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Robust Filtering for Discrete Nonlinear Fractional Transformation Systems

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Abstract

Robust filtering for uncertain discrete systems has been intensively studied in literature in recent years. Nonlinear fractional transformation (NFT) is an attractive tool, which effectively exploits partial linear structures of nonlinear systems. The paper gives viable linear matrix inequality (LMI) optimization formulations for uncertain NFT discrete systems with performance criteria based on generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm constraints. This is verified by thorough computer simulations and comparisons.

1 Introduction

In recent years, robust filtering has been intensively studied in the literature [6, 7, 8, 10, 11, 12, 13, 14, 16]. This is due to the introduction [3, 4] of linear matrix inequality (LMI) as the main tool toward effective solutions of robust control and filtering. Indeed, LMI setting is really fit to handle the robust optimization and estimation because most realistic uncertainty constraints can be adequately and accurately expressed by LMIs. Usually, the uncertain systems are assumed linear in uncertain parameters [6, 10, 13, 16]. When uncertain parameters enter continuous systems in nonlinear way, robust filtering have been addressed in [10, 14]. The result of [10] is given as matrix inequalities, which are still nonlinear in all scaling vector variables, while the result of [14] is given by completely LMIs. As shown by [14], it is crucial to express nonlinear parameter dependence of a system in a tractable form, which allows exploiting its partial linear structures that can be maximally used for LMI derivation. Nonlinear Fractional Transformation introduced in [14] seems to be the appropriate model for this purpose.

The aim of this paper is to extend the result of [14] to the case of discrete systems, that is the robust filtering for discrete uncertain linear system in the nonlinear fractional transformation (NFT) form

$$\begin{array}{rcl}\begin{bmatrix} x(k+1) \\ y(k) \\ z_\Delta(k) \\ z(k) \end{bmatrix} & = & \begin{bmatrix} A(\alpha) & B_\Delta(\alpha) & B(\alpha) \\ C(\alpha) & D_\Delta(\alpha) & D(\alpha) \\ C_\Delta(\alpha) & D_{\Delta z}(\alpha) & D_z(\alpha) \\ L(\alpha) & D_{\Delta\Delta}(\alpha) & M(\alpha) \end{bmatrix} \begin{bmatrix} x(k) \\ w_\Delta(k) \\ w(k) \end{bmatrix}, \\ w_\Delta(k) & = & \Delta(\alpha) z_\Delta(k), \end{array} \tag{1}$$

where $A(\alpha) \in \boldsymbol{R}^{n\times n}$, $B_\Delta(\alpha) \in \boldsymbol{R}^{n\times m_\Delta}$, $B(\alpha) \in \boldsymbol{R}^{n\times m}$, $D(\alpha) \in \boldsymbol{R}^{p\times m}$, $C_\Delta(\alpha) \in R^{m_\Delta \times n}$, $L(\alpha) \in \boldsymbol{R}^{q\times n}$ and $x \in \boldsymbol{R}^n$ is the state, $y \in \boldsymbol{R}^p$ is the measured output, $z \in \boldsymbol{R}^q$ is the output to be estimated and $w \in R^m$ is noise, $w_\Delta \in R^{m_\Delta}$ and $z_\Delta \in R^{m_\Delta}$ help manage the uncertainty component of the system. The uncertain parameter α is supposed to be in the unit simplex Γ:

$$\Gamma := \{(\alpha_1, ..., \alpha_s) : \sum_{j=1}^{s} \alpha_j = 1,\ \alpha_j \geq 0\}.$$

and the state-space matrix data in (1) are such that

$$\begin{bmatrix} A(\alpha) & B_\Delta(\alpha) & B(\alpha) \\ C(\alpha) & D_\Delta(\alpha) & D(\alpha) \\ C_\Delta(\alpha) & D_{\Delta z}(\alpha) & D_z(\alpha) \\ L(\alpha) & D_{\Delta\Delta}(\alpha) & M(\alpha) \\ 0 & \Delta(\alpha) & 0 \end{bmatrix} = \sum_{j=1}^{s} \alpha_j \begin{bmatrix} A_j & B_{\Delta j} & B_j \\ C_j & D_{\Delta j} & D_j \\ C_{\Delta j} & D_{\Delta z j} & D_{zj} \\ L_j & D_{\Delta\Delta j} & M_j \\ 0 & \Delta_j & 0 \end{bmatrix} \tag{2}$$

or, in short, they are linear in parameter α.

Such NFT has been introduced in [14] as a tool for representing uncertain continuous systems. The NFT representation (1) covers the linear fractional transformation (LFT) representation [17] in which only $\Delta(\alpha)$ is uncertain and the polytopic representation with $\Delta(\alpha) = 0$ as two particular classes. Though it is well known that LFT can be applied to almost all the uncertain systems, the advantages of NFT compared to LFT is that it results in substantial reduction in term of system dimensions and solutions to polytopic and LFT systems can be easily inferred from those to the NFT ones. Dimension reduction by NFT can lead to dramatically better analysis and synthesis whereas LFT may lead to performance deterioration. This will be clearly demonstrated in section 4.

On the other hand, it is obvious that the structure of the used filter class has much influence on the filter performance. The customary used filters [6, 8, 10, 13, 14, 16] usually take the strictly proper form

$$\begin{bmatrix} x_F(k+1) \\ z_F(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_F & \mathbf{B}_F \\ \mathbf{L}_F & 0 \end{bmatrix} \begin{bmatrix} x_F(k) \\ y(k) \end{bmatrix}, \qquad (3)$$

While strictly proper filters work well for continuous systems [10, 13, 14], they may not be the best candidate for discrete systems. This is due to the fact that according to (3), at each time instant k, one estimates the output $z(k)$ of system (1) based on information of the measured output y available up only to time $k-1$. Thus, naturally, the filtering performance can be essentially improved by using the following proper structure introduced in the paper

$$\begin{bmatrix} x_F(k+1) \\ z_F(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_F & \mathbf{B}_F \\ \mathbf{L}_F & \mathbf{D}_F \end{bmatrix} \begin{bmatrix} x_F(k) \\ y(k) \end{bmatrix}, \qquad (4)$$
$$\mathbf{A}_F \in \boldsymbol{R}^{n\times n}, \mathbf{L}_F \in \boldsymbol{R}^{q\times n}$$

which obviously updates the estimation $z_F(k)$ for the output $z(k)$ based on information on all states of the measured output $y(k)$ up to present instant k. Furthermore, the estimation criterion of filters is based on the mixed generalized $\mathcal{H}_2/\mathcal{H}_\infty$ criterion

$$\max_{\alpha\in\Gamma}[\rho||z-z_F||^2_{\text{pk}} + (1-\rho)||z-z_F||_2^2] \to \min, \qquad (5)$$

where $||.||_{\text{pk}}$ and $||.||_2$ denote signal norms inducing the discrete generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms respectively corresponding to the generalized $\mathcal{H}_2$ norm and the $\mathcal{H}_\infty$ norm of continuous systems. Like their counterparts for continuous systems, the generalized $\mathcal{H}_2$ norm constraint introduced in section 2 is the peak error amplitude criterion and $\mathcal{H}_\infty$ norm constraint is the error energy criterion so (5) makes a compromise between the two conflicting constraints with trade-off constant ρ $(0 \le \rho \le 1)$. So, solutions to the generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problems are on hand readily.

This paper develops an effective approach toward the posed robust filtering problems. We are then successful in:

- Making out a new characterization of the generalized $\mathcal{H}_2$ norm constraint for uncertain NFT systems.
- Forming new LMI formulations for uncertain NFT systems. Coherently, new LMI formulations for polytopic and LFT cases are available.

We organize the paper as follows. Section 2 outlines characterizations of the generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the above NFT systems. Section 3 transforms these characterizations into LMI formulations. Section 4 validates the effectiveness of our approach via thorough simulations and comparisons.

2 Characterizations for norm constraints

LMI based formulations will be provided for the generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm constraints to evaluate the corresponding performances of filters. This is done with the loop system (1) having the output to be minimized $z_{cl} = z - z_F$

$$\begin{aligned} \begin{bmatrix} x_{cl}(k+1) \\ z_\Delta(k) \\ z_{cl}(k) \end{bmatrix} &= \begin{bmatrix} \mathcal{A}_{cl} & \mathcal{B}_{\Delta cl} & \mathcal{B}_{cl} \\ \mathcal{C}_\Delta & D_{\Delta z} & D_z \\ \mathcal{L}_{cl} & \mathcal{D}_{cl} & \mathcal{M}_{cl} \end{bmatrix} \begin{bmatrix} x_{cl}(k) \\ w_\Delta(k) \\ w(k) \end{bmatrix}, \\ w_\Delta(k) &= \Delta(\alpha) z_\Delta(k) \end{aligned} \qquad (6)$$

where

$$\begin{aligned} & x_{cl}(k) = \begin{bmatrix} x(k) \\ x_F(k) \end{bmatrix}, \ \mathcal{A}_{cl}(\alpha) = \begin{bmatrix} A(\alpha) & 0 \\ \mathbf{B}_F C(\alpha) & \mathbf{A}_F \end{bmatrix}, \\ & \mathcal{B}_{\Delta cl}(\alpha) = \begin{bmatrix} B_\Delta(\alpha) \\ \mathbf{B}_F D_\Delta(\alpha) \end{bmatrix}, \ \mathcal{B}_{cl}(\alpha) = \begin{bmatrix} B(\alpha) \\ \mathbf{B}_F D(\alpha) \end{bmatrix}, \\ & \mathcal{C}_\Delta(\alpha) = [\, C_\Delta(\alpha) \quad 0 \,], \ \mathcal{D}_{cl}(\alpha) = D_{\Delta\Delta}(\alpha) - \mathbf{D}_F D_\Delta(\alpha), \\ & \mathcal{L}_{cl}(\alpha) = [\, L(\alpha) - \mathbf{D}_F C(\alpha) \quad -\mathbf{L}_F \,], \\ & \mathcal{M}_{cl}(\alpha) = M(\alpha) - \mathbf{D}_F D(\alpha). \end{aligned} \qquad (7)$$

2.1 Generalized $\mathcal{H}_2$-norm characterization

The generalized $\mathcal{H}_2$ norm of the (6) is defined as

$$\sup_{w,T} \frac{||z_{cl}(T)||}{\left(\sum_{k=0}^{T} ||w(k)||^2\right)^{1/2}} \qquad (8)$$

Thus, if the inequality

$$||z_{cl}(T)||^2 < \nu \sum_{k=0}^{T} ||w(k)||^2 \qquad (9)$$

holds for any input sequence $w(k)$ and its output sequence $z_{cl}(k)$, then the generalized $\mathcal{H}_2$ norm of system (6) is less than $\sqrt{\nu}$ and vice versa.

Theorem 1 *One has (9) guaranteeing the generalized $\mathcal{H}_2$-norm of system (6) less than $\sqrt{\nu}$ if for every $\alpha \in \Gamma$, there are symmetric matrices $\mathbf{X}(\alpha) > 0$ and scaling matrices $\mathbf{R}_i(\alpha)$, $\mathbf{S}_i(\alpha)$, a scalar μ and slack matrices $\mathbf{V}(\alpha)$, $\mathbf{H}_i(\alpha)$, $\mathbf{F}_i(\alpha)$ satisfying the following matrix inequalities*

$$\begin{bmatrix} \mathbf{R}_i & \Delta^T \mathbf{H}_i^T \\ \mathbf{H}_i \Delta & \mathbf{S}_i + (\mathbf{H}_i + \mathbf{H}_i^T) \end{bmatrix} (\alpha) \ge 0 \ \forall \alpha \in \Gamma \quad i = 1, 2; \qquad (10)$$

$$\begin{bmatrix} T_{11} & * & * & * \\ 0 & T_{22} & * & * \\ T_{31} & T_{32} & T_{33} & * \\ T_{41} & T_{42} & 0 & T_{44} \end{bmatrix} (\alpha) < 0 \ \forall \alpha \in \Gamma, \qquad (11)$$

$$\begin{bmatrix} U_{11} & * & * & * \\ 0 & U_{22} & * & * \\ U_{31} & U_{32} & U_{33} & * \\ U_{41} & U_{42} & 0 & -\nu I \end{bmatrix} (\alpha) < 0 \ \forall \alpha \in \Gamma \qquad (12)$$

where

$$T_{11}=-\mathbf{X},\ T_{22}=\begin{bmatrix}\mathbf{S}_1 & * \\ 0 & -(1-\mu)I\end{bmatrix},$$
$$T_{31}=\mathbf{V}^T\mathcal{A}_{cl},\ T_{32}=\mathbf{V}^T[\,\mathcal{B}_{\Delta cl}\quad \mathcal{B}_{cl}\,],$$
$$T_{33}=\mathbf{X}-(\mathbf{V}+\mathbf{V}^T),\ T_{41}=\mathbf{F}_1\mathcal{C}_\Delta,$$
$$T_{42}=\mathbf{F}_1[\,D_{\Delta z}\quad D_z\,],\ T_{44}=\mathbf{R}_1-(\mathbf{F}_1+\mathbf{F}_1^T), \tag{13}$$
$$U_{11}=-\mathbf{X},\ U_{22}=\begin{bmatrix}\mathbf{S}_2 & * \\ 0 & -\mu I\end{bmatrix},\ U_{31}=\mathbf{F}_2\mathcal{C}_\Delta,$$
$$U_{32}=\mathbf{F}_2[\,D_{\Delta z}\quad D_z\,],\ U_{33}=\mathbf{R}_2-(\mathbf{F}_2+\mathbf{F}_2^T),$$
$$U_{41}=\mathcal{L}_{cl},\ U_{42}=[\,\mathcal{D}_{cl}\quad \mathcal{M}_{cl}\,],$$

2.2 $\mathcal{H}_\infty$-norm characterization

As well defined, the $\mathcal{H}_\infty$ norm for system (6) is

$$\sup_{w,T}\frac{\left(\sum_{k=0}^{T}||z_{cl}(k)||^2\right)^{1/2}}{\left(\sum_{k=0}^{T}||w(k)||^2\right)^{1/2}} \tag{14}$$

Hence

$$\sum_{k=0}^{T}||z_{cl}(k)||^2<\gamma^2\sum_{k=0}^{T}||w(k)||^2 \tag{15}$$

always holds, meaning that the $\mathcal{H}_\infty$-norm of system (6) is less than γ.

Theorem 2 *One has (15) if for every $\alpha\in\Gamma$ there are matrices $\mathbf{Y}(\alpha)>0$, $\mathbf{V}(\alpha)$, $\mathbf{R}(\alpha)$, $\mathbf{S}(\alpha)$, $\mathbf{H}(\alpha)$ and $\mathbf{F}(\alpha)$ satisfying the following inequalities*

$$\begin{bmatrix}\mathbf{R} & \Delta^T\mathbf{H}^T \\ \mathbf{H}\Delta & \mathbf{S}+(\mathbf{H}+\mathbf{H}^T)\end{bmatrix}(\alpha)\geq 0\ \forall\alpha\in\Gamma, \tag{16}$$

$$\begin{bmatrix}P_{11} & * & * & * & * \\ 0 & P_{22} & * & * & * \\ P_{31} & P_{32} & P_{33} & * & * \\ P_{41} & P_{42} & 0 & P_{44} & * \\ P_{51} & P_{52} & 0 & 0 & -\gamma I\end{bmatrix}(\alpha)<0\ \forall\alpha\in\Gamma \tag{17}$$

where

$$P_{11}=-\mathbf{Y},\ P_{22}=\begin{bmatrix}\mathbf{S} & * \\ 0 & -\gamma I\end{bmatrix}$$
$$P_{31}=\mathbf{V}^T\mathcal{A}_{cl},\ P_{32}=\mathbf{V}^T[\,\mathcal{B}_{\Delta cl}\quad \mathcal{B}_{cl}\,]$$
$$P_{33}=\mathbf{Y}-(\mathbf{V}+\mathbf{V}^T),\ P_{41}=\mathbf{F}\mathcal{C}_\Delta,\ P_{42}=\mathbf{F}[\,D_{\Delta z}\quad D_z\,],$$
$$P_{44}=\mathbf{R}-(\mathbf{F}+\mathbf{F}^T),\ P_{51}=\mathcal{L}_{cl},\ P_{52}=[\,\mathcal{D}_{cl}\quad \mathcal{M}_{cl}\,]. \tag{18}$$

3 Robust filters for NFT

In (6), note that $\mathcal{A}_{cl}(\alpha),\mathcal{B}_{\Delta cl}(\alpha),\mathcal{B}_{cl}(\alpha)$ are functions of the variable $\mathbf{K}=[\,\mathbf{B}_F\quad \mathbf{A}_F\,]$

$$[\,\mathcal{A}_{cl}\quad[\,\mathcal{B}_{\Delta cl}\quad \mathcal{B}_{cl}\,]\,](\alpha)=$$
$$\sum_{j=1}^{s}\alpha_j([\,\Theta A_j\Theta^T\quad \Theta\mathcal{B}_j\,]+\Upsilon\mathbf{K}[\,\mathcal{C}_j\quad \Theta_p\mathcal{D}_j\,]) \tag{19}$$

where

$$\Upsilon=\begin{bmatrix}0_n \\ I_n\end{bmatrix},\ \Theta=\begin{bmatrix}I_n \\ 0_n\end{bmatrix},\Theta_p=\begin{bmatrix}I_p \\ 0_{np}\end{bmatrix},$$
$$\mathcal{I}=[\,I_n\quad I_n\,],\ \mathcal{B}_j=[\,B_{\Delta j}\quad B_j\,],\ \mathcal{C}_j=\begin{bmatrix}C_j & 0_{pn} \\ 0_n & I_n\end{bmatrix},$$
$$\mathcal{D}_j=[\,D_{\Delta j}\quad D_j\,],\ \mathcal{D}_{zj}=[\,D_{\Delta zj}\quad D_{zj}\,]. \tag{20}$$

This facilitates the linearization for the stated generalized $\mathcal{H}_2/\mathcal{H}_\infty$ norm characterizations .

3.1 Robust generalized $\mathcal{H}_2$ filter

In order to derive tractable LMI-based formulation for the posed filtering problems, we must impose the following structures for decision variables in (11), (12) and (10)

$$\mathbf{V}(\alpha)\equiv\mathbf{V},\ \mathbf{F}_i(\alpha)\equiv\mathbf{F}_i,\ \mathbf{H}_i(\alpha)\equiv\mathbf{H}_i\ \forall\alpha\in\Gamma,\ i=1,2. \tag{21}$$

$$\mathbf{X}(\alpha)=\sum_{j=1}^{s}\alpha_j\mathbf{X}_j,\ \mathbf{R}_i(\alpha)=\sum_{j=1}^{s}\alpha_j\mathbf{R}_{ij},$$
$$\mathbf{S}_i(\alpha)=\sum_{j=1}^{s}\alpha_j\mathbf{S}_{ij},\ i=1,2, \tag{22}$$

i.e. the basic variables are parameter-dependent while the slack variables are not.

Theorem 3 *There is a filter (4) which satisfies the estimation condition (9) whenever the following LMI constraints are feasible in $\hat{\mathbf{V}}$, $\hat{\mathbf{X}}_j$, $\mathbf{S}_{ij}$, $\mathbf{R}_{ij}$, $\hat{\mathbf{K}}$, $\hat{\mathbf{L}}_F$, $\mathbf{D}_F$, $\mathbf{H}_i$, $\mathbf{F}_i$ and μ*

$$\begin{bmatrix}\mathbf{R}_{ij} & \Delta_j^T\mathbf{H}_i^T \\ \mathbf{H}_i\Delta_j & \mathbf{S}_{ij}+(\mathbf{H}_i+\mathbf{H}_i^T)\end{bmatrix}\geq 0, \tag{23}$$

$$\begin{bmatrix}M_{11}^j & * & * & * \\ 0 & M_{22}^j & * & * \\ M_{31}^j & M_{32}^j & M_{33}^j & * \\ M_{41}^j & M_{42}^j & 0 & M_{44}^j\end{bmatrix}<0, \tag{24}$$

$$\begin{bmatrix}N_{11}^j & * & * & * \\ 0 & N_{22}^j & * & * \\ N_{31}^j & N_{32}^j & N_{33}^j & * \\ N_{41}^j & N_{42}^j & 0 & -\nu I\end{bmatrix}<0, \tag{25}$$
$$j=1,2,...,s.$$

Here,

$$M_{11}^j=-\hat{\mathbf{X}}_j,\ M_{22}^j=\begin{bmatrix}\mathbf{S}_{1j} & * \\ 0 & -(1-\mu I)\end{bmatrix},$$
$$M_{31}^j=\hat{\mathbf{V}}^T\Theta A_j\Theta^T+\mathcal{I}^T\hat{\mathbf{K}}\mathcal{C}_j,$$
$$M_{32}^j=\hat{\mathbf{V}}^T\Theta\mathcal{B}_j+\mathcal{I}^T\hat{\mathbf{K}}\Theta_p\mathcal{D}_j,$$
$$M_{33}^j=\hat{\mathbf{X}}_j-(\hat{\mathbf{V}}+\hat{\mathbf{V}}^T),\ M_{41}^j=\mathbf{F}_1C_{\Delta j}\Theta^T,$$
$$M_{42}^j=\mathbf{F}_1\mathcal{D}_{zj},\ M_{44}^j=\mathbf{R}_{1j}-(\mathbf{F}_1+\mathbf{F}_1^T),$$

$$N_{11}^j=-\hat{\mathbf{X}}_j,\ N_{22}^j=\begin{bmatrix}\mathbf{S}_{2j} & * \\ 0 & -\mu I\end{bmatrix},\ N_{31}^j=\mathbf{F}_2C_{\Delta j}\Theta^T,$$
$$N_{32}^j=\mathbf{F}_2\mathcal{D}_{zj},\ N_{33}^j=\mathbf{R}_{2j}-(\mathbf{F}_2+\mathbf{F}_2^T),$$
$$N_{41}^j=[\,L_j-\mathbf{D}_FC_j\quad -\hat{\mathbf{L}}_F\,],$$
$$N_{42}^j=[\,D_{\Delta\Delta j}\quad M_j\,]-\mathbf{D}_F\mathcal{D}_j. \tag{26}$$

The matrix data $\mathbf{A}_F$, $\mathbf{B}_F$, $\mathbf{L}_F$, $\mathbf{D}_F$ defining the filter (4) can be derived from the solutions of the matrix inequalities (23), (24), (25) according to

$$\mathbf{A}_F=\hat{\mathbf{A}}_F\hat{\mathbf{V}}_3^{-T},\ \mathbf{B}_F=\hat{\mathbf{B}}_F,\ \mathbf{L}_F=\hat{\mathbf{L}}_F\hat{\mathbf{V}}_3^{-T}. \tag{27}$$

3.2 $\mathcal{H}_\infty$ and mixed generalized $\mathcal{H}_2/\mathcal{H}_\infty$ filters

Theorem 4 *There is a filter (4) which satisfies the robust estimation condition (15) whenever the following LMIs are feasible in* $\hat{\mathbf{V}},\ \hat{\mathbf{Y}}_j,\ \mathbf{S}_j,\ \mathbf{R}_j,\ \hat{\mathbf{K}},\ \hat{\mathbf{L}}_F,\ \mathbf{D}_F,\ \mathbf{G},\ \mathbf{F}$.

$$\begin{bmatrix} \mathbf{R}_j & \Delta_j^T\mathbf{H}^T \\ \mathbf{H}\Delta_j & \mathbf{S}_j + (\mathbf{H}+\mathbf{H}^T) \end{bmatrix} \geq 0, \tag{28}$$

$$\begin{bmatrix} E_{11}^j & * & * & * & * \\ 0 & E_{22}^j & * & * & * \\ E_{31}^j & E_{32}^j & E_{33}^j & * & * \\ E_{41}^j & E_{42}^j & 0 & E_{44}^j & * \\ E_{51}^j & E_{52}^j & 0 & 0 & -\gamma I \end{bmatrix} < 0, \quad j = 1, 2..., s \tag{29}$$

with

$$\begin{array}{l} E_{11}^j = -\hat{\mathbf{Y}}_j,\ E_{22}^j = \begin{bmatrix} \mathbf{S}_j & * \\ 0 & -\gamma I \end{bmatrix} \\ E_{31}^j = \hat{\mathbf{V}}^T\Theta A_j\Theta^T + \mathcal{I}^T\hat{\mathbf{K}}\mathcal{C}_j,\ E_{32}^j = \hat{\mathbf{V}}^T\Theta B_j + \mathcal{I}^T\hat{\mathbf{K}}\Theta_p\mathcal{D}_j \\ E_{33}^j = \hat{\mathbf{Y}}_j - (\hat{\mathbf{V}}+\hat{\mathbf{V}}^T),\ E_{41}^j = \mathbf{F}C_{\Delta j}\Theta^T,\ E_{42}^j = \mathbf{F}\mathcal{D}_{zj}, \\ E_{44}^j = \mathbf{R}_j - (\mathbf{F}+\mathbf{F}^T),\ E_{51}^j = [\,L_j - \mathbf{D}_F C_j \quad -\hat{\mathbf{L}}_F\,], \\ E_{52}^j = [\,D_{\Delta\Delta j} \quad M_j\,] - \mathbf{D}_F\mathcal{D}_j. \end{array} \tag{30}$$

The filter data $\mathbf{A}_F,\ \mathbf{B}_F,\ \mathbf{L}_F,\ \mathbf{D}_F$ *defining the filter (4) can be derived from the solutions of the LMIs (28) and (29) according to the formulas in (27).*

The solution to the optimal mixed filter problem (5) is merely the combination of the theorems 3 and 4.

Theorem 5 *A sub-optimal robust filter (4) for problem (5) can be solved by the following optimization problem*

$$\min[\rho\nu + (1-\rho)\gamma^2] : (23), (24), (25), (28), (29). \tag{31}$$

with decision variables $\hat{\mathbf{V}},\ \hat{\mathbf{X}}_j,\ \hat{\mathbf{Y}}_j,\ \mathbf{S}_{ij},\ \mathbf{R}_{ij},\ \mathbf{S}_j,\ \mathbf{R}_j,\ \hat{\mathbf{K}},\ \hat{\mathbf{L}}_F,\ \mathbf{D}_F,\ \mathbf{H}_i,\ \mathbf{F}_i,\ \mathbf{H},\ \mathbf{F},\ \mu,\ \nu,\ \gamma$. *The matrix data* $\mathbf{A}_F, \mathbf{B}_F, \mathbf{L}_F, \mathbf{D}_F$ *defining the suboptimal filter (4) can be derived from the solutions of the optimization problem (31) according to the formulas in (27).*

4 Numerical examples

Different representations of the system model (NFT/LFT) as well as different filter structures (4/3) may result in dramatically different estimation performances. This is shown via the solutions the robust filtering problems for the system

$$\begin{bmatrix} x(k+1) \\ y(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\alpha) & B \\ C & D \\ L & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{32}$$

with

$$\begin{array}{l} A(\alpha) = Q_0 + \alpha_1^3 Q_1 + \alpha_2^3 Q_2 + \alpha_1\alpha_2^2 Q_3 + \alpha_1 Q_4 + \alpha_2 Q_5, \\ Q_0 = \begin{bmatrix} -0.3 & 0.5 \\ 0.2 & -0.1 \end{bmatrix}, Q_1 = \begin{bmatrix} 0.1 & 0.15 \\ 0.1 & 0.15 \end{bmatrix}, \\ Q_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.25 & 0.25 \end{bmatrix},\ Q_3 = \begin{bmatrix} 0.2 & 0.15 \\ 0.2 & 0.15 \end{bmatrix}, \\ Q_4 = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.2 \end{bmatrix},\ Q_5 = \begin{bmatrix} 0.1 & 0 \\ 0.25 & 0.1 \end{bmatrix}, \\ B = \begin{bmatrix} -2 & 0 \\ 1.5 & 0 \end{bmatrix},\ C = [\,-10 \quad 10\,], \\ D = [\,0 \quad 3\,],\ L = [\,1 \quad 0\,]. \end{array} \tag{33}$$

Both representations are used.

- NFT as in (1) with

$$\begin{array}{l} A(\alpha) = \alpha_1(Q_0+Q_4) + \alpha_2(Q_0+Q_5), \\ B_\Delta(\alpha) = [\,\alpha_1 I_2 \quad \alpha_2 I_2\,] \begin{bmatrix} Q_1 & Q_3 \\ O & Q_2 \end{bmatrix}, \\ \Delta(\alpha) = \begin{bmatrix} \alpha_1 I_2 & 0_2 \\ 0_2 & \alpha_2 I_2 \end{bmatrix},\ D_{\Delta z} = 0, \\ C_\Delta(\alpha) = \begin{bmatrix} \alpha_1 I_2 \\ \alpha_2 I_2 \end{bmatrix}, D_z = 0, D_\Delta = 0, D_{\Delta\Delta} = 0 \end{array} \tag{34}$$

- LFT as in (1) with

$$\begin{array}{l} A = Q_0, B_\Delta = [\,I_2 \quad 0_2 \quad 0_2 \quad I_2 \quad 0_2 \quad 0_2\,], \\ D_{\Delta z} = \begin{bmatrix} 0_2 & Q_1 & 0_2 & 0_2 & Q_3 & 0_2 \\ 0_2 & 0_2 & I_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & Q_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & I_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \end{bmatrix}, C_\Delta = \begin{bmatrix} Q_4 \\ 0_2 \\ I_2 \\ Q_5 \\ 0_2 \\ I_2 \end{bmatrix} \\ D_z = 0, D_\Delta = 0, D_{\Delta\Delta} = 0, \Delta(\alpha) = \begin{bmatrix} \alpha_1 I_6 & 0_6 \\ 0_6 & \alpha_2 I_6 \end{bmatrix} \end{array} \tag{35}$$

Note that using theorems 1 and 2, the upper bounds on the generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of this system are found equal to 2.6405 and 5.5908 respectively. The improvement (Im.) ratios are fractions having the upper bound on the generalized $\mathcal{H}_2$ ($\mathcal{H}_\infty$) norm of the to be estimated sequence $z(k)$ of the system as their numerators and the corresponding upper bounds on generalized $\mathcal{H}_2$ ($\mathcal{H}_\infty$) norms of error sequences $z(k) - z_F(k)$ by filters as their respective denominators

The dimension 12 of z_Δ in the LFT (35) is three time larger than that of the NFT in (34), severely affecting computational efficiency and estimation performances of the resulting filters as described in tables 1 and 2. In addition, running times of LMI programs for NFT (34) are short whereas their counterparts for LFT (35) are very long. Table 3 lists the trade-offs between the generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances.

Tracking performances of proper filters and that of the strictly proper generalized $\mathcal{H}_2$ one are taken within 100

Model/Filter	$\mathcal{H}_2$	Time	Im. ratio
NFT/Proper	0.5503	6.5 s	4.7983
LFT/Proper	1.2469	13428 s	2.1176
NFT/Str. proper	2.0402	4.8 s	1.2942

Table 1: Generalized $\mathcal{H}_2$ performances of different filter structures and system representations

Model/Filter	$\mathcal{H}_\infty$	Time	Im. ratio
NFT/Proper	0.7934	2.5 s	7.0466
LFT/Proper	2.3471	2348 s	2.3820
NFT/Str. proper	2.2576	2.7s	2.4764

Table 2: $\mathcal{H}_\infty$ performances of different filter structures and system representations

steps in the case that noise is zero mean white noise with the identity spectral density.

Figure 1 captures the real (to be estimated) sequence $z(k)$. The error sequences $|z(k) - z_F(k)|^2$ by proper filters (fig. 2-4) are small in sample amplitude as compared to the real sequence, confirming that proper filters achieve good tracking performances. The error $|z(k) - z_F(k)|^2$ by the strictly proper generalized $\mathcal{H}_2$ filter (fig. 5) is nearly equal to the real sequence in absolute value, showing that the strictly proper generalized $\mathcal{H}_2$ filter is unacceptable. This well agrees with their improvement ratios listed in tables 3 and 4 as well as highlights the effectiveness of the proper filter structure.

The error sequence by the proper generalized $\mathcal{H}_2$ filter (fig. 2) shows peak sample amplitudes smaller than those of the proper $\mathcal{H}_\infty$ filter. As a compensation, the error sequence by the proper $\mathcal{H}_\infty$ filter (fig. 3) is smoother in amplitude change of samples than that of the proper generalized $\mathcal{H}_2$ filter. This reflects the physics nature of the two norm constraints as mentioned earlier. The error sequence by the proper mixed filter with the trade off constant $\rho = 0.9$ (fig. 4) is smoother in amplitude change of samples as compared to those of the proper generalized $\mathcal{H}_2$ filter and has peak sample amplitudes smaller than those of the proper $\mathcal{H}_\infty$ one. Thus, it realizes a compromise between the two conflicting constraints as desired.

ρ	Mixed	$\mathcal{H}_2$	$\mathcal{H}_\infty$
0.5	0.7830	0.9040	0.8653
0.7	0.6612	0.7901	0.8645
0.9	0.4939	0.6831	0.8599

Table 3: Performances of mixed proper filters by different trade-off constants (ρ) for the NFT model

5 Conclusions

We have proposed a new approach toward robust filtering for time invariant uncertain NFT systems. In this paper, NFT once again shows its advantages against LFT not only via its generality but also, of utmost importance, via the computation efficiency it results in. Our norm constraint characterizations using parameter dependent Lyapunov functions together with the proper filter structure bring about effective LMI optimization formulations for generalized $\mathcal{H}_2$, $\mathcal{H}_\infty$ and mixed filtering problems. Finally, the viability of these formulations is manifested by careful simulations and analysis.

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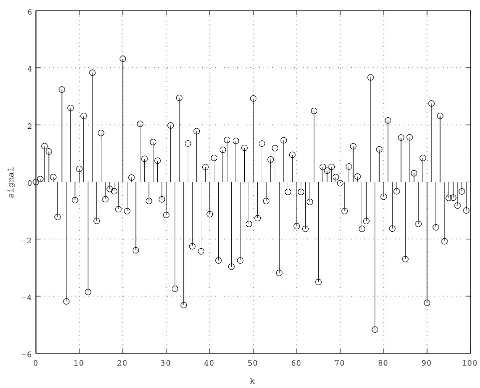

Figure 1: Real signal

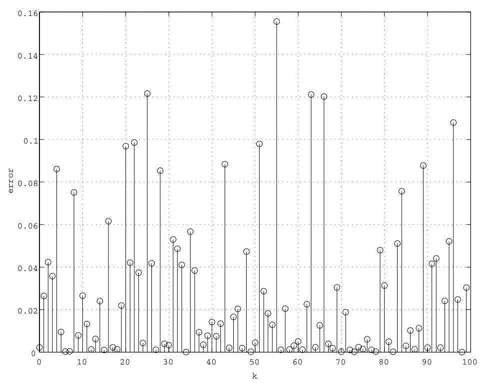

Figure 2: Error $|z(k) - z_F(k)|^2$ of the proper $\mathcal{H}_2$ filter

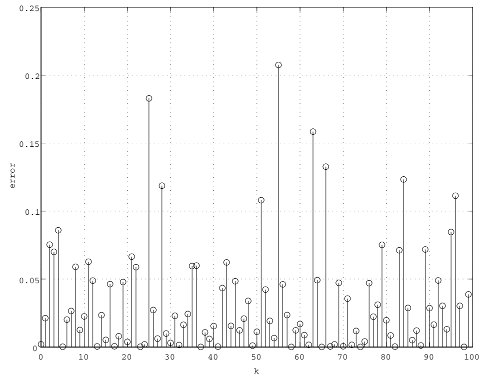

Figure 3: Error $|z(k) - z_F(k)|^2$ of the proper $\mathcal{H}_\infty$ filter

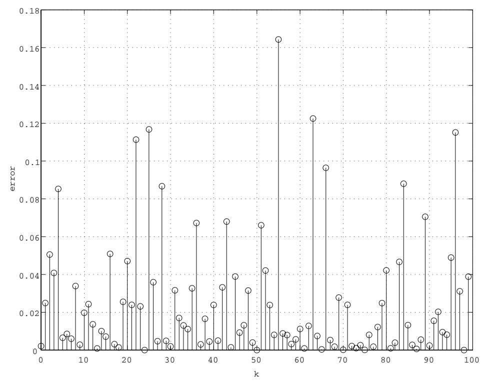

Figure 4: Error $|z(k) - z_F(k)|^2$ of the proper mixed filter

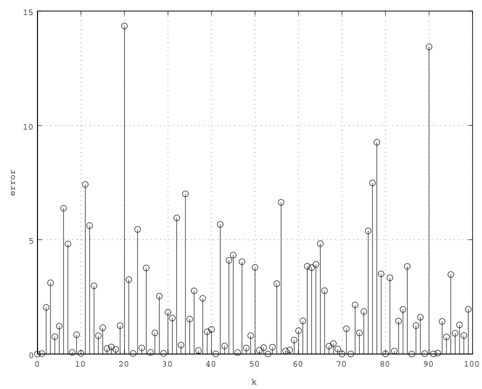

Figure 5: Error $|z(k) - z_F(k)|^2$ of the strictly proper $\mathcal{H}_2$ filter