Robust Control via Concave Minimization
Local and Global Algorithms

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Abstract

This paper is concerned with the robust control problem of LFT (Linear Fractional Representation) uncertain systems depending on a time-varying parameter uncertainty. Our main result exploits an LMI (Linear Matrix Inequality) characterization involving scalings and Lyapunov variables subject to an additional essentially nonconvex algebraic constraint. The nonconvexity enters the problem in the form of a rank deficiency condition or matrix inverse relation on the scalings only. It is shown that such problems but also more generally rank inequalities and bilinear constraints can be formulated as the minimization of a concave functional subject to Linear Matrix Inequality constraints. First of all, a *local* FW (Frank and Wolfe) feasible direction algorithm is introduced in this context to tackle this hard optimization problem. Exploiting the attractive concavity structure of the problem, several efficient *global* concave programming methods are then introduced and combined with the local feasible direction method to secure and certify global optimality of the solutions. Computational experiments indicate the viability of our algorithms, and that in the worst case they require the solution of a few LMI programs.

Keywords: Robust Control, Linear Matrix Inequalities, Concave Programming, Time-Varying uncertainties.

1 Introduction

A number of challenging problems in robust control theory fall within the class of rank minimization subject to LMI (convex) constraints (see e.g. [20, 9]) which are very hard nonconvex optimization problems with the NP-hardness shown in [24, 8].

As it plays a central role in robust control theory, many researchers have devoted their efforts to developing algorithms for determining solutions to this class of problems including heuristic methods [13, 16, 10], relaxation [18], and simple adaptation from general-purpose global optimization [21, 12]. A closely related algorithm, referred to as the *cone complementary linearization algorithm* is elaborated in [11, 7]. In [25], we developed a global optimization technique based upon d.c. (difference of convex functions/sets) optimization techniques exploiting the fact that the reverse convex constraints are of relatively low-rank, which is of primary importance to ensure practicality of the algorithm.

The contribution of this paper is twofold.

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• It is first shown that several important problems in robust control theory which involve bilinear constraints, equality and inequality rank constraints or matrix inverse constraints, can be recast as finding zero optimal solutions to generalized concave programs. These generalized concave programs consist in the minimization of a concave functional subject to convex constraints consisting of LMIs. A distinguished characteristic of these problems is that only *zero* solutions are of interest. This significantly reduces the difficulty of the search and thus makes the problems much more computationally attractive and painless than the conventional concave programs which seek an arbitrary minimum of a concave function over a convex set. A sample list of control applications of this new formulation includes robust control and robust multi-objective problems based on any kind of scalings or multipliers, robust fixed- or reduced-order control problems, multi-objective Linear Parameter-Varying (LPV) control, reduction of LFT representations, and more generally combinations of such problems. Starting from this viewpoint, the work here provides first a full generalization of the technique in [11] to handle robust control problems for plants subject to time-varying LFT (Linear Fractional Transformation) uncertainties. More precisely, we show that the robust synthesis problems involving either pairs of symmetric and skew-symmetric scalings or full generalized scalings as discussed in [22] are equivalent to zero-seeking concave programming problems where the convex constraints express in terms of LMIs. Although, this is not the central object of this paper, we reveal that BMI (Bilinear Matrix Inequality) problems can also be formulated in the same fashion, so that in this respect, concavity appears to be the most prominent feature of a very vast array of problems in control theory.

• We develop generalizations of local and global optimization methods for solving these zero-seeking concave programs. In this respect, we indicate how the FW algorithm must be modified to handle our problems. However, the FW algorithm is a local method and is not guaranteed to provide a global solution. This naturally leads us to combining recently available global search techniques with the FW algorithm to certify global optimality of the solutions or invalidate feasibility of the problem.

As concave programming is the best studied class of problems in global optimization [14, 15, 17, 27], we have exploited several key basic concepts for developing efficient and practical algorithms suitably generalized to the matrix context of our problems. Namely, we have paid special attention for developing extensions of the simplicial and conical BB (Branch and Bound) concave minimization methods which work with matrices and over the positive semidefinite cone. These methods respectively divide the feasible set into matrix simplices and matrix cones of decreasing sizes. Their main thrust is that they rely heavily on our specific matrix structures, on concavity and convexity geometric concepts which make them particularly appropriate for our problems. Therefore, an important target of this paper is to maintain a reasonable computational cost by combining local and global techniques. Hence, the global concave programming techniques are used either to refine a local solution issued from the FW algorithm until global optimality is achieved or to provide a certificate of global optimality. Due to space limitation, details on algorithms and experiments can be found in an extended version [3].

The following definitions and notations are used throughout the paper. For Hermitian or symmetric matrices, $M > N$ means that $M - N$ is positive definite and $M \geq N$ means that $M - N$ is positive semidefinite. The notation co $\{p_1, \ldots, p_L\}$ stands for the convex hull. The notation $|(P)$ is used to denote the set of vertices of a polyhedron P. In long matrix expressions, the symbol $\bigstar$ replaces terms that are induced by symmetry. Finally, in algorithm descriptions the notation X^k is used to designate the k-th iterate of the variable X. The notations $\mathrm{int}\, S$ and ∂S are used for the relative interior and the boundary of the set S.

2 Problem presentation and motivations

This section provides a brief review of a basic result that will be exploited throughout the paper. We are concerned with the robust control problem of an uncertain plant subject to LFT uncertainty. In other words, the uncertain plant is described as

$$\begin{array}{rcl}\begin{bmatrix} \dot{x} \\ z_\Delta \\ z \\ y \end{bmatrix} & = & \begin{bmatrix} A & B_\Delta & B_1 & B_2 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ C_1 & D_{1\Delta} & D_{11} & D_{12} \\ C_2 & D_{2\Delta} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w_\Delta \\ w \\ u \end{bmatrix} \\ w_\Delta & = & \Delta(t)\, z_\Delta \,, \end{array} \tag{1}$$

where $\Delta(t)$ is a time-varying matrix-valued parameter and is usually assumed to have a block-diagonal structure in the form

$$\Delta(t) = \operatorname{diag}(\ldots, \delta_i(t) I, \ldots, \Delta_j(t), \ldots) \in \mathbf{R}^{N\times N} \tag{2}$$

and normalized such that

$$\Delta(t)^T \Delta(t) \leq I, \qquad t \geq 0\,. \tag{3}$$

Blocks denoted $\delta_i I$ and Δ_j are generally referred to as repeated-scalar and full blocks according to the μ analysis and synthesis literature [6, 5]. Hereafter, we are using the following notation: u for the control signal, w for exogenous inputs, z for controlled or performance variables and y for the measurement signal.

For the uncertain plant (1)-(3) the robust control problem consists in seeking a Linear Time-Invariant (LTI) controller

$$\begin{array}{rcl} \dot{x}_K & = & A_K x_K + B_K y\,, \\ u & = & C_K x_K + D_K y\,, \end{array} \tag{4}$$

such that

- the closed-loop system (1)-(3) and (4) is internally stable,
- the L_2-induced gain of the operator connecting w to z is bounded by γ,

for all parameter trajectories $\Delta(t)$ defined by (3).

The characterization of the solutions to the robust control problem for LFT plants requires the definitions of scaling sets compatible with the parameter structure given in (2). Denoting this structure as $\boldsymbol{\Delta}$, the following scaling sets can be introduced. The set of symmetric scalings associated with the parameter structure $\boldsymbol{\Delta}$ is defined as

$$S_{\boldsymbol{\Delta}} := \left\{ S : S^T = S, \quad S\Delta = \Delta S, \quad \forall \Delta \text{ with structure } \boldsymbol{\Delta} \right\}.$$

Similarly, the set of skew-symmetric scalings associated with the parameter structure $\boldsymbol{\Delta}$ is defined as

$$T_{\boldsymbol{\Delta}} := \left\{ T : T^T = -T, \quad T\Delta = \Delta^T T, \quad \forall \Delta \text{ with structure } \boldsymbol{\Delta} \right\}.$$

Equivalently, it is easily verified that with $S > 0$, the uncertain matrix Δ satisfies the quadratic constraints

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} -S & T^T \\ T & S \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0, \quad \forall \Delta \text{ s. t. } \Delta^T\Delta \leq I, \quad \text{with structure } \boldsymbol{\Delta}. \tag{5}$$

With the above definitions and notations in mind, it is now well-known that such problems can be handled via a suitable generalization of the Bounded Real [19, 1, 2] which expresses as the existence of a Lyapunov matrix $X_{c\ell} > 0$ and scalings S and T such that

$$\begin{bmatrix} A_{c\ell}^T X_{c\ell} + X_{c\ell} A_{c\ell} & \star & \star \\ B_{c\ell}^T X_{c\ell} + \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} C_{c\ell} & -\begin{bmatrix} S & 0 \\ 0 & \gamma I \end{bmatrix} + \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} D_{c\ell} + D_{c\ell}^T \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}^T & \star \\ C_{c\ell} & D_{c\ell} & -\begin{bmatrix} S & 0 \\ 0 & \gamma I \end{bmatrix}^{-1} \end{bmatrix} < 0,$$

where the state-space data $A_{c\ell}$, $B_{c\ell}$, $C_{c\ell}$ and $D_{c\ell}$ determine the closed-loop system (1)-(4) with the loop $w_\Delta = \Delta(t)\, z_\Delta$ open.

Then, using the Projection Lemma [9] as the basic tool for reducing nonconvex terms and variables the following algebraically constrained LMI characterization for the solvability of the problem can be established.

Theorem 2.1 *Consider the LFT plant governed by (1) and (3) with Δ assuming a block-diagonal structure as in (2). Let $\mathcal{N}_X$ and $\mathcal{N}_Y$ denote any bases of the null spaces of $[C_2, D_{2\Delta}, D_{21}, 0]$ and $[B_2^T, D_{\Delta 2}^T, D_{12}^T, 0]$, respectively. Then, there exists a controller such that the (scaled) Bounded Real Lemma conditions hold for some L_2 gain performance γ if and only if there exist pairs of symmetric matrices (X, Y), (S, Σ) and a pair of skew-symmetric matrices (T, Γ) such that the structural constraints*

$$S,\ \Sigma \in S_{\boldsymbol{\Delta}} \text{ and } T,\ \Gamma \in T_{\boldsymbol{\Delta}} \tag{6}$$

hold and the matrix inequalities

$$\textbf{LMI}\,[1]\ : \mathcal{N}_X^T \begin{bmatrix} A^T X + XA & XB_\Delta + C_\Delta^T T^T & XB_1 & C_\Delta^T S & C_1^T \\ B_\Delta^T X + TC_\Delta & -S + TD_{\Delta\Delta} + D_{\Delta\Delta}^T T^T & TD_{\Delta 1} & D_{\Delta\Delta}^T S & D_{1\Delta}^T \\ B_1^T X & D_{\Delta 1}^T T^T & -\gamma I & D_{\Delta 1}^T S & D_{11}^T \\ SC_\Delta & SD_{\Delta\Delta} & SD_{\Delta 1} & -S & 0 \\ C_1 & D_{1\Delta} & D_{11} & 0 & -\gamma I \end{bmatrix} \mathcal{N}_X \ < \ 0\,, \tag{7}$$

$$\textbf{LMI}\,[2]\ : \mathcal{N}_Y^T \begin{bmatrix} AY + YA^T & YC_\Delta^T + B_\Delta \Gamma^T & YC_1^T & B_\Delta \Sigma & B_1 \\ C_\Delta Y + \Gamma B_\Delta^T & -\Sigma + \Gamma D_{\Delta\Delta}^T + D_{\Delta\Delta}\Gamma^T & \Gamma D_{1\Delta}^T & D_{\Delta\Delta}\Sigma & D_{\Delta 1} \\ C_1 Y & D_{1\Delta}\Gamma^T & -\gamma I & D_{1\Delta}\Sigma & D_{11} \\ \Sigma B_\Delta^T & \Sigma D_{\Delta\Delta}^T & \Sigma D_{1\Delta}^T & -\Sigma & 0 \\ B_1^T & D_{\Delta 1}^T & D_{11}^T & 0 & -\gamma I \end{bmatrix} \mathcal{N}_Y \ < \ 0\,, \tag{8}$$

$$\textbf{LMI}\,[3]\ : -\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \ < \ 0\,, \tag{9}$$

$$\textbf{LMI}\,[4]\ : -\begin{bmatrix} S & 0 \\ 0 & \Sigma \end{bmatrix} \ < \ 0 \tag{10}$$

subject to the algebraic constraints

$$(S+T)^{-1} = (\Sigma + \Gamma)\,, \ \textit{or alternatively} \ \begin{bmatrix} S & T \\ T^T & -S \end{bmatrix}^{-1} = \begin{bmatrix} \Sigma & \Gamma^T \\ \Gamma & -\Sigma \end{bmatrix}, \tag{11}$$

are feasible.

Note that due to the algebraic constraints (11), the problem under consideration is nonconvex and has been even shown to be NP-hard. See [4] and references therein. This feature is in stark contrast with the associated Linear Parameter-Varying control problem for which the LMI constraints (7)-(10) are the same but the nonlinear conditions (11) fully disappears.

3 Rank constraints, BMIs and concave programs

For tractability reasons, it is interesting to find alternate formulations that are amenable to numerical computations. A potential technique was introduced in [11] and amounts to constructing a nonlinear functional whose feasible optimal points satisfy the algebraic constraints (11). Hereafter, we develop different extensions of this technique that is applicable to structured μ-scalings (5), to full-block generalized scalings as considered in [22] but also more importantly to bilinearly constrained LMI problems. We begin the presentation by a more general result which reveals the close connections between BMIs, rank constrained LMI problems and concave programming. Hereafter, we shall use L, R, W to denote a subset of matrix variables which are involved in LMI constraints.

Lemma 3.1 (Concave representation) *Let C be a matrix with minimal rank factorization of rank r, that is $C = UV^T$ where the column dimension of U is r and let L, R and W be matrices of suitable dimensions, where without loss of generality it is assumed that $W \in \mathbf{R}^{l\times c}$ with $l \geq c$.*

The bilinear constraint

$$W = LCR \tag{12}$$

is equivalent to the existence of a symmetric (slack) matrix Z and x satisfying the following LMI

$$Z := \begin{bmatrix} Z_1 & Z_3 \\ Z_3^T & Z_2 \end{bmatrix}, \quad \begin{bmatrix} Z_1 & Z_3 & W^T & R^TV \\ Z_3^T & Z_2 & U^TL^T & I \\ W & LU & I & 0 \\ V^TR & I & 0 & I \end{bmatrix} \geq 0, \tag{13}$$

with the additional Schur complement constraint,

$$\mathrm{Tr}\,(Z_1 - Z_3Z_2^{-1}Z_3^T) = 0\,. \tag{14}$$

Moreover, the trace function in (14) is concave over the cone of positive semidefinite matrices and is bounded below by zero.

Proof: Necessity is trivial from the choice

$$\begin{bmatrix} Z_1 & Z_3 \\ Z_3^T & Z_2 \end{bmatrix} = \begin{bmatrix} W & LU \\ V^TR & I \end{bmatrix}^T \begin{bmatrix} W & LU \\ V^TR & I \end{bmatrix}$$

Sufficiency: It follows from (13) and (14) that Z has a loss of rank of dimension c and that Z_2 is invertible. Select a basis $\mathcal{N} \in \mathbf{R}^{(c+r)\times c}$ of the nullspace of Z. We infer by a Schur complement argument with respect to the identity term in the inequality (13) that

$$\begin{bmatrix} W & LU \\ V^TR & I_r \end{bmatrix} \mathcal{N} = 0\,.$$

But since $\mathcal{N}$ is a full rank matrix, we deduce that

$$\mathrm{Rank} \begin{bmatrix} W & LU \\ V^TR & I_r \end{bmatrix} = r\,.$$

Thus (12) follows from the rank-preserving transformations

$$\begin{bmatrix} I & -LU \\ 0 & I_r \end{bmatrix} \begin{bmatrix} W & LU \\ V^TR & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -V^TR & I_r \end{bmatrix} = \begin{bmatrix} W - LCR & 0 \\ 0 & I_r \end{bmatrix}.$$

The concavity of the trace function in (14) is easily seen by looking at its hypograph. Also, the trace function is trivially bounded below by zero. This terminates the proof. ■

Similarly, the inversion constraint

$$W = L^{-1}\,,$$

can be given the concave programming representation

$$\begin{bmatrix} Z_1 & Z_3 & W & I \\ Z_3^T & Z_2 & I & L \\ W^T & I & I & 0 \\ I & L^T & 0 & I \end{bmatrix} \geq 0, \quad \mathrm{Tr}\,(Z_1 - Z_3 Z_2^{-1} Z_3^T) = 0\,. \tag{15}$$

One important consequence of Lemma 3.1 is that BMI problems can be equivalently formulated as the search of zero optimal solutions of concave programs. These problems however exhibit a high degree of nonconvex dimensionality and consequently are generally harder to solve than the problems investigated in this paper. Important advantages lie in the simplicity of this new formulation but also in the fact that matrix structures are preserved in the concave program. This is an important factor for efficient implementation of algorithms that we shall consider in the sequel. Because of the special properties of concave programs, it is possible to develop algorithms local or global which take advantage of the problem properties to enhance efficiency. A discussion on concave programs is provided in Sections 4 and 5. Before going further, we would like to note that feasibility problems involving LMIs and rank inequalities can be handled in the same fashion. The outcome of this discussion is that a non-exhaustive list of potential control applications of the proposed algorithms include also

- reduced- and fixed-order robust control,
- multi-objective robust and Linear Parameter-Varying control,
- reduction of LFT representations.

An immediate application of Lemma 3.1 leads to a concave programming formulation of the robust control problem introduced in Section 2 and characterized in Theorem 2.1.

Corollary 3.2 *Introduce the concave LMI-constrained minimization program*

$$\textbf{Pb1:} \qquad \textit{minimize}\ \mathrm{Tr}\,(Z_1 - Z_3 Z_2^{-1} Z_3^T) \tag{16}$$

subject to LMIs (7)-(10) and

$$\textbf{LMI}\,[5]\ :-\begin{bmatrix} Z_1 & Z_3 & S+T & I \\ Z_3^T & Z_2 & I & \Sigma+\Gamma \\ (S+T)^T & I & I & 0 \\ I & (\Sigma+\Gamma)^T & 0 & I \end{bmatrix} \leq 0\,. \tag{17}$$

Then, any feasible point to **Pb1** *which further satisfies*

$$\mathrm{Tr}\,(Z_1 - Z_3 Z_2^{-1} Z_3^T) = 0, \tag{18}$$

is optimal and is a solution to the problem described in Theorem 2.1 and conversely.

Note that without loss of generality, it can be assumed that the matrix

$$Z := \begin{bmatrix} Z_1 & Z_3 \\ Z_3^T & Z_2 \end{bmatrix}$$

has a structure conformable with that of the particular block-diagonal structure of the scalings. This simple observation reduces the number of "nonconvex variables" Z_1, Z_2 and Z_3 and avoids a wasteful search in an unduly large space. The number of nonconvex variables is also reduced when some subblocks T_i and Γ_i in the skew-symmetric matrices T and Γ vanish. This is the case when the corresponding Δ_i in Δ is scalar or is considered as a complex block. In such case, one can remove this block from both LMI (17) and the objective functional (16). The sizes of Z_1, Z_2 and Z_3 are then reduced accordingly and the (concave) objective functional becomes

$$\mathrm{Tr}\,(Z_1 - Z_3 Z_2^{-1} Z_3^T) + \mathrm{Tr}\,(S_i - \Sigma_i^{-1}) \tag{19}$$

with the additional LMI

$$\begin{bmatrix} S_i & I \\ I & \Sigma_i \end{bmatrix} \geq 0\,.$$

One advantage of the formulation of the problem as in Corollary 3.2 is that one completely gets rid of the hard set constraints (11) and the nonconvexity is reflected in the functional to be optimized. A central target of this paper is to point out and discuss adequate algorithms for solving this class of problems. Before going into the details of the algorithm, we must stress out that the proposed concave reformulations apply with the same degree of simplicity to other classes of scalings such as the full block scalings introduced in [23] and also to dynamic scalings or multipliers hence providing a complete concave formulation of the μ synthesis problem.

4 A local search: Frank and Wolfe algorithm

In this section, we discuss a Frank and Wolfe algorithm for finding solutions to the Problem **Pb1** in Corollary 3.2. Analogous algorithms can be derived in the context of any of the control problems mentioned previously. Such algorithms are of local nature in the sense that they cannot guarantee global optimality but have proven very efficient in practice [4, 11].

Consider the minimization problem

$$\text{minimize} \;\; f(Z) \;\; \text{subject to} \;\; Z \in \mathcal{X} \tag{20}$$

where the function f has continuous first-order partial derivatives on $\mathcal{X}$ and is bounded below on the matrix set $\mathcal{X}$, a convex subset of the space of symmetric matrices. The general algorithm of Frank and Wolfe (FW) [7, 4] can be detailed as follows:

1. Find a steepest descent direction by solving the convex programming problem

$$D^k \in \arg\min_{D \in \mathcal{X}} \; \mathrm{Tr}\,(\nabla f(Z^k)\,D)$$

2. Perform a line search on the segment $[Z^k, D^k]$ to get

$$\begin{aligned} Z^{k+1} &= (1-\alpha^k) Z^k + \alpha^k D^k, \\ \text{where } \alpha^k &\in \arg\min_{0 \leq \alpha \leq 1} f((1-\alpha)Z^k + \alpha D^k) \end{aligned} \tag{21}$$

Interestingly, the concavity of the objective function (16) yields the following benefits in the implementation of FW:

- The line search can be completely bypassed since $\alpha^k \in \{0,1\}$. Thus one can perform a *full step size of length one* ($Z^{k+1} = D^k$) hence reducing the overall computational overhead.

- As $f(Z) - f(Z^k) \leq \mathrm{Tr}\,(\nabla f(Z^k)(Z - Z^k)), \quad \forall Z \in \mathcal{X}$, it can be shown that the algorithm generates *strictly decreasing* sequences ($f(Z^{k+1}) < f(Z^k)$) that can only terminate to a point satisfying the minimum principle local optimality conditions.

For the problem examined in this paper, the gradients are given as

$$G_1 := \frac{\partial J}{\partial Z_1} = I,\ G_2 := \frac{\partial J}{\partial Z_2} = Z_2^{k^{-1}} Z_3^{k^T} Z_3^k Z_2^{k^{-1}},\ G_3 := \frac{\partial J}{\partial Z_3} = -2 Z_2^{k^{-1}} Z_3^{k^T},$$

and the FW step can be described by the following LMI program:

$$\begin{array}{l} \text{minimize } \mathrm{Tr}\,(G_1 Z_1 + G_2 Z_2 + G_3 Z_3) \\ \text{subject to } \ \mathbf{LMI}\,[i]\ < 0,\ i = 1,2,3,4;\ \mathbf{LMI}\,[5]\ \leq 0. \end{array}$$

5 Global concave programming based methods

Concave programming constitutes a class of well-developed methods in global optimization whose foundations were mostly layed in [26]. It offers a wealth of practically efficient techniques for solving difficult problems which seem, however, to have been overlooked by the control community. Reasons for this disinterest lie in the fact that most successfully developed concave programming algorithms [15, 17, 27] deal with (linear) polytopic constraints, thus having a finite number of extreme points, and are restricted to the usual vector space $\mathbf{R}^n$. In this section, we shall show that several important basic concepts of concave programming carry over matrix spaces and the positive semidefinite cone of symmetric matrices and that these generalizations can be exploited to handle our problems. The discussion here is deliberately very short and avoids the abstract convergence theory that can be found in textbooks. The reader is referred to the recent book of Tuy [27] for further details on concave programming.

Return to the problem of checking whether there exists

$$Z^* \in \mathcal{X} = \{(Z_1, Z_2, Z_3) :\ \exists (X, Y, S, T, \Sigma, \Gamma) \text{ s.t. } \mathbf{LMI}\,[i]\ < 0,\ i = 1, \ldots, 4;\ \mathbf{LMI}\,[5]\ \leq 0\} \tag{22}$$

satisfying $f(Z^*) = 0$ where $f(Z) := \mathrm{Tr}\,(Z_1 - Z_3 Z_2^{-1} Z_3^T)$ is concave. Such a Z^* when it exists will be called a *zero* of f. It is important to note that since f satisfies $f(Z) \geq 0, \forall Z \in \mathcal{X}$, any zero of f is also a global optimal solution of (20), and consequently, our problem is much more computationally attractive than conventional concave programs in which minimal values of the cost function are unknown. In the methods presented hereafter, we can stop the search as soon as either such a zero is found in which case global optimality is ensured, or the minimum cost value is strictly positive in which case our problem has no solution.

In view of the recent developments in global optimization, it seems that a BB method is the most suitable for our global search. Our intention in the present work is to maximally exploit the structure and properties of the problem to make our search algorithms much more efficient than general BB schemes. The overall scheme goes as follows.

Branching: The function f is not only concave in (Z_1, Z_2, Z_3) but is also linear in Z_1 with (Z_2, Z_3) held fixed, i.e. only (Z_2, Z_3) are the "complicating" variables, responsible for the nonconvexity/hardness of the problem. The global search thus is concentrated on the reduced-dimensional space $\mathcal{Z}$ of variables (Z_2, Z_3). Accordingly, the feasible set can be interpreted as the projection of the convex set defined by the LMIs (7)-(10) and (17) on the space $\mathcal{Z}$. This space is partitioned into finitely many matrix polyhedrons of the same kind (simplices, cones etc.). At each iteration, a partition polyhedron M is selected and subdivided further into several subpolyhedrons according to a specified rule.

Bounding: With the branching strategy determined and given a partition set M, the convexity of $\mathcal{X}$, the concavity of f and its linearity in Z_1 are further exploited in the search of a zero of f over $(Z_2, Z_3) \in M$. This is carried out through computing a lower bound $\beta(M)$ of f on M by a convex program such that

$$\beta(M) \leq \nu(M) := \inf\{f(Z_1, Z_2, Z_3) : \ (Z_1, Z_2, Z_3) \in \mathcal{X}, (Z_2, Z_3) \in M\}. \tag{23}$$

Clearly, the partition sets M with $\beta(M) > 0$ cannot contain any zero of f and therefore are discarded from further consideration. On the other hand, the partition set with smallest $\beta(M) < 0$ can be considered the most promising one. To concentrate further investigation on this set, we subdivide it into more refined subsets. With a given tolerance $\epsilon > 0$, the stop criterion of the BB algorithm is

$$\min_M \beta(M) \geq \epsilon. \tag{24}$$

Stopping rule: The branching operation is devised for speeding up the convergence. The optimal solution $Z(M)$ of the problem for computing $\beta(M)$ is used for the stopping test developed above to reduce the time of global search.

Based on the kind of polyhedrons which are used in branching, we develop 2 different BB algorithms called the simplicial and conical algorithms which use different characterizations of concave functionals. As one may see, each of them has its own advantage depending on the more specific structure of the objective $f(Z)$. It is important to mention that all branching and bounding operations must be developed consistently to secure global convergence of the search to a global solution. Global convergence is often a delicate issue in BB techniques. Proofs are provided in [3].

5.1 Simplicial algorithm

In the simplicial algorithm, the space $\mathcal{Z}$ is partitioned into simplices. From now on, N will denote the dimension of $\mathcal{Z}$. For every simplex M with vertices $u^1, u^2, ..., u^{N+1}$ in $\mathcal{Z}$, the affine function $\phi_M(Z)$ defined for every Z_1 and $x = (Z_2, Z_3) = \sum_{i=1}^{N+1} \lambda_i u^i, \ \lambda_i \geq 0, \ \sum_{i=1}^{N+1} \lambda_i = 1$ by

$$\phi_M(Z_1, x) := \mathrm{Tr}\,(Z_1) + \phi_M(\sum_{i=1}^{N+1} \lambda_i u^i) = \mathrm{Tr}\,(Z_1) + \sum_{i=1}^{N+1} \lambda_i f(0, u^i),$$

satisfies $\phi_M(Z_1, x) = f(Z_1, x), \ \forall x \in \mathrm{vert}M$, and any Z_1 and thus $\phi_M(Z_1, x) \leq f(x) \ \forall x \in M, Z_1$, i.e. $\phi_M(Z)$ is an affine minorant of f in M (in fact the convex envelope of $f(Z)$ over M). On the other hand, if there is a zero (Z_1, Z_2, Z_3) with $(Z_2, Z_3) \in M$ then again by the concavity of f, one must have

$$\min_{i=1,2,...,N+1} f(Z_1, u^i) \leq 0 \Leftrightarrow \mathrm{Tr}\,(Z_1) + \min_{i=1,2,...,N+1} f(0, u^i) \leq 0\,. \tag{25}$$

Thus a lower bound $\beta(M)$ satisfying (23) is defined by the convex (LMI) program

$$\beta(M) := \min\{\phi_M(Z_1, \sum_{i=1}^{N+1} \lambda_i u^i) : \ (25), \ \sum_{i=1}^{N+1} \lambda_i = 1, \ \lambda_i \geq 0, \ (Z_1, \sum_{i=1}^{N+1} \lambda_i u^i) \in \mathcal{X}\} \tag{26}$$

Of course, we can use the optimal solution $Z_1(M)$ and $\omega(M) = \sum_{i=1}^{N+1} \lambda_i(M) u^i$ of (26) for updating the best current value (upper bound).

With these specialized branching and bounding with subdivision according to the normal rule [27], keeping in mind that the algorithm will stop when the current best value is 0 or there is evidence that the lower bound of (20) is positive (infeasibility), the simplicial algorithm can be implemented.

5.2 Conical algorithm

Close scrutiny of the objective function properties ($\mathrm{Tr}\,(Z_1 - Z_3 Z_2^{-1} Z_3^T)$) in Corollary 3.2 reveals the following.

(*i*) If (Z_1, Z_2, Z_3) is the solution of **Pb1** with the zero optimal value then (tZ_1, tZ_2, tZ_3) with $t \geq 1$ is also a solution satisfying the same conditions. Thus, without loss of generality, we can set $\mathrm{Tr}\,(Z_1) = L$, with L a constant large enough.

(*ii*) $Z_2 \geq I$ which means that we can use the change of variable $Z_2 \to Z_2 + \epsilon I$ with $Z_2 \geq 0$ instead of $Z_2 > 0$.

As a consequence, problem **Pb1** can be reduced to minimizing the objective function

$$f(Z_2, Z_3) = L - \mathrm{Tr}\,(Z_3 (Z_2 + \epsilon I)^{-1} Z_3^T) \tag{27}$$

and LMIs (7)-(17) are changed accordingly using the substitution $Z_2 \to Z_2 + \epsilon I$. The function f in (27) is concave in the cone $\mathcal{C}_+^{m_2} \times \mathcal{C}^{m_3}$ where $\mathcal{C}_+^{m_2}$ is the cone of nonnegative definite matrices with the same structure as Z_2 and $\mathcal{C}^{m_3}$ is the space of symmetric matrices having the same structure as Z_3. It is sufficient to take $\bar{\mathcal{Z}}$ as a large enough finite family of canonical cones approximating $\mathcal{C}_+^{m_2} \times \mathcal{C}^{m_3}$ with some tolerance. Perhaps, the most essential property of a concave function f is that its level sets $C_0 = \{Z = (Z_2, Z_3) \in \bar{\mathcal{Z}} : \ f(Z) \geq 0\}$ are convex and therefore an alternative formulation of our problem is to find $Z \in \mathcal{X} \setminus \mathrm{int}\, C_0$ or else prove that $\mathcal{X} \subset \mathrm{int}\, C_0$, where both $\mathcal{X}, C_0$ are convex sets. All these facts are taken into account in the following search based upon the so-called concavity cut or Tuy cut [26]. In what follows, by a cone we mean a cone with vertex at 0 and exactly N edges. Consider an initial family $\mathcal{P}_0$ of cones covering $\bar{\mathcal{Z}}$ and with pairwise disjoint interiors. For each initial cone in $\mathcal{P}_0$ take a fixed hyperplane cutting all its edges. Then, the intersection of each subcone of this initial cone with the above mentioned hyperplane is a simplex with N vertices and is called the base of the subcone. Let M be a cone with base $[u^1, u^2, ..., u^N]$. Since $f(0) = L > 0$, we have $0 \in \mathrm{int}\, C_0$ and by the convexity of C_0 the ray from 0 through u^i meets the boundary of C_0 at a unique point $\bar{u}^i = \theta_i u^i$ with $\theta_i > 0$ determined by

$$\theta_i = \sup\{\theta > 0 : \ f(\theta u^i) \geq 0\}. \tag{28}$$

Then, since the convex set C_0 is closed, $\bar{u}^i \in C_0$ and

$$\mathrm{co}\{0, \bar{u}^i, \ i = 1, 2, ..., N\} \subset C_0. \tag{29}$$

Consider then the convex (LMI) program

$$\max\{\sum_{i=1}^{N}\lambda_i: \ \lambda_i \geq 0, \ \sum_{i=1}^{N}\lambda_i\bar{u}^i \in \mathcal{X}\}, \tag{30}$$

and let $\mu(M)$ and $\lambda(M)$ be the optimal value and the optimal solution of this program. Also let $\omega(M) = \sum_{i=1}^{N}\lambda_i(M)\bar{u}^i \in \mathcal{X}$. Only one of the following mutually exclusive possibilities can occur:

(i) $\mu(M) < 1$. Then it easily follows that $M \cap \mathcal{X} \subset \operatorname{int} C_0$, i.e. there is no zero optimal solution in $M \cap \mathcal{X}$ and so M can be discarded from further consideration;

(ii) $\omega(M) \in \mathcal{X} \setminus \operatorname{int} C_0$ (i.e. $f(\omega(M)) = 0$): then we have obtained a zero optimal solution;

(iii) $\mu(M) \geq 1$ and $\omega(M) \in C_0$. In this case $\omega(M)$ does not lie on any edge of M (so that the subdivision of M by the ray through $\omega(M)$ is possible). Indeed, if $\omega(M)$ lies on some edge u^i of M then we must have $\omega(M) = \mu(M)\bar{u}^i = \mu(M)\theta_i u^i$ with $\mu(M)\theta_i > \theta_i$ and $f(\mu(M)\theta_i u^i) = f(\omega(M)) \geq 0$, which contradicts the definition (28) for θ_i.

Actually, $\mu(M)$ is not a lower bound for $f(x)$ on $M \cap \mathcal{X}$ but because of the above property, $1 - \mu(M)$ plays essentially the same role as a lower bound for eliminating portions of the constraint set. Therefore, with cone partitioning via the ray through a point in its simplex base defined according to the normal subdivision rule, the conical algorithm can be described. Its convergence proof is similar to that of the simplicial algorithm.

5.3 Trade-off of two global searches

Clearly, by concentrating the search on the boundary of the feasible set, the conical algorithm better exploits the fact that the global minimum is attained at an extreme point and is therefore more efficient than the simplicial algorithm in the case of problem **Pb.1**.

However, the simplicial algorithm is convenient for exploiting the partial linearity of the objective. For instance, in the case when all skew-symmetric matrices T and Γ vanish, the objective for (19) can be reduced to the form

$$\operatorname{Tr}(S) - \operatorname{Tr}(\Sigma^{-1}), \tag{31}$$

which means that it is concave in Σ and linear in S. The simplicial algorithm can then be applied directly, with branching operations in the reduced $\Sigma-$space as previously. Thus in this case, the simplicial algorithm might be preferred.

6 Numerical experiments

Due to space limitation, details on algorithms and experiments are provided in the extended version of the paper [3]. Here is a brief summary of our observations. As mentioned in the introduction, the overall algorithm can be detailed as follows. The FW algorithm is computationally cheaper than simplicial and conical global techniques, and hence is used first to find a good suboptimal value γ. Then, the simplicial/conical algorithm are employed to further reduce γ, or to certify global optimality. In realistic and randomly generated examples, the FW algorithm is able to locate a suboptimal solution, up to 8% of the global optimal value, after only a few iterations. The simplicial/conical algorithms starting from this good initial guess find a global optimal solution very quickly, less than 5 iterations when the problem is feasible. For infeasible problems, they obtain a positive lower bound of **Pb.1** after less than 10 iterations. It is also important to emphasize that for feasible γ, the use of stopping criteria (see [3]) substantially reduces the computational cost since only an approximate solution is required for termination. This fact and the power of simplicial/conical techniques explains why so few iterations (LMI runs) are needed and thus the relatively cheap cost of our global algorithms.

7 Concluding remarks

In this paper, we show that many important problems in robust control theory can be formulated as the minimization of a concave functional over a convex set determined by LMI constraints. In this respect, concavity appears to play a central role in a broad class of problems. This is the departure point which motivates the development of a comprehensive technique which provides a global solution of robust control problems admitting scaling-based characterizations. Although, we do not pursue the vein further, it appears that the technique is applicable with only modest changes to many other difficult problems encompassing fixed-order robust control, multi-objective LPV control, and any aggregation of these problems. We also derive new results, interesting in their own, which clarify the equivalence between BMIs, rank constrained LMI problems and zero-seeking concave programs. Extensive numerical experiments indicate that the algorithms behave well as theoretically expected.

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