} \Pi$ designates the set of vertices of Π, $\operatorname{vert} \Pi := V$. The affine hull, $\operatorname{aff} S$, of a set S is defined as the set of all affine combinations of elements of S, i.e.

$$\operatorname{aff} S := \left\{ \sum_{i=1}^{k} \alpha_i s_i : \ s_i \in S, \quad \sum_{i=1}^{k} \alpha_i = 1 \right\} .$$

The direction space associated to $\operatorname{aff} S$ is defined as $\operatorname{aff} S - s_0$, where s_0 is any point of $\operatorname{aff} S$. The notation $\#S$ stands for the number of elements in a set S.

3. Extreme point results. This section introduces some useful tools that permit to convert the maximization of a function over a polytope Π into the combinatorial problem of maximizing f over $\operatorname{vert} \Pi$. We begin with a general result which is the core of the subsequent derivations.

THEOREM 3.1 (Central result). *Consider a polytope Π and assume that for any x in Π, there exists a direction d in the direction space of $\operatorname{aff} \Pi$ such that f is quasi-convex on the line segment*

$$L_d(x) := \{z \in \Pi : \ z = x + \lambda d,\ \lambda \in \mathbf{R}\} .$$

Then, f has a maximum over Π in $\operatorname{rbd} \Pi$.

Proof. Assume f has a maximum $\hat{x}$ in $\operatorname{ri} \Pi$. Consider a line segment $L_d(\hat{x})$ where f is quasi-convex. From this property, we infer that f has a maximum point in $\operatorname{rbd} \Pi \cap L_d(\hat{x})$, and therefore

$$f(\hat{x}) \leq f(\bar{x}),$$

for some $\bar{x}$ in $\operatorname{rbd} \Pi \cap L_d(\hat{x})$. □

By virtue of Theorem 3.1, the search of a maximum point is reduced to exploring the relative boundary of Π. This result is analogous to the well-known *maximum principle* for analytic functions of complex variables. Although this constitutes an appealing result which might find applications, it is still hardly tractable for our particular purpose. A stronger result is obtained by forcing the directions d to be parallel to the edges of the polytope. The corollary below clarifies this fact.

COROLLARY 3.2 (Multi-quasi-convexity). *Consider a polytope Π and the directions $d_1, \ldots, d_q$ determined by the edges of Π. Assume that for any x in Π, the function f is quasi-convex on the line segments $L_{d_i}(x)$ for $i = 1, \ldots, q$. Then, f has a maximum over Π at a vertex of Π.*

Proof. Immediate by application of Theorem 3.1 to Π and to the (polytopic) faces and edges of Π. □

An obvious consequence of Theorem 3.1 is the following.

COROLLARY 3.3. *Under the hypotheses of Corollary 3.2, the following conditions are equivalent:*

(i) $f(x) < 0, \quad \forall x \in \Pi.$

(ii) $f(x) < 0, \quad \forall x \in \operatorname{vert}\Pi$.

As claimed previously, the maximization problem in Corollary 3.2 and the sign verification problem in Corollary 3.3 are turned into simpler combinatorial problems of lower complexity. This is a consequence of the multi-quasi-convexity property defined in Corollary 3.2. Note that the term *multi-quasi-convex* emphasizes the fact that f is separately quasi-convex along parallels to the edges of the polytope. This property is attached to the function f, but is also intimately related to the particular geometry of the polytope.

Quasi-convexity is a less stringent requirement that usual convexity, the counterpart being the difficulty of its verification even for differentiable functions. Alternative conditions that are more easily amenable to numerical computation are derived by replacing quasi-convexity with convexity in Theorem 3.1, Corollaries 3.2 and 3.3. For twice continuously differentiable functions, Corollary 3.2 then becomes.

COROLLARY 3.4 (Multi-convexity). *With the definitions in Corollary 3.2, f has a maximum over Π in* $\operatorname{vert}\Pi$ *whenever it holds that*

$$\frac{\partial^2 f(x+\lambda d_i)}{\partial \lambda^2} \geq 0, \quad \forall x \in \Pi, \quad i = 1, \ldots, q\,. \tag{4}$$

Affine functions are trivially multi-quasi-convex functions so that any of the above results is applicable. It is instructive to consider the case where f is a quadratic function and Π is a hyper-rectangle.

COROLLARY 3.5 (Quadratic functions). *Consider a quadratic functions, $f(x) = x^TQx + c^Tx + a$, and assume Π is a hyper-rectangle with edges paralleling the axes of coordinates, that is, $x = [x_1, \ldots, x_n]^T$ with*

$$\alpha_i \leq x_i \leq \beta_i, \quad i = 1, \ldots, n\,.$$

Assume further that

$$Q_{ii} \geq 0, \quad i = 1, \ldots, n \tag{5}$$

then, f has a maximum over Π in $\operatorname{vert}\Pi$.

Proof. From Corollary 3.4, the conditions (5) express multi-convexity of the quadratic function. □

Clearly, the conditions (5) are less demanding than (global) convexity which requires $Q \geq 0$. When such conditions hold, the maximization of f over the polytope Π reduces to a boolean programming problem [35] which is much simpler (though possibly costly) than the maximization of a general f. One possible advantage is that some costly but practically useful concave minimization techniques such as simplicial and conical partitioning (branch and bound) techniques such as those of Tuy and Thach might be used to find a global optimal solution. The reader is referred to the book of Tuy [41] for a thorough treatment.

4. Relaxation of PLMIs. This section presents some applications of these results to PLMIs whose coefficients are dependent on a parameter evolving in a polytopic set. To emphasize the fact that these parameters might be interpreted as uncertainties or scheduled variables of robust control or LPV control problems, the free variable x is denoted θ or α, hereafter.

Before proceeding further, it is instructive to have in mind the following important facts from [4]. Consider the "robust counterpart" (parameterized convex program in our terminology) of a general uncertain convex program :

$$\begin{array}{c} \text{minimize } c^Tz\,, \\ \text{subject to } F(z,\theta) \in K, \forall \theta \in H \end{array} \tag{6}$$

where K is a closed convex cone, H is a generalized ellipsoidal set including as instances standard ellipsoids but also ellipsoidal cylinders and polyhedras, $F(z,\theta)$ is K-concave with respect to z. A key additional assumption is that $F(z,\theta)$ must be K-concave with respect to θ. With these assumptions in place, Ben-Tal and Nemirovski established the following.

(i) The robust counterpart of an uncertain linear program is a conic quadratic program, thus is perfectly tractable.

(ii) The robust counterpart of an uncertain quadratically constrained convex quadratic program is a semidefinite program, hence tractable, but is NP-hard for intersections of ellipsoidal uncertainty sets.

(iii) The robust counterpart of an uncertain semidefinite program is generally NP-hard even for a single ellipsoidal uncertainty set.

The problems examined in the sequel fall within the latter class, so that they are generally NP-hard. They also generally fail to satisfy the K-concavity in θ, mentioned above. By virtue of its inherent complexity, one must, as a last resort, use relaxation techniques to end up with tractable "approximate" programs. Again with reference to [4], we take advantage of some directional K-concavity instead of complete K-concavity in the uncertain parameter to derive such relaxations.

4.1. PLMIs with quadratic parameter dependence. We consider PLMIs in the class

$$\mathcal{L}(z,\alpha) := M_0(z) + \sum_{i=1}^{L} \alpha_i M_i(z) + \sum_{i,j=1}^{L} \alpha_i \alpha_j M_{ij}(z) < 0, \tag{7}$$

where z stands for the decision variable and $M_0(.)$, $M_i(.)$ and $M_{ij}(.)$ are real symmetric matrix-valued and linear functions of z. In addition, it is supposed that the parameter $\alpha = [\alpha_1, \ldots, \alpha_L]^T$ evolves in the simplex

$$\Gamma := \left\{ \alpha : \sum_{i=1}^{L} \alpha_i = 1,\ \alpha_i \geq 0 \right\}. \tag{8}$$

Note that the problem presented in (7) involves infinitely many LMIs associated with each value of the parameter α and is known to be intractable [4]. By enforcing some constraints of geometric nature on the functional dependence in α, it is however possible to reduce, potentially conservatively, the problem to solving a finite number of LMIs. This is established in the next proposition.

PROPOSITION 4.1. *The infinite set of LMIs (7) is feasible for some z whenever the finite set of LMIs*

$$M_0(z) + M_k(z) + M_{kk}(z) < 0, \tag{9}$$

$$M_{ii}(z) + M_{jj}(z) - (M_{ij}(z) + M_{ji}(z)) \geq 0, \tag{10}$$

where $1 \leq k \leq L$ and $1 \leq i < j \leq L$, is feasible for some z.

Proof. Note first that the conditions (7) are equivalent to $x^T \mathcal{L}(z,\alpha) x < 0$, for all $x \neq 0$. For fixed $x \neq 0$, consider $x^T \mathcal{L}(z,\alpha) x$ as function of α. By virtue of Corollary 3.4, it is negative whenever it is multi-convex along lines paralleling the edges of Γ and furthermore is negative over $\operatorname{vert}\Gamma$. The remainder of the proof follows from the fact that $\operatorname{vert}\Gamma$ is composed of the canonical basis of $\mathbf{R}^L$, and the directions of the edges of Γ are determined by vectors with all but two zero coordinates, the non-zero coordinates having opposite sign :

$$d_1 := [1, -1, 0, \ldots, 0]$$
$$d_2 := [1, 0, -1, 0, \ldots, 0], \ \ldots$$

Repeating the reasoning for all $x \neq 0$, yields the condition (9) and (10), as desired. □

Remarks: By strengthening the conditions in (9), one can slightly relax the multi-convexity requirement in (10). As an example, the solutions (z, Z_i) to the LMI feasibility problem

$$M_0(z) + \sum_{i=1}^{L} \alpha_i M_i(z) + \sum_{i,j=1}^{L} \alpha_i \alpha_j M_{ij}(z) < - \sum_{i=1}^{L} \alpha_i^2 Z_i\,, \qquad \forall \alpha \in \Gamma$$
$$Z_i \geq 0, \quad i = 1, \ldots, L,$$

give solutions z to the feasibility problem (7). Arguing as in proposition 4.1, associated sufficient solvability conditions are easily obtained as

$$M_0(z) + M_k(z) + M_{kk}(z) < -Z_k, \tag{11}$$

$$M_{ii}(z) + M_{jj}(z) - (M_{ij}(z) + M_{ji}(z)) \geq -(Z_i + Z_j) \tag{12}$$

$$Z_k \geq 0\,, \tag{13}$$

where $1 \leq k \leq L$ and $1 \leq i < j \leq L$. Due to the strict nature of (11), the non-strict inequalities in (12) and (13) can be changed into strict inequalities without any loss of generality. In the strict form, such problems are readily solved using interior-point semi-definite programming techniques as those in [8, 42, 27]. Note also that the Z_i's can be chosen as general symmetric matrices whose size is that of the LMI condition (7). Less costly characterizations are obtained by using instead diagonal or scalar matrices, that is,

$$Z_i = \operatorname{diag} \lambda_i, \qquad \text{or} \qquad Z_i = \lambda_i I .$$

When Γ is a hyper-rectangle and the LMIs (7) are expressed in terms of the Cartesian coordinates of α (as opposed to polytopic ones) the main result in [18] is recovered as a special case. Assume $\theta := [\theta_1, \ldots, \theta_N]^T$ ranges over a hyper-rectangle, denoted H, that is,

$$\underline{\theta}_i \le \theta_i \le \bar{\theta}_i , \tag{14}$$

then

$$\mathcal{L}(z, \theta) := M_0(z) + \sum_{i=1}^{N} \theta_i M_i(z) + \sum_{i,j=1}^{N} \theta_i \theta_j M_{ij}(z) < 0, \quad \forall \theta \in H$$

whenever

$$\begin{array}{lcl} \mathcal{L}(z, \theta) < 0, & \theta \in & \operatorname{vert} H \\ M_{ii}(z) \ge 0, & i = & 1, \ldots, N \end{array}$$

As before, one can relax the multi-convexity requirement above by replacing these conditions with

$$\mathcal{L}(z, \theta) < -\sum_{i=1}^{N} \theta_i^2 \lambda_i I, \quad \theta \in \operatorname{vert} H, \tag{15}$$

$$M_{ii}(z) \ge -\lambda_i I, \quad i = 1, \ldots, N \tag{16}$$

More generally, any non-positive matrix-valued function of θ is a good candidate for the right-hand side of (15). More complicated polynomial functions lead naturally to more costly characterizations. From our practical experience, a reasonable compromise between computational efficiency and tightness of the test is obtained with non-homogeneous functions of the form $-(\lambda_0 + \sum_{i=1}^{N} \theta_i^2 \lambda_i) I$.

4.2. Gridding techniques. The techniques developed in Sections 3 and 4 provide sufficient and computationally simple conditions for checking the sign of a function or the feasibility of a PLMI problem. These conditions may introduce conservatism though it turns out to be small from our practical experience. A different technique which is guaranteed to provide a non-conservative answer but is potentially optimistic and generally computationally intensive, is to use a fine gridding of the parameter range and solve a finite set of LMIs corresponding to each point on the grid. Denoting the grid as G, the PLMI problem (7) is then replaced with the finite set of LMIs

$$M_0(z) + \sum_{i=1}^{L} \alpha_i M_i(z) + \sum_{i,j=1}^{L} \alpha_i \alpha_j M_{ij}(z) < 0, \quad \alpha \in G .$$

Such a technique is currently used in stability analysis and LPV control. It is however limited to problems of reasonable size, say less than 3 parameters. There is also the risk to miss a critical value of the parameter, hence leading to overly optimistic answers. With the approaches presented earlier these difficulties are inherently ruled out. These techniques can be mixed with gridding approaches hence offering alternative possibilities. Indeed, instead of gridding the entire parameter range, there is only need to grid a surface of lower dimension whenever the function is quasi-convex or convex along some direction. This is an immediate consequence of Theorem 3.1. A simple illustration of this fact is given below. For the sake of simplicity, we restrict the discussion to 2 parameters θ_1 and θ_2 evolving in the normalized square

$$|\theta_1| \le 1, \qquad |\theta_2| \le 1 , \tag{17}$$

and we consider the PLMI problem

$$M_0(z) + \sum_{i=1}^{2} \theta_i M_i(z) + \sum_{i,j=1}^{2} \theta_i \theta_j M_{ij}(z) < 0 \,. \tag{18}$$

A potential technique for checking the feasibility of this problem consists first in enforcing convexity in the direction of θ_1. This is equivalent to the LMI constraint

$$M_{11}(z) \geq 0 \,. \tag{19}$$

Thanks to this condition, it is then enough to grid the line segments

$$\theta_1 = \pm 1, \qquad |\theta_2| \leq 1 \,,$$

to check the feasibility of (18).

Finally, let us note that the approaches presented in the previous subsections are also very useful for developing a global optimization algorithm solving PLMIs. Indeed, the main difficulty in global optimization is "the curse of dimensionality", i.e. the size of the space where the global search is performed. Thus exploiting convexity properties such as directional convexity is very important for developing an efficient global optimization algorithm (see e.g. [24, 41]) since it allows us to drastically simplify the problem by limiting the global search to a restricted region of the feasible domain. For instance, with condition (19) it is sufficient to perform a global search for (18) just on the line segment $|\theta_2| \leq 1$ instead on the square (17) in R^2.

4.3. PLMIs with polynomial parameter dependence . In this section, we are considering polynomially θ-dependent PLMIs of the form

$$\mathcal{L}(\theta, z) := \sum_{\nu \in J} \theta^{[\nu]} M_\nu(z) < 0 \,, \tag{20}$$

where the terms $M_\nu(z)$ denote symmetric matrix-valued functions of the decision variable z that are linear in z. The notation $[\nu]$ is the vector of partial degrees $[\nu] = [\nu_1, \ldots, \nu_N]$ associated with the lexicographically ordered term

$$\theta^{[\nu]} = \theta_1^{\nu_1} \theta_2^{\nu_2} \ldots \theta_N^{\nu_N} \,,$$

with the convention $\theta^{[0]} = 1$. It is assumed that θ ranges over an hyper-rectangle H as in (14). J is a set of N-tuples of partial degrees describing the polynomial expansion (20). Exploiting again Corollary 3.4, it is possible to reduce (conservatively) this problem to a finitely constrained LMI problem. The symbols d_k and d designate the partial and total degrees in the matrix polynomial expansion.

LEMMA 4.2. *Consider the PLMI (20), where θ ranges over a hyper-rectangle. Then the LMI conditions*

$$\mathcal{L}(\theta, z) < 0, \qquad \forall \theta \in H \tag{21}$$

hold for some z, whenever the finite family of LMI conditions:

$$\mathcal{L}(\theta, z) < 0, \qquad \forall \theta \in \operatorname{vert} H \tag{22}$$

$$(-1)^m \frac{\partial^{2m}}{\partial \theta_{l_1}^2 \ldots \partial \theta_{l_m}^2} \mathcal{L}(\theta, z) \leq 0, \quad \forall \theta \in \operatorname{vert} H \,, \tag{23}$$

where

$$1 \leq l_1 \leq l_2 \leq \ldots \leq l_m \leq N, \quad 1 \leq m \leq \frac{d}{2}$$

$$2\#\{l_j = k : \; j \in \{1, \ldots, m\}\} \leq d_k, \quad k = 1, 2, \ldots, N.$$

are feasible for some z.

Proof. The proof is obtained by a repeated use of Corollary 3.4. □

As an example, consider the PLMI feasibility problem

$$\mathcal{L}(z,\theta) := M_0(z) + \theta_1^2\theta_2 M_{112}(z) + \theta_2^3 M_{222}(z) \ < \ 0, \qquad |\theta_i| \leq 1\,.$$

Replacing this problem with, for instance,

$$\widetilde{\mathcal{L}}(z,\theta) := M_0(z) + \theta_1^2\theta_2 M_{112}(z) + \theta_2^3 M_{222}(z) + \lambda_0 I + \lambda_1\theta_1^2 I + \lambda_2\theta_2^2 I \ < \ 0, \qquad |\theta_i| \leq 1$$

and using Lemma 4.2, yields the LMI conditions:

$$\widetilde{\mathcal{L}}(z,\theta) < 0, \qquad \forall\theta \in \operatorname{vert} H$$

$$-\frac{\lambda_1}{2} I \leq M_{112}(z) \leq \frac{\lambda_1}{2} I, \qquad -\frac{\lambda_2}{3} I \leq M_{222}(z) \leq \frac{\lambda_2}{3} I, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0\,.$$

It is of interest to note that when PLMIs also involve full-matrix parameters Δ_j, it is more appropriate for computational reasons to take advantage of a combined use of directional convexity concepts and of the **S**-procedure to formulate a feasibility test.

4.4. Algebraically constrained PLMI problems . PLMI problems with algebraic constraints are described as

$$\mathcal{L}(z,\theta) < 0, \qquad \forall\theta \in H \tag{24}$$

subject to

$$g_1(\theta) = 0, \ldots, g_q(\theta) = 0\,, \tag{25}$$

where $g_1,\ldots\ g_q$ are polynomials in θ. Note that for consistency, the algebraic surface (25) should have a non-void intersection with the hypercube H.

It is readily verified that solutions to the unconstrained PLMI problem

$$\mathcal{L}(z,\theta) < \sum_{i=1}^{q} g_i(\theta)^2 \lambda_i I, \qquad \forall\theta \in H \tag{26}$$

$$\lambda_i \geq 0, \quad i = 1, \ldots, q \tag{27}$$

also solves (24)-(25). Recast as the sufficient conditions (26)-(27), the hard problem (24)-(25) can be handled with the technical machinery developed in Section 4 and is therefore amenable to a conventional LMI problem. Once again, there is some practically useful flexibility for selecting the right-hand side of the first inequality in (26).

4.5. Linear objective minimization under PLMI constraints . The directional convexity concepts introduced previously are applicable with minor changes to linear objective minimization problems subject to PLMI constraints. This means problems of the form

$$\begin{array}{c} \text{minimize } c^T z \\ \text{subject to } \mathcal{L}(z,\theta) < 0, \quad \theta \in H\,, \end{array} \tag{28}$$

where c is a given vector and the inequalities constitute a PLMI constraint. It is also possible to handle min-max problems of the form

$$\begin{array}{c} \text{minimize } \max\limits_{\theta \in H} c(\theta)^T z \\ \text{subject to } \mathcal{L}(z,\theta) < 0, \quad \theta \in H\,, \end{array} \tag{29}$$

using standard manipulations [4, 29]. Defining

$$\widetilde{z} = \begin{bmatrix} z \\ \lambda \end{bmatrix}, \ \widetilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \widetilde{\mathcal{L}}(\widetilde{z}, \theta) = \begin{bmatrix} \mathcal{L}(z, \theta) & 0 \\ 0 & c(\theta)^T z - \lambda \end{bmatrix},$$

problem (29) is equivalently formulated as

$$\begin{array}{c} \text{minimize } \widetilde{c}^T \widetilde{z} \\ \text{subject to } \widetilde{\mathcal{L}}(\widetilde{z}, \theta) < 0, \quad \theta \in H, \end{array} \tag{30}$$

which has a form similar to problem (28).

In this form and provided that the parameter dependence is polynomial, such problems are easily converted into standard LMI problems using directional convexity concepts. This is left to the reader. Finally, we note that since these concepts amounts to shrinking the z-feasible set, the optimal value of the relaxed LMI optimization problem is an upper bound for problems (28) or (29).

5. Applications in Control Theory . The techniques and tools presented in Sections 3 and 4 enjoy a wide scope of applications. They are useful for the analysis of both the stability and the performance of uncertain systems. Potentially, all Lyapunov-based stability and performance measures can be handled with the proposed techniques which are more general and less conservative than single quadratic function approaches [7, 3]. They also have implications in the context of μ analysis where some upper bounds can be refined into less conservative upper bounds. Another important domain of application concerns LPV control techniques. For brevity, we only report a few of these applications.

5.1. Robust stability. We consider the linear uncertain system

$$\dot{x} = A(\alpha)x, \qquad A(\alpha) := \alpha_1 A_1 + \ldots + \alpha_L A_L, \tag{31}$$

where α is a fixed uncertain parameter evolving in the simplex (8). It follows that the uncertain matrix $A(\alpha)$ ranges over a matrix polytope

$$A(\alpha) \in \text{co}\ \{A_1, \ldots, A_L\}$$

We are seeking a quadratic parameter-dependent Lyapunov function with similar structure

$$V(x, \alpha) := x^T\ (\alpha_1 X_1 + \ldots + \alpha_L X_L\ )\ x$$

establishing stability of the uncertain system for all admissible dynamics. If we explicitate the Lyapunov conditions for stability

$$V(x, \alpha) > 0, \qquad \frac{d}{dt} V(x, \alpha) < 0, \qquad \forall x \neq 0,$$

we obtain

$$\begin{array}{rcl} \alpha_1 X_1 + \ldots + \alpha_L X_L & > & 0 \\ A(\alpha)^T (\alpha_1 X_1 + \ldots + \alpha_L X_L) + (\alpha_1 X_1 + \ldots + \alpha_L X_L) A(\alpha) & < & 0, \end{array}$$

which constitutes a PLMI problem. Thus, Proposition 4.1 can be used to convert the problem into a finite number of LMI feasibility conditions, the following sufficient test for robust stability is derived.

PROPOSITION 5.1. *Assume one of the A_i's is stable. Then, the uncertain system (31) is stable whenever there exist symmetric matrices X_1, ... , X_L and scalars λ_1, ..., λ_L such that the following LMI conditions hold*

$$\begin{array}{rcl} A_k^T X_k + X_k A_k & < & -\lambda_k I, \\ A_i^T X_i + X_i A_i + A_j^T X_j + X_j A_j - (A_i^T X_j + X_j A_i + A_j^T X_i + X_i A_j) & \geq & -(\lambda_i + \lambda_j) I \\ \lambda_k & \geq & 0, \end{array}$$

for $k = 1, \ldots, L$ and $1 \leq i < j \leq L$.

In such case the Lyapunov function $V(x, \alpha)$ establishes stability of the uncertain system (31).

Proof. The above conditions ensure that $\frac{d}{dt} V(x, \alpha) < 0$ for all admissible values of the parameter. Moreover, $V(x, \alpha)$ is a candidate Lyapunov function since at least one the A_i's is stable and since $\alpha_1 X_1 + \ldots + \alpha_L X_L$ cannot be singular, we infer that $\alpha_1 X_1 + \ldots + \alpha_L X_L > 0$ for all α in the simplex (8). ◻

5.2. Robust performance. As claimed earlier, the proposed techniques are potentially applicable to any Lyapunov-based performance measure. We illustrate this claim with the H_2 performance criterion. See [9] for a Lyapunov characterization of the H_2 norm.

Consider the uncertain system

$$\begin{array}{rcl} \dot{x} & = & A(\alpha)x + B(\alpha)w \\ z & = & C(\alpha)x \end{array} \tag{32}$$

where

$$\begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & 0 \end{bmatrix} \in \text{co}\left\{ \begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} A_L & B_L \\ C_L & 0 \end{bmatrix} \right\}.$$

By virtue of Proposition 4.1, and paralleling the argument in Proposition 5.1, we deduce that the H_2 norm from w to z of the uncertain system (32) is bounded by ν for all values of α in the simplex (8) whenever there exist symmetric matrices X_1, ... , X_L, Q and scalars λ_1, ..., λ_L such that the following LMI conditions hold

$$\begin{array}{rcl} U_k(X_k) & < & -\lambda_k I \\ V_i(X_i) + V_j(X_j) - (V_i(X_j) + V_j(X_i)) & \geq & -(\lambda_i + \lambda_j)I \\ \lambda_k \geq 0, \quad \begin{bmatrix} P_k & C_k^T \\ C_k & Q \end{bmatrix} > 0, \quad \operatorname{Tr} Q & < & \nu \end{array}$$

for $k = 1, \ldots, L$ and $1 \leq i < j \leq L$, with the definitions

$$U_i(X_j) := \begin{bmatrix} A_i^T X_j + X_j A_i & X_j B_i \\ B_i^T X_j & -I \end{bmatrix}, \quad V_i(X_j) := \begin{bmatrix} A_i^T X_j + X_j A_i & X_j B_i \\ B_i^T X_j & 0 \end{bmatrix}.$$

Extensions and reformulations for time-varying uncertain parameters, pole clustering in LMI regions, H_∞ and passivity constraints, and many others are straightforward. When the dependence on the parameter is polynomial the same line of attack is still valid with the assistance of Lemma 4.2.

5.3. μ analysis. The structured singular value (SSV) or μ is an important linear algebra tool to study a class of matrix perturbation problems [11, 12, 36]. Since many robust stability/performance problems can be recast as one of computing μ with respect to an appropriate block-diagonal structure, it is also particularly useful in control theory and practice. The computation of μ involves an optimization problem which is not convex and known to be NP-complete [32], so that it is difficult to compute μ exactly. Fortunately, it is possible to compute lower and upper bounds for μ with reasonable computational effort [13, 46]. This is the approach considered in this section.

The computation of μ can be formulated as computing the smallest norm perturbation for which the matrix $I - \Delta M$ becomes singular, where M denotes the plant's transfer function at some given frequency and Δ stands for uncertainties which are generally assumed to have a specific block-diagonal structure. In this section, we assume without loss of generality that uncertainties are real, $\Delta_{ij} \in \mathbf{R}$, and range over a polytope

$$\Delta \in \text{co}\,\{\Delta_1, \ldots, \Delta_L\}.$$

Extensions to mixed real/complex uncertainties are readily derived.

Our goal is to determine sufficient conditions for which $I - \Delta M$ remains non-singular for all admissible uncertainties. Our approach is inspired by the work in [15, 26] and goes as follows. A necessary and sufficient condition for the non-singularity of $(I - \Delta M)$ is the existence of a parameter-dependent matrix $F(\Delta)$, such that

$$F(\Delta)(I - \Delta M) + (I - \Delta M)^* F(\Delta)^* < 0. \tag{33}$$

The awkward condition (33) is simplified by restricting the search of $F(\Delta)$ matrices to those having the form

$$F(\Delta) := \sum_{i=1}^{L} \alpha_i F_i,$$

where the α_i's are the coordinates of Δ in the convex decomposition

$$\Delta := \sum_{i=1}^{L} \alpha_i \Delta_i \, .$$

With these restrictions, it is not difficult to see that inequality (33) takes a form similar to (7), that is, a PLMI feasibility problem. Therefore, by a direct application of Proposition 4.1 or its refined version (11)-(13), sufficient conditions for the non-singularity of $I - \Delta M$ express as the existence of suitably dimensioned complex matrices F_1, F_2, ... F_L such that

$$\begin{array}{rcl} F_k + F_k^* - (F_k \Delta_k M + (F_k \Delta_k M)^*) & < & -\lambda_k I \\ (F_i \Delta_i M + F_j \Delta_j M) - (F_i \Delta_j M + F_j \Delta_i M) + (\star)^* & \leq & (\lambda_i + \lambda_j) I \\ \lambda_k & \geq & 0 \, . \end{array}$$

This new upper bound for μ reduces to the upper bound proposed by M. Fu and N. Barabanov in [15] when $F = F_1 = \ldots = F_L$, thus is less conservative but also more costly. It is also less conservative than the more classical upper bound in [13], as is easily proved by choosing $F = F_1 = \ldots = F_L = -D - jM^* G$. A similar approach though somewhat more conservative has been proposed by Chen and Sugie in [10].

5.4. Linear Parameter-Varying Control. In this section, we more thoroughly investigate how the concepts and tools introduced can be utilized in the context of LPV control. For clarity, we recall the general statement of the problem.

We are considering an LPV plant with state-space realization

$$\begin{array}{rcl} \dot{x} & = & A(\theta)x + B_1(\theta)w + B_2(\theta)u \\ z & = & C_1(\theta)x + D_{11}(\theta)w + D_{12}(\theta)u \\ y & = & C_2(\theta)x + D_{21}(\theta)w \, , \end{array} \tag{34}$$

where

$$A \in \mathbf{R}^{n\times n}, \quad D_{12} \in \mathbf{R}^{p_1 \times m_2}, \text{and} \quad D_{21} \in \mathbf{R}^{p_2 \times m_1}$$

define the problem dimension. It is assumed that

(A1) the state-space data $A(\theta)$, $B_1(\theta)$, ... are bounded continuous functions of θ,

(A2) the time-varying parameter $\theta(t) := [\theta_1(t), \cdots, \theta_N(t)]^T$ and its rate of variation $\dot{\theta}(t)$, defined at all times and continuous, evolve in hyper-rectangles H and H_d, that is,

$$\theta_i(t) \in [\underline{\theta}_i, \bar{\theta}_i], \quad \forall t \geq 0 \, , \tag{35}$$

$$\dot{\theta}_i(t) \in [\underline{\nu}_i, \bar{\nu}_i], \quad \forall t \geq 0 \, . \tag{36}$$

The assumptions **(A1)** and **(A2)** are general. They secure existence and uniqueness of the solutions to (34) for given initial conditions and also specify the parameter trajectories under consideration.

With these assumptions in place, the general LPV control problem with guaranteed L_2-gain performance consists of finding a dynamic LPV controller with state-space equations

$$\begin{array}{rcl} \dot{x}_K & = & A_K(\theta, \dot{\theta}) x_K + B_K(\theta, \dot{\theta}) y \\ u & = & C_K(\theta, \dot{\theta}) x_K + D_K(\theta, \dot{\theta}) y \, , \end{array} \tag{37}$$

which ensures internal stability and a guaranteed L_2-gain bound γ for the closed-loop operator (34)-(37) from the disturbance signal w to the error signal z, that is,

$$\int_0^T z^T z \, d\tau \leq \gamma^2 \int_0^T w^T w \, d\tau, \quad \forall T \geq 0$$

for all admissible parameter trajectories $\theta(t)$.

Sufficient solvability conditions for this problem can be derived using a suitable extension of the Bounded Real Lemma [44], and by confining the search of (Lyapunov) variables to some finite-dimensional subspace of functions of θ. The next theorem provides such a set of conditions for the general LPV control problem. An alternative approach, based on polytopic covering techniques, is proposed by Yu and Sideris in [47]. For technical reasons that are clarified in the proof, we also assume that

(A3) the matrices $[\,B_2^T(\theta) \quad D_{12}^T(\theta)\,]$, $[\,C_2(\theta) \quad D_{21}(\theta)\,]$ have full row-rank over H.

Note that the dependence of data and variables on θ, or $\dot\theta$ is generally dropped for simplicity.

THEOREM 5.2. *With the assumptions* **(A1)**-**(A3)** *in force, the following conditions are equivalent:*

(i) The Bounded Real Lemma conditions with L_2-gain performance level γ hold for some quadratic Lyapunov function

$$V(x, x_K, \theta) := \begin{bmatrix} x \\ x_K \end{bmatrix}^T P(\theta) \begin{bmatrix} x \\ x_K \end{bmatrix},$$

where $P(\theta)$ is continuously differentiable, and for some LPV controller (37).

(ii) There exist continuously differentiable parameter-dependent symmetric matrices $X(\theta)$ and $Y(\theta)$ such that the following PLMI problem is feasible :

$$\begin{bmatrix} \mathcal{N}_X & 0 \\ \hline 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} \dot X + XA + A^T X & XB_1 & C_1^T \\ B_1^T X & -\gamma I & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma I \end{array}\right] \begin{bmatrix} \mathcal{N}_X & 0 \\ \hline 0 & I \end{bmatrix} < 0 \tag{38}$$

$$\begin{bmatrix} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} -\dot Y + YA^T + AY & YC_1^T & B_1 \\ C_1 Y & -\gamma I & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma I \end{array}\right] \begin{bmatrix} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{bmatrix} < 0 \tag{39}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{40}$$

for all $(\theta, \dot\theta)$ on $H \times H_d$, and where $\mathcal{N}_X$ and $\mathcal{N}_Y$ designate any bases of the nullspaces of $[\,C_2 \quad D_{21}\,]$ and $[\,B_2^T \quad D_{12}^T\,]$, respectively.

(iii) There exist continuously differentiable parameter-dependent symmetric matrices $X(\theta)$ and $Y(\theta)$ and a scalar σ solving the PLMI problem:

$$\begin{bmatrix} \dot X + XA + A^T X & XB_1 & C_1^T \\ B_1^T X & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} C_2^T \\ D_{21}^T \\ 0 \end{bmatrix} [C_2 \quad D_{21} \quad 0] < 0 \tag{41}$$

$$\begin{bmatrix} -\dot Y + YA^T + AY & YC_1^T & B_1 \\ C_1 Y & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix} [B_2^T \quad D_{12}^T \quad 0] < 0, \tag{42}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{43}$$

for all $(\theta, \dot\theta)$ on $H \times H_d$.

Proof. See Appendix. □

Equipped with Theorem 5.2, it is relatively straightforward to show how multi-convexity concepts can be used to reduce complexity in LPV control problems with polynomial parameter-dependence.

For simplicity of the presentation, it is first assumed the the state-space data in (34) and the Lyapunov variables are affine functions of the parameter θ, that is,

(A4) $\qquad A(\theta) := A_0 + \sum_{i=1}^{N} \theta_i A_i, \quad B_1(\theta) := B_{10} + \sum_{i=1}^{N} \theta_i B_{1i} \;\ldots$

THEOREM 5.3. *With the assumptions* **(A1)**-**(A4)** *above, there exists an LPV controller (37) solution to the LPV control problem with guaranteed L_2-gain performance with level γ whenever there exist symmetric matrices X_0, X_1, ... ,X_N and Y_0, Y_1, ...,Y_N and scalars λ_0, λ_1,... λ_N, μ_0, μ_1,... μ_N and σ such that*

$$(44)\quad \begin{bmatrix} \dot{X}+XA+A^TX & XB_1 & C_1^T \\ B_1^TX & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} C_2^T \\ D_{21}^T \\ 0 \end{bmatrix} \begin{bmatrix} C_2 & D_{21} & 0 \end{bmatrix} < -(\lambda_0 + \sum_{i=1}^N \theta_i^2 \lambda_i) I$$

$$(45)\quad \begin{bmatrix} -\dot{Y}+YA^T+AY & YC_1^T & B_1 \\ C_1Y & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix} \begin{bmatrix} B_2^T & D_{12}^T & 0 \end{bmatrix} < -(\mu_0 + \sum_{i=1}^N \theta_i^2 \mu_i) I$$

$$(46)\quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0$$

for $(\theta, \dot{\theta}) \in \operatorname{vert} H \times \operatorname{vert} H_d$ *and*

$$(47)\quad \begin{bmatrix} X_iA_i + A_i^TX_i & X_iB_{1i} \\ B_{1i}^TX_i & 0 \end{bmatrix} - \sigma \begin{bmatrix} C_{2i}^TC_{2i} & C_{2i}^TD_{21i} \\ D_{21i}^TC_{2i} & D_{21i}^TD_{21i} \end{bmatrix} \geq -\lambda_i I$$

$$(48)\quad \begin{bmatrix} Y_iA_i^T + A_iY_i & Y_iC_{1i}^T \\ C_{1i}Y_i & 0 \end{bmatrix} - \sigma \begin{bmatrix} B_{2i}B_{2i}^T & B_{2i}D_{12i}^T \\ D_{12i}B_{2i}^T & D_{12i}D_{12i}^T \end{bmatrix} \geq -\mu_i I\,,$$

$$(49)\quad \lambda_0 \geq 0, \quad \lambda_i \geq 0, \quad \mu_0 \geq 0, \quad \mu_i \geq 0\,.$$

for $i = 1, \ldots, N$ *with the notations*

$$X := X_0 + \sum_{i=1}^N \theta_i X_i, \qquad Y := Y_0 + \sum_{i=1}^N \theta_i Y_i\,.$$

Proof. The proof is a direct consequence of Theorem 5.2 combined with an application of Proposition 4.1 or Lemma 4.2 to the particular case where data and variables are affine with respect to θ. □

Remarks: The conditions in Theorem 5.3 constitute a standard semi-definite programming problem. The linear objective γ should be minimized subject to a finite number of LMI constraints, and a number of softwares are available for this purpose. The characterization is easily modified to encompass any polynomial parameter dependence for both the state-space data and the variables $X(\theta)$ and $Y(\theta)$ by direct application of Lemma 4.2. The multi-convexity requirements in equations (47) and (48) can be relaxed using the simple techniques in Section 4. When either the multi-convexity approach is too conservative or brute force gridding of the parameter range is too costly (more than two parameters), it might be appropriate to enforce multi-convexity along some direction and to grid a surface of lower dimension. See the examples in Section 6 for illustrations.

5.4.1. LPV controller construction. The PLMI conditions *(ii)* and *(iii)* in Theorem 5.2 are equivalent and provide lossless solvability conditions for problem *(i)*. The characterization in Theorem 5.3 may be conservative but give tractable conditions for solving the same problem. Clearly, when any of the latter problems is feasible, the state-space data (37) of an LPV controller solving the problem can be constructed for any pair $(\theta, \dot{\theta})$ in $H \times H_d$ from any solutions $X(\theta)$, $Y(\theta)$ and σ by the very same algebraic formulae. For completeness, we provide the following sequential scheme:

(i) compute D_K solution to

$$(50)\quad \overline{\sigma}(D_{11} + D_{12}D_KD_{21}) < \gamma,$$

and set $D_{c\ell} := D_{11} + D_{12}D_KD_{21}$.

(ii) compute $\widehat{\boldsymbol{B}}_K$ and $\widehat{\boldsymbol{C}}_K$ solutions to the linear matrix equations

$$(51)\quad \begin{bmatrix} 0 & D_{21} & 0 \\ D_{21}^T & -\gamma I & D_{c\ell}^T \\ 0 & D_{c\ell} & -\gamma I \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{B}}_K^T \\ \star \end{bmatrix} = - \begin{bmatrix} C_2 \\ B_1^TX \\ C_1 + D_{12}D_KC_2 \end{bmatrix};$$

$$(52)\quad \begin{bmatrix} 0 & D_{12}^T & 0 \\ D_{12} & -\gamma I & D_{c\ell} \\ 0 & D_{c\ell}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{C}}_K \\ \star \end{bmatrix} = - \begin{bmatrix} B_2^T \\ C_1Y \\ (B_1 + B_2D_KD_{21})^T \end{bmatrix}.$$

(iii) compute

$$\widehat{\boldsymbol{A}}_K = -(A+B_2D_KC_2)^T + \begin{bmatrix} XB_1+\widehat{\boldsymbol{B}}_KD_{21} & (C_1+D_{12}D_KC_2)^T \end{bmatrix} \begin{bmatrix} -\gamma I & D_{c\ell}^T \\ D_{c\ell} & -\gamma I \end{bmatrix}^{-1} \begin{bmatrix} (B_1+B_2D_KD_{21})^T \\ C_1Y+D_{12}\widehat{\boldsymbol{C}}_K \end{bmatrix}. \tag{53}$$

(iv) solve for N, M, the factorization problem

$$I - XY = NM^T.$$

(v) finally, compute A_K, B_K and C_K with the help of

$$A_K = N^{-1}(X\dot{Y} + N\dot{M}^T + \widehat{\boldsymbol{A}}_K - X(A-B_2D_KC_2)Y - \widehat{\boldsymbol{B}}_KC_2Y - XB_2\widehat{\boldsymbol{C}}_K)M^{-T} \tag{54}$$

$$B_K = N^{-1}(\widehat{\boldsymbol{B}}_K - XB_2D_K) \tag{55}$$

$$C_K = (\widehat{\boldsymbol{C}}_K - D_KC_2Y)M^{-T}. \tag{56}$$

The reader might consult [17, 16, 22, 37, 1] for details on this construction.

6. Numerical examples

6. Numerical examples . In this section, the concepts and tools developed above are illustrated by some numerical examples. All LMI-related computations were performed on a *ULTRA* 1 *SUN* station using the LMI Control Toolbox [19].

6.1. Stability analysis. We consider the following example from [39]. The A matrix of the uncertain system is given in the form:

$$A(\theta_1,\theta_2) = \begin{bmatrix} -2+\theta_1 & 0 & -1+\theta_1 \\ 0 & -3+\theta_2 & 0 \\ -1+\theta_1 & -1+\theta_2 & -4+\theta_1 \end{bmatrix}.$$

We are seeking the maximum rectangle in the (θ_1,θ_2)- space for which stability is guaranteed. In this context, Proposition 5.1 is directly applicable to the polytope of extreme values of the parameters θ_1 and θ_2. The uncertain system is found stable for all values of θ_1 and θ_2 in the rectangle

$$-1e6 \le \theta_1 \le 1.7499, \qquad -1e6 \le \theta_2 \le 2.99\,.$$

This result is consistent with the true domain of stability ($\theta_1 < 1.75$, $\theta_2 < 3$), and is markedly superior to existing results [39].

6.2. LPV synthesis example. The following example provides an illustration of the proposed LPV control synthesis techniques. The discussion emphasizes the complexity and cost associated with various LPV synthesis strategies. The problem setup comes from [33]. It has been slightly complicated to incorporate two time-varying parameters for illustration purpose, while retaining the same design specifications.

The LPV model of the longitudinal dynamics of the missile are given as:

$$\begin{bmatrix} \dot\alpha \\ \dot q \end{bmatrix} = \begin{bmatrix} -0.89 & 1 \\ -142.6 & \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 & -0.89 \\ 178.25 & 0 \end{bmatrix} \begin{bmatrix} w_{\theta_1} \\ w_{\theta_2} \end{bmatrix} + \begin{bmatrix} -0.119 \\ -130.8 \end{bmatrix} \delta_{\text{fin}}$$

$$\begin{bmatrix} w_{\theta_1} \\ w_{\theta_2} \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix}$$

$$\begin{bmatrix} \eta_z \\ q \end{bmatrix} = \begin{bmatrix} -1.52 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix}$$

where α, q, η_z and δ_{fin} denote the angle of attack, the pitch rate, the vertical accelerometer measurement, the fin deflection, respectively; and θ_1, θ_2 are two time-varying parameters, measured in real time, resulting from changes in missile aerodynamic conditions (angle of attack from 0 up to 20 degrees). The synthesis structure used in this problem is depicted in Figure 1.

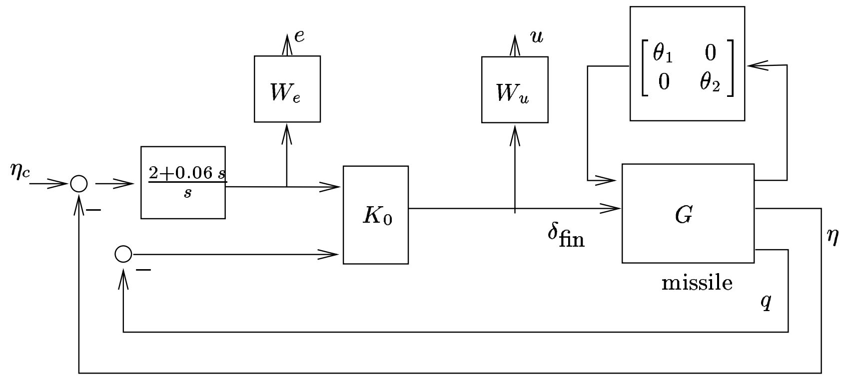

FIG. 1. *Synthesis structure*

The problem specifications are as follows:

(i) A settling time of 0.2 second with minimal overshoot and zero steady-state error for the vertical acceleration η_z in response to a step command η_c.

(ii) The controller must achieve an adequate high-frequency roll-off for noise attenuation and to withstand neglected dynamics and flexible modes. Magnitude constraints of 2 are also imposed to the control signal δ_{fin}.

Moreover, those specifications must be met for all parameter values:

$$|\theta_1| \le 1, \qquad |\theta_2| \le 1\,.$$

An integrator has been introduced on the acceleration channel to ensure zero steady-state error. It turns out that the resulting LPV controller K is obtained as the composition of the operators K_0 and

$$\begin{bmatrix} \frac{2+0.06\,s}{s} & 0 \\ 0 & 1 \end{bmatrix}.$$

The weighting functions W_e and W_u were chosen to be

$$W_e = 0.8, \qquad W_u = \frac{0.001s^3 + 0.03s^2 + 0.3s + 1}{1\text{e-}5s^3 + 3\text{e-}2s^2 + 30s + 10000}\,.$$

The design synthesis consists in the computation of a parameter-dependent controller, $K_0(\theta_1, \theta_2)$ such that all specifications above are met. For simplicity of the discussion, we assume that the LPV model can be considered as a parameterized family of linear time-varying models. Similar conclusions can be drawn with time-varying parameters with bounded rates of variation. The synthesis problem is attacked via three different strategies with increasing conservatism and decreasing computational effort:

(i) The full gridding approach makes use of a 6×6 point gridding of the parameter range of (θ_1, θ_2).

(ii) The mixed strategy uses a grid in the θ_2 direction and enforces multi-convexity along the θ_1 direction.

(iii) The multi-convexity approach enforces multi-convexity in both directions θ_1 and θ_2.

Results and numerical features of each technique are collected in Table 1.

	# of gridding points	# of LMIs	cputime	achieved perf. level
full gridding	36	108	17 min. 24 sec.	0.1265
mixed strategy	12	30	6 min. 20 sec.	0.1282
multi-convexity	0	16	3 min. 12 sec.	0.1293

TABLE 1
Numerical comparisons of LPV synthesis techniques

It is instructive to see that all techniques provide about the same performance level. This indicates that there is no significant growth of conservatism when using multi-convexity concepts to reduce or eradicate the gridding points. This is confirmed by the time-domain simulations in Figures 2-4 which correspond, for each derived LPV controller, to parameter values at the vertices and the center of the (θ_1, θ_2)-range. Performance specs. as well as the roll-off property of the controllers have been checked to be satisfactory for each technique. In spite of the consistency in the results, it must be bore in mind that the multi-convex synthesis is the only one to provide theoretical stability and performance guarantees at any operating condition of the parameter range. The full-gridding technique gives similar guarantees solely at the grid points and the achieved performance γ is necessarily a lower bound of the actual performance. The situation is slightly more embarrassing for the mixed strategy since the performance level is over-estimated in the direction of θ_2 and under-estimated in the direction of θ_1. So that we cannot decide whether the result is conservative or optimistic on the whole parameter range. Nevertheless, the approach is of practical interest for computational reasons. It appears clearly in Table 1 that LPV syntheses exploiting either partial or complete multi-convexity are significantly cheaper than full-gridding techniques. This difference is likely to be even more dramatic for problems involving more than two parameters for which full gridding is practically prohibited. This is a direct consequence of the exponential growth of the number of LMIs in the full-gridding approach. Note that we do not account for scalar dimensional constraints of the type λ_i, $\mu_i \geq 0$ in Table 1, as they negligibly affect the overall computational time.

Any of the LPV synthesis techniques considered in this section turns out to be less conservative than LFT (Linear Fractional Transformation) gain-scheduling techniques [30, 2, 20, 38] which disregard the parameter variation rates. About 10 percents degradation of the performance level has been observed in this simple application. Finally, the techniques behave as theoretically expected and provide valuable LPV synthesis alternatives.

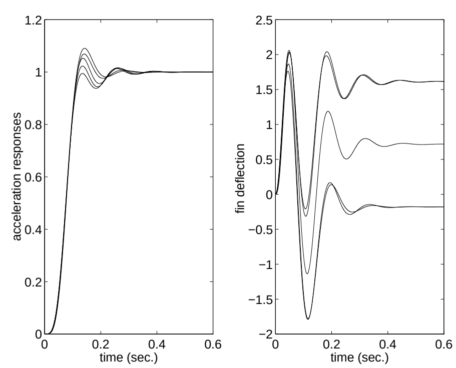

FIG. 2. *Time domain responses - LPV controller #1 full-gridding technique*

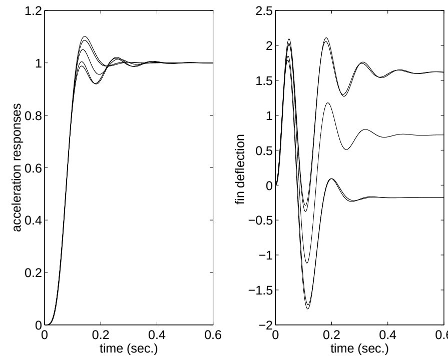

FIG. 3. *Time domain responses - LPV controller #2 mixed technique*

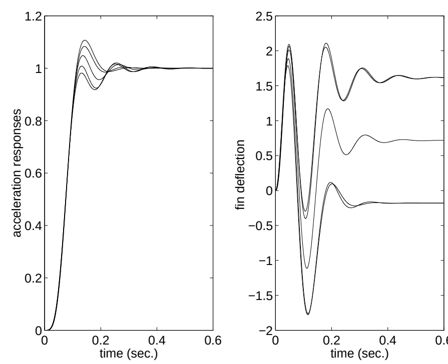

FIG. 4. *Time domain responses - LPV controller #3 multi-convexity technique*

7. Conclusion. A general framework for relaxing PLMI problems into conventional LMI problems has been introduced. The techniques are simple and exploit or enforce directional convexity properties of PLMI problems. A non-exhaustive list of implications of the proposed techniques in control theory have been examined with a particular focus on LPV synthesis, a most important emerging technique in the recent years.

This work raises some open questions some of which might be beyond reach but also suggests some directions for future research:

(i) For affine PLMI problems, directional convexity concepts are less conservative than the **S**-procedure but a theoretical comparison is still lacking in the general case. From the viewpoint of computational efforts, one can hardly draw definitive conclusions but the proposed approach is better exploited by using primal SDP interior-point techniques since it involves less decision variables than the **S**-procedure. We note experimentally that the multi-convexity approach is more efficient for problems with significantly more states than parameters which is a common situation in control applications.

(ii) An unsolved issue is the following. Is it possible to exploit directional quasi-convexity instead of directional convexity for some classes of PLMI problems ?

(iii) Other topics not examined in this paper and for which the proposed techniques might prove useful are robust least-squares and robust interpolation and approximation. The relaxation of some intractable generic robust convex programs is also of interest.

Appendix. Proof of Theorem 5.2. Following [44], the LPV control problem with guaranteed L_2-gain performance γ is solvable whenever one can find a LPV controller such that a suitable extension of the

Bounded Real Lemma is satisfied with a quadratic parameter-dependent Lyapunov function, continuously differentiable with respect to θ. This is nothing else than statement *(i)* of the theorem.

In turn, the latter conditions are equivalent to the existence of continuously parameter-dependent symmetric matrices $X(\theta)$ and $Y(\theta)$ such that LMI conditions (38)-(40) hold for all $(\theta, \dot\theta)$ on $H \times H_d$. This assertion is a slight extension of the main result in [44, 17, 22]. Assume a (closed-loop) Lyapunov function establishing L_2-gain performance is

$$V(x, x_K, \theta) := \begin{bmatrix} x \\ x_K \end{bmatrix}^T P(\theta) \begin{bmatrix} x \\ x_K \end{bmatrix},$$

where

$$P := \begin{bmatrix} X & N \\ N^T & \star \end{bmatrix}, \qquad P^{-1} := \begin{bmatrix} Y & M \\ M^T & \star \end{bmatrix}.$$

It trivially holds that

$$I - XY = NM^T.$$

Then, defining

$$\Pi_Y := \begin{bmatrix} Y & I \\ M^T & 0 \end{bmatrix}, \qquad \Pi_X := \begin{bmatrix} I & X \\ 0 & N^T \end{bmatrix}$$

yields the identities $P\Pi_Y = \Pi_X$, and

$$\Pi_Y^T \frac{d}{dt} P \Pi_Y = \begin{bmatrix} -\dot Y & -(X\dot Y + N\dot M^T)^T \\ -(X\dot Y + N\dot M^T) & \dot X \end{bmatrix}$$

which is the only additional term with respect to the customary H_∞ control problem in [17]. This establishes statement *(ii)*.

To prove assertion *(iii)*, we first note that the rates of variation $\dot\theta_i$ are involved linearly in (38) and (39), and thus it suffices to assess feasibility of these LMIs over $H \times \operatorname{vert} H_d$. By virtue of Finsler's Lemma [31, 22], the LMIs (38)-(39) with $(\theta, \dot\theta)$ ranging over $H \times \operatorname{vert} H_d$ are feasible if and only if there exist a function $\sigma(.)$ of θ such that

$$\begin{bmatrix} \dot X + XA + A^T X & XB_1 & C_1^T \\ B_1^T X & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} - \sigma(\theta) \begin{bmatrix} C_2^T \\ D_{21}^T \\ 0 \end{bmatrix} \begin{bmatrix} C_2 & D_{21} & 0 \end{bmatrix} < 0 \tag{57}$$

$$\begin{bmatrix} -\dot Y + YA^T + AY & YC_1^T & B_1 \\ C_1 Y & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} - \sigma(\theta) \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix} \begin{bmatrix} B_2^T & D_{12}^T & 0 \end{bmatrix} < 0. \tag{58}$$

Therefore, we end up with an infinite set of LMIs whose members are all of the form

$$\Psi(\theta) - \sigma(\theta) R(\theta) R(\theta)^T < 0, \qquad \theta \in H. \tag{59}$$

Denote $R_\perp$ a continuous basis of the nullspace of R^T. This is always possible [21, 25] by virtue of assumption **(A3)**. It follows that $[R(\theta) \quad R_\perp(\theta)]$ is a continuous invertible matrix over H. With the *proviso* that the LMIs (59) are feasible, it is not difficult to see using Schur complements that admissible σ are described as

$$\sigma(\theta) > l(\theta) := \overline{\lambda}\{(R^T\Psi R - (R^T\Psi R_\perp)(R_\perp^T \Psi R_\perp)^{-1}(R^T\Psi R_\perp)^T)(R^T R R^T R)^{-1}\}, \qquad \theta \in H \tag{60}$$

where $\overline{\lambda}(.)$ stands for the maximum eigenvalue of a matrix. From the continuity of both the plant's state-space data (assumption **(A1)**) and the variables $X(\theta)$ and $Y(\theta)$, we deduce that $l(\theta)$ in (60) is also continuous with respect to θ. Now since H is a compact set, the choice

$$\sigma(\theta) := \sigma > \sup_{\theta \in H} l(\theta)$$

is again a valid choice for σ. It follows that the LMIs (57) and (58) are feasible if and only if this is so for a *constant* sufficiently large σ. This completes the proof of the theorem. ☐

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