Overview of LTI interval observer design: towards a non-smooth optimisation-based approach

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Abstract: *Some applications in control require the state vector of a system to be expressed in the appropriate coordinates so as to satisfy to some mathematical properties which constrain the studied system dynamics. This is the case with the theory of linear interval observers which are trivial to implement on cooperative systems, a rather limited class of control systems. The available literature shows how to enforce this limiting cooperativity condition for any considered system through a state-coordinate transformation. This article proposes an overview of the existing numerical techniques to determine such a transformation. It is shown that in spite of being practical, these techniques have some limitations. Consequently, a reformulation of the problem is proposed so as to apply non-smooth control design techniques. A solution is obtained in both the continuous- and discrete-time frameworks. Interestingly, the new method allows to formulate additional control constraints. Simulations are performed on three examples.*

1 Introduction

To be applicable, many observation and control strategies applied to linear systems are required to comply with some constraints on the designed dynamics. These constraints are expressed through mathematical properties which should be satisfied by the resulting system state-space representation matrices. For example, considering the case of interval observers [1], a constructive cooperativity condition [2] on the observation error dynamics is looked for, which is linked to properties of the resulting state matrix. In the continuous-time case, this matrix should be Metzler which means that its off-diagonal elements should be positive. For a given system, this property is rarely satisfied in the original state coordinates. However, a change of coordinates can be calculated so that cooperativity is satisfied in the new coordinates [3, 4]. Various techniques – which are detailed in this paper – currently exist to determine such a transformation. Their advantages lie in their simplicity. However, they rely on "first guess" and are hardly compatible with additional constraints related to control performance. In the case of interval observers, it may be expected that the error dynamics should satisfy to additional constraints like disturbances rejection. To the authors opinion, the fact that the observer gain and the state-coordinate transform cannot be simultaneously computed to satisfy to the Metzler property and to an additional disturbance rejection constraint is a limit to these methods.

An overview of the aforementioned existing techniques is presented in this article. Their limitations motivate a reformulation of the problem. This is presented in this article both in the continuous- and discrete-time frameworks. This leads to an optimization problem which is solved using non-smooth optimization techniques usually dedicated to control design. Using such techniques, additional control constraints can be taken into account in the determination of the appropriate state-coordinate transformation. This article extends the preliminary results presented in [5]. This approach was motivated by the interval observers cooperativity requirement but it is expected to be generalizable to any problem requiring specific structures of the matrices of the studied system. It is important to understand that no theoretical addition is brought to the existing interval observer framework

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but rather on a computation-related sub-problem. Note that the use of time-varying changes of coordinates [3] is not considered since our method is not applicable in this case.

The core of the technique is to reformulate the mathematical conditions on a matrix – like the Metzler requirement on the state-matrix – into a stabilization control problem. Using recent theoretical results [6] of the structured controller synthesis field, a control law synthesis is then performed on multiple models in view of the specified control requirements. The approach is applied to various examples of interval observer problems. Numerical and simulation results are compared to the ones obtained using the existing techniques.

This paper is organized as follows. The notations and definitions required to understand the motivations and the resulting computation-related problems are presented in Sect. 2. The theory of interval observers is recalled in Sect. 3. In the same section, an overview of the existing techniques to compute the necessary state-coordinate transformation is performed. This raises a difficult numerical problem which is described in Sect. 4. A solution based on multiple models control law synthesis is proposed in Sect. 5 for both the continuous- and discrete-time frameworks. The approach is then applied on various examples in Sect. 6 before concluding in Sect. 7.

2 Notations and definitions

The Laplace transform is denoted s and the Z-transform is denoted z. Let denote $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+$ the set of positive real numbers. Let $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. If not otherwise stated, the indices (i, j, k, l, m, n) are integers in $\mathbb{Z}_+$. The H_2-norm of a transfer function $T(s)$ (resp. $T(z)$) is denoted $\|T(s)\|_2$ (resp. $\|T(z)\|_2$) and is defined as

$$\|T(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left[T\left(\mathrm{j}\omega\right)^\star T\left(\mathrm{j}\omega\right)\right] \mathrm{d}\omega}$$

resp. $\|T(z)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\left[T\left(e^{\mathrm{j}\omega}\right)^\star T\left(e^{\mathrm{j}\omega}\right)\right] \mathrm{d}\omega}$, where "tr" is the trace operator and $T\left(\mathrm{j}\omega\right)^\star$ the conjugate transpose of $T\left(\mathrm{j}\omega\right)$. The elements of a matrix $A \in \mathbb{R}^{n\times m}$ are referred to using the lowercase notation a_{ij} where $i \leq n$ and $j \leq m$. The ordering operators $>$, $<$, $\leq$ and $\geq$ are understood component-wise when applied to multi-dimensional signals. Known (or computed) time-varying bounds on a multi-dimensional signal $x \in \mathbb{R}^n$ are denoted $\underline{x}(t)$ and $\overline{x}(t)$:

$$\forall t,\ \underline{x}(t) \leq x(t) \leq \overline{x}(t) \tag{1}$$

In this article, only linear time-invariant (LTI) systems are considered. In the continuous-time case, let an LTI system (G) be defined by

$$(G)\left\{\begin{array}{ccl} \dot{x} & = & Ax + B_1 u + B_2 d \\ y & = & Cx \\ x(0) & = & x_0 \end{array}\right. \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $d \in \mathbb{R}$ and matrices are of appropriate dimensions.

Definition 1 (Interval observer, continuous-time). *Suppose quantities $\underline{x}_0$, $\overline{x}_0$ and signals $\underline{d}(t)$, $\overline{d}(t)$ are known such that $\underline{x}_0 \leq x_0 \leq \overline{x}_0$ and $\underline{d}(t) \leq d(t) \leq \overline{d}(t)$, $\forall t$. Then, the dynamical system*

$$\left(G^\sharp\right)\left\{\begin{array}{cclr} \dot{z} & = & A^\sharp z + B_1^\sharp u + B_2^\sharp \begin{bmatrix}\underline{d} \\ \overline{d}\end{bmatrix} & \in \mathbb{R}^{2n} \\ \begin{bmatrix}\underline{x} \\ \overline{x}\end{bmatrix} & = & C^\sharp z & \text{with rank}\left(C^\sharp\right) = 2n \end{array}\right. \tag{3}$$

associated with the initial condition $z_0 = \left(C^{\sharp\top} C^\sharp\right)^{-1}\begin{bmatrix}\underline{x}_0 & \overline{x}_0\end{bmatrix}^\top$, where the newly defined matrices are of appropriate dimensions, is a linear interval observer of the system (G) in Eq. (2) if, see [7, Definition 4]:

1. *$\left(G^\sharp\right)$ is Input-to-State Stable;*

2. *The solutions to Eqs.* (2) *and* (3) *with respectively* x_0 *and* z_0 *as initial conditions are defined* $\forall t \in \mathbb{R}_+$ *and fulfil*
$$\underline{x}(t) \leq x(t) \leq \overline{x}(t),\, \forall t$$

3. *If* $\left|\overline{d}(t) - \underline{d}(t)\right|$ *is uniformly bounded then* $\|\overline{x}(t) - \underline{x}(t)\|$ *is also uniformly bounded and if* $\underline{d}(t) = \overline{d}(t),\, \forall t \in \mathbb{R}_+$ *then* $\|\overline{x}(t) - \underline{x}(t)\| \to 0$.

A similar definition is obtained in the discrete-time case. This definition means that any system can be an interval observer as long as the two conditions are satisfied. However, this is quite hard in general to find the required matrices $A^\sharp$, $B_1^\sharp$, $B_2^\sharp$ and $C^\sharp$. To do so, some constructive assumptions are gradually made on the considered systems structures so as to easily satisfy both conditions. For a given matrix $A \in \mathbb{R}^{n\times m}$, let define $A^+ = \max(A, 0)$ and $A^- = A^+ - A$. To determine matrix $B_2^\sharp$ and $C^\sharp$, the following Lemma will be used.

Lemma 1. *(see [8, Lemma 1] and the associated proof) Let* $x \in \mathbb{R}^n$ *s.t.* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ *and* $A \in \mathbb{R}^{m\times n}$ *a constant matrix s.t.* $A = A^+ - A^-$*. Then* $\forall t$,

$$A^+\underline{x}(t) - A^-\overline{x}(t) \leq Ax(t) \leq A^+\overline{x}(t) - A^-\underline{x}(t) \tag{4}$$

The following definitions will be useful to define $A^\sharp$ and obtain interval observers on linear systems.

Definition 2 (Metzler matrix)**.** Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. The matrix A is said to be Metzler if

$$\forall i \neq j,\, a_{ij} \geq 0 \tag{5}$$

Definition 3 (Non-negative matrix)**.** Let $A_d = (a_{ij}) \in \mathbb{R}^{n\times n}$. The matrix A_d is said to be non-negative if

$$\forall\,(i,j)\,,\, a_{ij} \geq 0 \tag{6}$$

The theory of cooperative systems is extensively used when designing interval observers. Such systems have interesting properties as far as trajectory ordering is considered. This is used to fulfil the second condition in Def. 1 straightforwardly.

Definition 4 (Cooperative system, continuous-time)**.** An autonomous continuous-time linear system (G) is said to be cooperative if its state matrix A is a Metzler matrix.

Remark 1. *In the case of a continuous-time linear system with inputs (and* $D = 0$*), it is said to be cooperative if* A *is Metzler,* $B \in \mathbb{R}_+^{n\times l}$ *and* $C \in \mathbb{R}_+^{m\times n}$*. A cooperative continuous-time system is thus a positive system.*

Of course, discrete-time systems can also be cooperative under slightly different conditions.

Definition 5 (Cooperative system, discrete-time)**.** An autonomous discrete-time linear system (G_d) is said to be cooperative if its state matrix A_d is a non-negative matrix.

3 Motivations

In many control applications, it is required to enforce specific properties on a system state-space representation. This is the case when designing interval observers as defined in Def. 1.

3.1 Motivating problem

The linear interval observer framework and existing techniques to compute a state-coordinate transformation such that the state-matrix is Metzler or non-negative are now presented.

3.1.1 Interval observer literature

The theory of interval observers was first introduced in [1] and [9] as a deterministic state "framing" method in the case of cooperative systems [2] disturbed by unknown disturbances with known bounds. When the original system is not cooperative, a state-coordinate transformation is considered. In [10] the authors show (on a precise class of systems) exponential stability of a time-varying interval observer obtained using a time-varying state-coordinate change. This is applied in [3] on a broader class. A similar transformation is also proposed in [11]. The approach is extended in [12] to Linear Time Varying (LTV) systems. In other works, a more classical approach leads to time-invariant linear interval observers with weaker stability guarantees but broader system classes. In [4], a time-invariant state coordinate transform is used to obtain a cooperative observation error dynamics on which an interval observer is designed. This is extended to a class of nonlinear and LPV[1]/LTV systems in [13] and [8]. Another formulation is proposed in [14]. An extensive study of these techniques is available in [15] and [16].

The theory of interval observers applied to discrete-time systems [17] is very similar to the continuous-time case. Extensions to discrete-time LPV systems have been proposed in [18]. An extension to continuous-time systems with discrete measurements is proposed in [16].

Note there exists other approaches based on Internal Positive Representations of systems [7, 17]

In this paper, the considered systems are linear time-invariant systems and the results in [4] are used.

3.1.2 Interval observer design using a classical observer

Let consider an LTI continuous-time system with a state-space representation as in Eq. (2) where $u \in \mathbb{R}$ is the control signal, $d \in \mathbb{R}$ is an unknown state disturbance and $x_0 \in \mathbb{R}^n$ is the initial condition. The following hypotheses are considered

Assumption 1. *Couples* $(\underline{x}_0, \overline{x}_0)$ *and* $(\underline{d}, \overline{d})$ *are known s.t.* $\underline{x}_0 \leq x_0 \leq \overline{x}_0$ *and*

$$\forall t \geq 0,\ \underline{d}(t) \leq d(t) \leq \overline{d}(t) \tag{7}$$

Assumption 2. *The pair* (A, C) *in Eq.* (2) *is detectable.*

To fulfil the first condition in Def. 1, a classical observer of the system in Eq. (2) is used:

$$(G_{\mathrm{obs}}) \left\{ \begin{array}{lcl} \dot{\widehat{x}} & = & A\widehat{x} + B_1 u + L\left(y - C\widehat{x}\right) \\ \widehat{y} & = & C\widehat{x} \\ \widehat{x}(0) & = & \widehat{x}_0 \end{array} \right. \tag{8}$$

where $L \in \mathbb{R}^{n\times m}$ is defined such that $A - LC$ is Hurwitz. The underlying estimation error dynamics is deduced from both Eqs. (2) and (8):

$$(G_e) \left\{ \begin{array}{lcl} \dot{e} & = & (A - LC)\, e + B_2 d \\ e_0 & = & x_0 - \widehat{x}_0 \end{array} \right. \tag{9}$$

where $e = x - \widehat{x}$. If $A - LC$ is not Metzler, (G_e) is not cooperative and it is not possible to fufil the second condition in Def. 1 without a state coordinate change. Under Hyp. 1 one obtains bounds on the initial error e_0: $\underline{e}_0 = \underline{x}_0 - \widehat{x}_0$ and $\overline{e}_0 = \overline{x}_0 - \widehat{x}_0$.

Assumption 3. *A state coordinate transformation* $P \in \mathbb{R}^{n\times n}$ *is known s.t.* $M = P\left(A - LC\right)P^{-1}$ *is Metzler.*

The error dynamics in Eq. (9) expressed in the new coordinates becomes

$$(G_{e_z}) \left\{ \begin{array}{lcl} \dot{e}_z & = & M e_z + B_2' d \\ e_z(0) & = & P e_0 \end{array} \right. \tag{10}$$

[1] Linear Parameter Varying

where $e_z = Pe$ and $B_2' = PB_2$. According to Def. 1 and using Lemma 1 and Hyp. 3, an interval observer $\left(G^\sharp\right)$ on the system in Eq. (10) is defined by the following dynamic system

$$\left(G^\sharp_{e_z}\right)\begin{cases} \dot{\underline{e}}_z & = & M\underline{e}_z + B_2'^+\underline{d} - B_2'^-\overline{d} \\ \dot{\overline{e}}_z & = & M\overline{e}_z + B_2'^+\overline{d} - B_2'^-\underline{d} \\ \underline{e}_z(0) & = & P^+\underline{e}_0 - P^-\overline{e}_0 \\ \overline{e}_z(0) & = & P^+\overline{e}_0 - P^-\underline{e}_0 \end{cases} \tag{11}$$

Let $T = P^{-1}$. To obtain bounds in the original coordinates $e = Te_z$, the following relations are used

$$\begin{cases} \underline{e} & = & T^+\underline{e}_z - T^-\overline{e}_z \\ \overline{e} & = & T^+\overline{e}_z - T^-\underline{e}_z \end{cases} \tag{12}$$

which preserve state ordering under Lemma 1. Thus, for any $e_0 \in [\underline{e}_0, \overline{e}_0]$ the following framing is obtained:

$$\forall t \geq 0,\, \underline{e}(t) \leq e(t) \leq \overline{e}(t) \tag{13}$$

which in turn leads to

$$\forall t \geq 0,\, \underline{x}(t) = \widehat{x}(t) + \underline{e}(t) \leq x(t) \leq \widehat{x}(t) + \overline{e}(t) = \overline{x}(t) \tag{14}$$

Using an appropriate state coordinate transformation, the state of a generic disturbed linear time-invariant continuous-time system can thus be bounded.

In the discrete-time case, the formalism is very close to the continuous-time case with a slight difference on the properties to be satisfied by the state matrix, since it should be Schur-stable and non-negative, as stated in Def. 5:

Assumption 4. *A state coordinate transformation $P \in \mathbb{R}^{n\times n}$ is known s.t. $M = P\,(A - LC)\,P^{-1}$ is non-negative.*

3.2 An overview of existing techniques to find an appropriate state coordinate transform

In the framework presented in Sect. 3.1.2, Hyp. 3 and 4 are strong hypotheses. In the case of high-order systems, choosing a satisfying state coordinate transformation matrix P is not obvious:

Problem 1. *(continuous-time case)* Find $P \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{n\times m}$ such that $M = P\,(A - LC)\,P^{-1}$ is Hurwitz Metzler.

Problem 2. *(discrete-time case)* Find $P \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{n\times m}$ such that $M = P\,(A - LC)\,P^{-1}$ is Schur-stable and non-negative.

In this section, the existing methods to determine a solution to Pbs. 1 and 2 are presented along with their advantages and limitations.

3.2.1 Real-constrained pole placement (trivial solution)

This trivial solution is obtained by noticing that diagonal matrices satisfy to Def. 2. Supposing that the pair (A, C) is observable (or that the unobservable eigenvalues are real negative, which is stronger than simple detectability), the matrix L is chosen such that $A - LC$ only has negative real eigenvalues. Then, choosing P as the matrix of the right-hand eigenvectors means $M = P\,(A - LC)\,P^{-1}$ is diagonal hence is a Metzler matrix.

The advantage of this technique is its simplicity since it only requires a pole placement and an eigenstructure determination algorithm. Three main limitations can however be mentioned:

- placing real poles can result in very large observation gains L resulting in sensitivity problems;

- an observer with complex poles may achieve better results;
- due to floating-point computations precision errors, the computed matrix M is Metzler up to the machine precision.

Remark 2. *In the discrete-time case, the poles have to be placed on real values between* 0 *and* 1 *to comply with the definitions of Schur-stability and non-negativity.*

3.2.2 Resolution of a Sylvester equation

To find a solution to Pb. 1, another approach can be used

- choose $M \in \mathbb{R}^{n\times n}$ to be a Metzler matrix;
- choose $Q \in \mathbb{R}^{n\times m}$, this will represent the matrix product PL;
- using algorithms for example proposed in [19] or [20], solve the following Sylvester equation in P:
$$-MP + PA = PLC = QC \tag{15}$$
If M and A have no common eigenvalues, this equation has a unique solution for any Q.

One thus obtains a Metzler matrix $M = P\,(A - LC)\,P^{-1}$ where the observer gain is defined by $L = P^{-1}Q$. This method was proposed in [4] along with a constructive lemma [4, Lemma 1, p. 261] to solve the equation using observability criteria. However this Lemma is difficult to apply in case complex eigenvalues are looked for since there is no trivial (i.e. diagonal or triangular) Metzler matrix M with complex eigenvalues. Numerical errors may also occur in case of badly-conditioned matrices. A solution to the Sylvester equation can be obtained for example using [20].

Remark 3. *In the discrete-time case, Remark 2 is still valid here.*

3.3 Conclusions on the interests of an alternative approach

Both techniques proposed in the literature bring the advantage of simplicity and only use built-in functions of any numerical computation software. This will be sufficient in most considered cases.

However, in more complex cases – high-order systems, ill-conditioned system matrices, additional constraints like observer error dynamics decay rate optimization, etc. – it seems more difficult to obtain satisfying results due to numerical precision problems as well as to the method philosophy. Indeed, the chosen matrices M and Q could lead to non-optimal results in view of other control constraints. Moreover, it seems difficult to obtain an expression of a Metzler matrix from expected complex eigenvalues.

An alternative technique is hereby proposed which could complement the existing methods by taking these more complex cases into account.

4 Problems statement

In this section, the different problems which arise in the proposed approach are exposed. As far as the state coordinate transformation determination is concerned, a reformulation into a control law synthesis problem is proposed, as exposed in [5]. A numerical approach to find a solution simultaneously satisfying to these problems is then proposed in Sect. 5. It uses existing controller synthesis techniques as permitted by the reformulation.

4.1 State coordinate transformation determination

In Sect. 3, the design of interval observers has motivated the need for state coordinate transformation determination methods. This article proposes a new approach based on solving a control design problem.

4.1.1 Initial problem

Let recall the initial state coordinate transformation problem as stated in Pb. 1.

Problem 3. *(Mathematical formulation)* Given (A, C) detectable, determine simultaneously $P \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{n\times m}$ such that $M = P\,(A - LC)\,P^{-1}$ is

- Metzler and Hurwitz (continuous-time case);
- non-negative and Schur-stable (discrete-time case).

In other words, finding a solution to Pb. 3 in the continuous-time case is equivalent to finding P and L under $n(n-1)$ positivity requirements on the off-diagonal elements of $M = (m_{ij}) \in \mathbb{R}^{n\times n}$. This directly comes from the definition 2 of a Metzler matrix:

$$m_{ij} \geq 0,\, \forall i \neq j \Leftrightarrow -m_{ij} \leq 0,\, \forall i \neq j$$

In terms of control theory, these last inequalities are reminiscent of the stabilisation condition on a "fictitious" mono-dimensional system having its real pole located at $-m_{ij}$.

4.1.2 Reformulation into a control design problem

A reformulation of Pb. 3 into a control design problem is done in the following proposition.

Proposition 1. *(Equivalent control design problem, continuous-time case) Let (A, C) detectable and*

$$m_{ij}\,(P, L) = \left[P\,(A - LC)\,P^{-1}\right]_{ij}$$

for $P \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{n\times m}$. If the following system

$$(G_c)\left\{\begin{array}{lcl} \dot{x}_1 & = & (A - LC)\,x_1 \\ \dot{x}_{ij} & = & -m_{ij}\,(P, L)\,x_{ij} \quad \forall i \neq j \end{array}\right. \tag{16}$$

is Hurwitz-stable for given P and L then $P\,(A - LC)\,P^{-1}$ is Metzler and Hurwitz.

Proof. Straightforward considering the definition of a Metzler matrix, see Def. 2, and the definitions of the coefficients $m_{ij}\,(P, L)$. In particular, ensuring positiveness of $m_{ij}\,(P, L)$ is equivalent to stabilizing $\dot{x}_{ij} = -m_{ij}\,(P, L)\,x_{ij}$. □

The states $x_{ij} \in \mathbb{R}$ are called "fictitious" states since they do not have any physical meaning. For an initial matrix $A \in \mathbb{R}^{n\times n}$, $n\,(n-1)$ "fictitious" states are considered.

Remark 4. *Prop. 1 is reminiscent of a structured control design problem which is NP-hard in general. It is usually solved using algorithms with local optimality certificates which are known to work in practice. There is however no guarantee that the locally optimal solution stabilizes Eq.* (16)*.*

Remark 5. *In the discrete-time case, n^2 "fictitious" states are considered since the diagonal elements of matrix M should also be non-negative. However, due to the definition of Schur-stability, using Prop. 1 would lead to $-m_{ij}(P, L) \in [-1, 1]\,, \forall (i, j)$ which does not comply with the definition of non-negativity. In that case, the system in Eq.* (16) *is replaced by the following system:*

$$(G_{c,d})\left\{\begin{array}{lcl} x_{1,t+1} & = & (A - LC)\,x_{1,t} \\ x_{ij,t+1} & = & f\left(-m_{ij}\,(P, L)\right)x_{ij,t} \quad \forall\,(i, j) \end{array}\right. \tag{17}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a linear function which maps $\left[-m_{ij}^{\max}, 0\right]$ onto $[-1, 1]$ with $m_{ij}^{\max} > 0$ a design parameter which will be useful later:

$$f(x) = \frac{2}{m_{ij}^{\max}}x + 1 \tag{18}$$

Using this function has proved to be effective in practice.

4.2 Adding control requirements: resulting optimization problem

As already mentioned in Sect. 3.3, choosing P and L simultaneously such that additional control constraints are fulfilled is difficult. For example, considering the observation error dynamics in Eq. (9), it may be beneficial in terms of estimation quality to minimize a normalized value of the H_2-norm of the transfer from the disturbance d to the observation error e:

$$\begin{array}{l} \min_{P,L} \max\left\{\|W(s)T_{d\to e}(s,L)\|_2, \|T_2(s,P,L)\|_2, \|T_3(s,P,L)\|_\infty, \ldots\right\} \\ \text{subject to } G_c\,(P,L)\ \text{(Eq. (16)) is stable.} \end{array} \tag{19}$$

where $W(s)$ is an appropriate weighting and $T_2(s,P,L)$ and $T_3(s,P,L)$ are additional design transfer functions. Note that the cost function is nonsmooth. This type of optimization problem is typical of the kind of problem encountered when synthesizing structured controllers against multiple requirements over multiple models. The idea is thus to use such formalism to find at least a local minimizer to this problem.

4.3 Structured controller synthesis

The synthesis of full-order controllers through H_∞ methods has been widely studied and used in the past two decades. Solutions to the problem of H_∞ synthesis in the case of MIMO systems were provided for example in [21] and [22]. When considering controllers with a fixed order much lower than the original plant, the problem of finding a controller is a non-convex optimization problem. A local solution to this problem was proposed in [23] and implemented in consecutive works [24]. Other implementations for computing fixed-order controllers include [25]. More recent works [26, 6, 27] consider the case of finding a controller satisfying to multiple requirements on multiple models. Considering these techniques, it appears possible to solve the minimisation problem in Eq. (19).

5 Nonsmooth optimization-based approach

In this section, the *structured controller synthesis* approach is detailed to solve the optimization problem in Sect. 4.2. An illustration of the technique to determine an interval observer state coordinate transformation is also proposed, which is then applied to three examples in Sect. 6.

5.1 Structured controller synthesis algorithm

As mentioned in Sect. 4.2, it appears that the reformulation proposed in Prop. 1 joined to additional control requirements leads to a structured controller synthesis problem. As already mentioned, this problem has been solved in [6, 23]. In this article, the `systune` function – along with the `TuningGoal` structure – from the MATLAB Robust Control Toolbox [28] is used to solve the problem. This function uses nonsmooth optimization algorithms to solve nonsmooth non-convex optimization problems. The method is provably convergent and computes certified locally optimal solutions from any remote starting point.

In the following sections, the models and requirements needed to solve the optimization problem in Sect. 4.2 using these tools are detailed.

5.2 Models definition

As was already mentioned, recent control synthesis techniques allow to set multiple requirements on multiple models, be they describing an existing physical system or mathematical constraints, as is the case with the "fictitious" states used in Prop. 1. In the continuous-time case, let define $n(n-1)$ fictitious systems:

$$\left(G_M^{ij}\right)\ \ \dot{x}_{ij} = -m_{ij}\,(P,L)\,x_{ij},\ \forall i \neq j \tag{20}$$

and, in the discrete-time case, let define the following n^2 fictitious systems:

$$\left(G_{M,d}^{ij}\right) \quad x_{ij,t+1} = f\left(-m_{ij}\left(P,L\right)\right) x_{ij,t},\ \forall\left(i,j\right) \tag{21}$$

The required synthesis models are enumerated here in the form of *collections* of models. Note that some collections only apply to the continuous- (**CT**) or discrete-time case (**DT**).

(C1–CT) $n(n-1)$ unidimensional fictitious systems $\left\{G_M^{ij}\right\}_{i,j,i\neq j}$ as defined in Eq. (20);

(C1–DT) n^2 unidimensional fictitious systems $\left\{G_{M,d}^{ij}\right\}_{i,j}$ as defined in Eq. (21) with f a linear function mapping $\left[-m_{ij}^{\max},0\right]$ onto $[-1,1]$ where $m_{ij}^{\max}$ is a design parameter playing the same role as in **(C2–CT)** (see Remark 5);

(C2–CT) (*optional*) $n(n-1)$ unidimensional fictitious systems $\left\{\overline{G}_M^{ij}\right\}_{i,j,i\neq j}$ with state matrix equal to $m_{ij}(P,L)-m_{ij}^{\max}\in\mathbb{R}$. This helps to restrict the set of acceptable solutions when $0<m_{ij}^{\max}<+\infty$. The initialising variables excursion is thus limited around a smaller set of potential local optima;

(C3) any other model in state-space representation, describing the control or observation problem, thus including tunable controller or observer parameters.

A series of requirements are now expressed on each collection.

5.3 Requirements

For each collection in Sect. 5.2, a set of requirements (or constraints, see Sect. 4.3) is expressed.

(R1–CT) a requirement on the closed-loop poles location is expressed on **(C1–CT)** in the form of a constraint on the minimum decay rate. This ensures negativity of $-m_{ij}(P,L)$ hence positivity of m_{ij}, $\forall i\neq j$ (CT case);

(R1–DT) similar requirement to **(R1–CT)** but expressed on the n^2 models in **(C1–DT)**;

(R2–CT) similar to previous requirements sets but applied on **(C2–CT)**. This ensures negativity of $m_{ij}-m_{ij}^{\max}$, $\forall i\neq j$;

(R3) any control requirement to enforce on **(C3)** like disturbance rejection, closed-loop poles, etc.

6 Applications

In this section, the method proposed in Sect. 5 is applied to three examples, including an unstable system, using the interval observers formalism presented in Sect. 3. A comparison with the methods proposed in the literature is also provided.

6.1 Continuous-time stable sixth-order example

The formalism of interval observers in the continuous-time case is presented in Sect. 3.1.2. The method presented in this article proposes to enforce Hyp. 3 by computing P and L such that M is *Hurwitz Metzler*.

The following example is inspired from [3]. The considered system dynamics is given by

$$\left\{\begin{array}{rcl}\dot{x} & = & Ax+B_1u\\ y & = & Cx+d\end{array}\right. \tag{22}$$

where $x\in\mathbb{R}^6$, $u(t)=\sin(t)$ and $d(t)$ is an unknown but bounded measurement disturbance with $\underline{d}=-2\leq d(t)\leq 1=\overline{d}$, $\forall t$. A random number generator with limited range is used in simulation. The system matrices are numerically given by

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 0 \\ -1 & -2 & 0 & -1 & 0 & 1 \\ -2 & 0 & -3 & -2 & 0 & 0 \\ -1 & 0 & -2 & -3 & 0 & 1 \\ -1 & 0 & 2 & 0 & -4 & 0 \\ -1 & -1 & 0 & 1 & 0 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} -18 \\ -13 \\ -5 \\ -4 \\ -10 \\ 22 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

The system is stable and it is initialized at $x_0 = \begin{bmatrix} 20 & 10 & 6 & 20 & 30 & 40 \end{bmatrix}^\top$. The Luenberger observer is initialized at $\widehat{x}_0 = 0$ and the initial state is supposed to lie in $[-\overline{x}_0, \overline{x}_0]$ where $\overline{x}_0 = 50 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$. Thus, the initial observation error e_0 lies in $[\underline{e}_0, \overline{e}_0] = [-\overline{x}_0 - \widehat{x}_0, \overline{x}_0 - \widehat{x}_0]$.

The observation error dynamics is defined by

$$\begin{cases} \dot{e} &= (A - LC)e - Ld \\ e_0 &= x_0 - \widehat{x}_0 \end{cases} \tag{24}$$

which is not cooperative. Under Hyp. 3, the following system is an interval observer for the observation error dynamics in Eq. (24), where $P_L^+ = \max(PL, 0)$ and $P_L^- = P_L^+ - PL$.

$$\begin{cases} \underline{e}_z &= M\underline{e}_z + P_L^+\underline{d} - P_L^-\overline{d} \\ \overline{e}_z &= M\overline{e}_z + P_L^+\overline{d} - P_L^-\underline{d} \\ \underline{e}_z(0) &= P^+\underline{e}_0 - P^-\overline{e}_0 \\ \overline{e}_z(0) &= P^+\overline{e}_0 - P^-\underline{e}_0 \end{cases} \tag{25}$$

Reverting back to the initial coordinates is done using Eq. (12). The three following methods are considered to determine P and L:

- real pole placement allowing state matrix diagonalization;
- resolution of a Sylvester equation;
- application of the nonsmooth control-based method.

6.1.1 Existing techniques

The numerical results obtained using existing techniques are presented here.

Real pole placement The following pole placement gives rather good results in simulation as far as the convergence speed of the observer is concerned

$$p = \begin{bmatrix} -4.7 & -4.6 & -2.1 & -1.2 & -1.1 & -0.3 \end{bmatrix} \tag{26}$$

Using the pole placement and diagonalization approach in Sect. 3.2.1, one obtains the following P and L matrices

$$P = \begin{bmatrix} -2.5932 & 1.9801 & -9.9547 & -12.1661 & -3.7046 & 2.7530 \\ -2.8007 & 2.1553 & -9.4734 & -12.2466 & -4.6679 & 2.8031 \\ 1.6458 & -0.1905 & 1.6282 & -1.5989 & -0.8662 & 1.6268 \\ -20.6249 & -16.3609 & -11.6690 & 17.8681 & 7.3660 & -7.5362 \\ 19.8606 & 16.2310 & 10.4293 & -16.7563 & -6.8485 & 5.2526 \\ 0.0250 & 0.4350 & 0.6878 & -0.9352 & -0.0068 & -0.7146 \end{bmatrix}, L = \begin{bmatrix} -0.0000 \\ -1.2981 \\ -1.1800 \\ -1.0496 \\ -0.0981 \\ -0.3113 \end{bmatrix} \tag{27}$$

which leads to the following matrix $M = P\,(A - LC)\,P^{-1}$

Table 1: Control-based approach parameters for the example described in Sect. 6.1, $p_{(A-LC)}$ refers to the poles of the observer error dynamics (24)

Syn. model	*Requirement*	*Settings*
(C1–CT)	**(R1–CT)**	-
(C2–CT)	**(R2–CT)**	$m_{ij}^{\max} = 200$
(C3)	**(R3)**-1	$\mathrm{Re}(p_{A-LC}) \in \left]-\infty, -3.10^{-3}\right]$
	(R3)-2	$\left\Vert \frac{1}{0.4} T_{d\to e}(s) \right\Vert \le 1$

$$M = \begin{bmatrix} -4.7000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & -4.6000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & -2.1000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -0.0000 & -1.2000 & -0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.1000 & -0.0000 \\ -0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.3000 \end{bmatrix} \tag{28}$$

Due to numerical errors, the observation error dynamics is however not certified to be cooperative.

Sylvester equation The approach in Sect. 3.2.2 is applied to the considered system. The following upper-triangular Metzler matrix M is proposed

$$M = \begin{bmatrix} -4.7000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0 & -4.6000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0 & 0 & -2.1000 & 1.0000 & 1.0000 & 1.0000 \\ 0 & 0 & 0 & -1.2000 & 1.0000 & 1.0000 \\ 0 & 0 & 0 & 0 & -1.1000 & 1.0000 \\ 0 & 0 & 0 & 0 & 0 & -0.3000 \end{bmatrix} \tag{29}$$

along with $Q = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}^\top$. Solving Eq. (15) for these values, one obtains

$$P = \begin{bmatrix} 11.1557 & -23.0016 & -41.8920 & 58.6245 & -3.1677 & 54.8212 \\ 8.6278 & -19.1428 & -35.4258 & 48.3810 & -2.9658 & 45.4833 \\ 15.8575 & -28.1624 & -53.0607 & 70.9787 & -6.0558 & 71.5056 \\ 11.3015 & -26.6410 & -53.7302 & 70.7881 & -3.8611 & 64.4377 \\ 1.8712 & -16.6031 & -28.5216 & 38.4000 & -0.7267 & 32.0343 \\ -0.8742 & -15.2203 & -24.0624 & 32.7206 & 0.2363 & 25.0003 \end{bmatrix}, \; L = \begin{bmatrix} 0.0000 \\ -1.2981 \\ -1.1800 \\ -1.0496 \\ -0.0981 \\ -0.3113 \end{bmatrix} \tag{30}$$

which leads to the following matrix $M = P\,(A - LC)\,P^{-1}$

$$M = \begin{bmatrix} -4.7000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.0000 & -4.6000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ -0.0000 & -0.0000 & -2.1000 & 1.0000 & 1.0000 & 1.0000 \\ 0.0000 & -0.0000 & 0.0000 & -1.2000 & 1.0000 & 1.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & -1.1000 & 1.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.3000 \end{bmatrix} \tag{31}$$

In this case, numerical errors also appear in the final result.

6.1.2 Control-based approach

The method proposed in Sect. 4 and 5 is used. The synthesis models and requirements used are recalled in Table 1 and refer to notations used in Sect. 5.

Using the control law synthesis algorithm presented in Sect. 5.1, a locally optimal solution is found in $287s$ after 3 restarts and requiring 1107 iterations. The following results are obtained, where $M = P(A - LC)P^{-1}$ is Hurwitz Metzler.

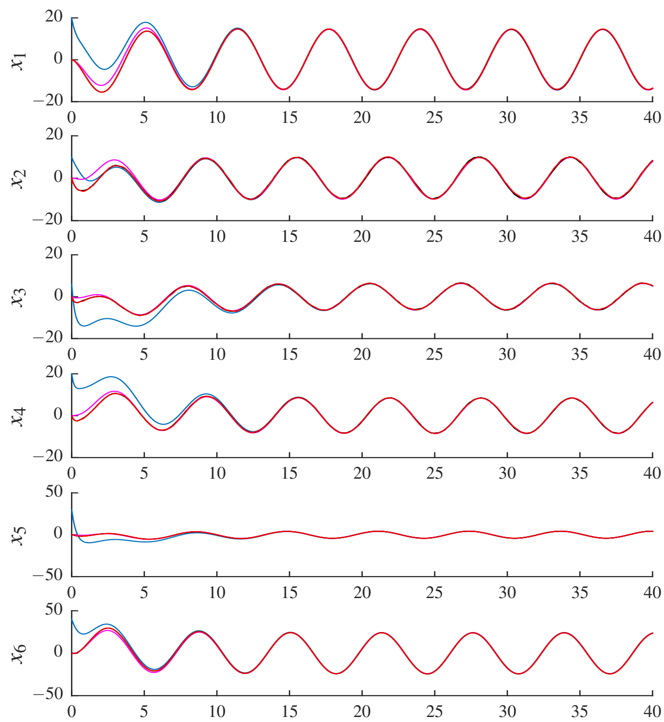

Figure 1: In blue, simulation results for the system in Eq. (22) and in magenta (resp. black/red), for the classical observer in Eq. (8) obtained using the observer gain L in Eq. (32) (resp. Eq. (27) or Eq. (30)) produced by the control-based approach (resp. real pole placement or Sylvester equation resolution).

$$M = \begin{bmatrix} -0.9015 & 0.0000 & 22.5484 & 0.0002 & 0.0054 & 0.0001 \\ 0.0231 & -2.5523 & 0.1372 & 0.0000 & 0.4259 & 0.0589 \\ 0.0000 & 0.0114 & -1.4533 & 0.0001 & 0.0622 & 0.0000 \\ 0.0831 & 26.4981 & 0.0000 & -4.4748 & 4.3388 & 2.7824 \\ 0.0000 & 0.0000 & 0.0050 & 0.5871 & -3.4814 & 0.0335 \\ 1.3946 & 0.0000 & 0.5513 & 0.0010 & 0.0032 & -1.0506 \end{bmatrix},$$

$$P = \begin{bmatrix} -0.5886 & -0.9537 & -1.2910 & 1.7496 & 0.3504 & 0.3355 \\ 0.0110 & 0.0364 & -0.1391 & 0.1713 & -0.0840 & 0.0965 \\ 0.0398 & 0.0055 & -0.0037 & 0.0087 & -0.0221 & 0.0337 \\ 1.4422 & -0.3846 & -0.7972 & 3.8482 & -0.4497 & 2.3833 \\ 0.1262 & 0.0438 & -0.8724 & 0.3191 & 0.2818 & 0.4526 \\ 0.2464 & -1.0668 & -0.7884 & 1.4262 & -0.2453 & 2.5864 \end{bmatrix}, L = \begin{bmatrix} -0.0862 \\ 0.0583 \\ -0.1903 \\ 0.0349 \\ 0.0366 \\ -0.1631 \end{bmatrix} \quad (32)$$

6.1.3 Simulations and comparisons

The system is initialized at x_0 and simulated over $40s$ using $u(t) = \sin(t)$ and a random-number-generated signal $d(t) \in [-2, 1]$. Results are shown on Fig. 1 and 2.

The results obtained using the control-based approach are highly satisfying. Although the obtained interval observer seems slower than the one obtained using real pole placement, it converges closer to the system state. Additional requirements could be used to deal with the convergence speed. The results obtained using the Sylvester equation approach are very bad due to the difficulty of selecting appropriate matrices M and Q. Moreover, P is used to determine L while this is not the case in the other methods.

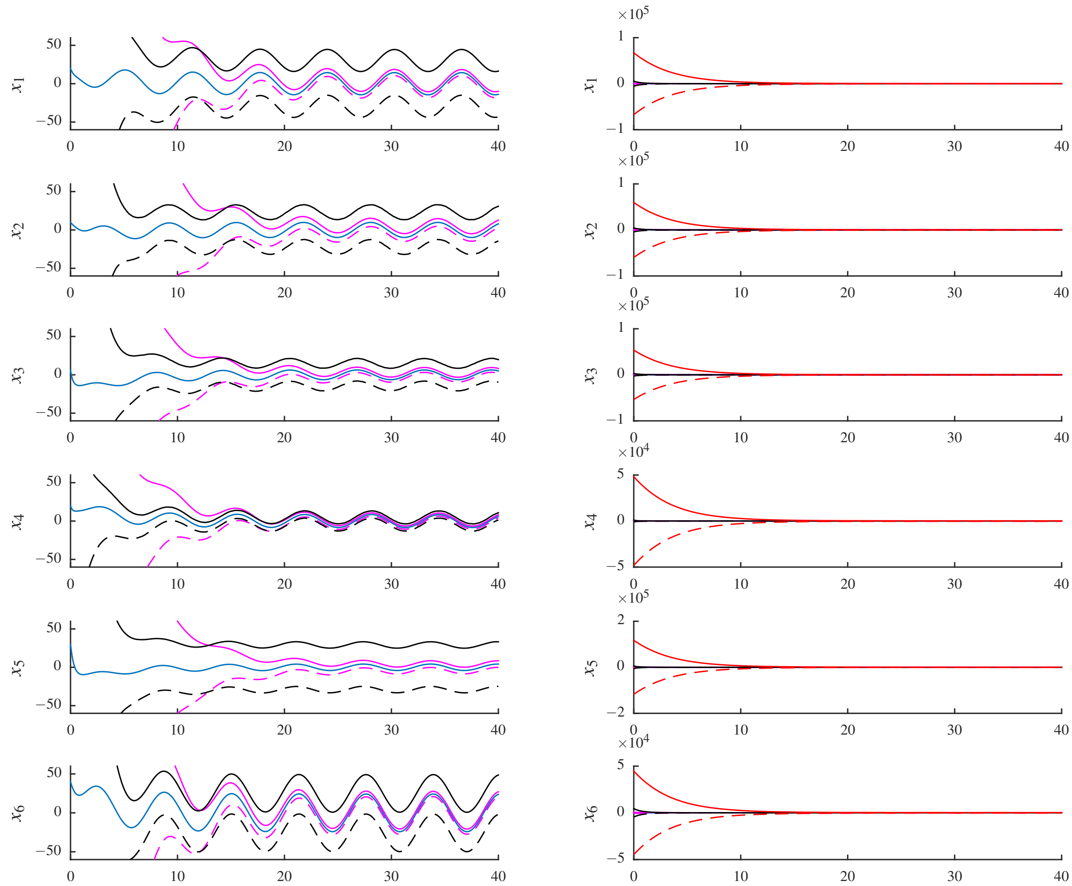

Figure 2: In blue, simulation results for the system in Eq. (22) and in magenta (resp. black/red), bounds of the interval observer obtained using the control-based approach (resp. real pole placement/Sylvester equation resolution). The lower bounds are represented using a dashed line. The plots on the left are the same as the plots on the right but with limited ordinate excursion.

6.2 Continuous-time unstable third-order example

The following example is adapted from [4]. The considered system dynamics is given by

$$\begin{cases} \dot{x} &= Ax + Bd \\ y &= Cx \end{cases} \tag{33}$$

where $x \in \mathbb{R}^3$ and $d(t)$ is an unknown but bounded state disturbance with $\underline{d}(t) = -1 \leq d(t) \leq 1 = \overline{d}(t)$, $\forall t$. The signal $d(t) = \sin(2t)$ will be used in simulation. The system matrices are defined as follows

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & \sqrt{3} \\ -1 & -\sqrt{3} & -4 \end{bmatrix}, \; B = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{34}$$

This system is unstable and detectable. The initial state is given by $x_0 = \begin{bmatrix} -2 & 1.5 & -1 \end{bmatrix}^\top$ and the initial observer state by $\widehat{x}_0 = 0$. Bounds on the initial state are supposed to equal $\underline{x}_0 = \begin{bmatrix} -2 & -2 & -2 \end{bmatrix}^\top$ and $\overline{x}_0 = -\underline{x}_0$. The initial observation error e_0 lies in $[\underline{e}_0, \overline{e}_0] = [\underline{x}_0 - \widehat{x}_0, -\underline{x}_0 - \widehat{x}_0]$.

The observation error dynamics is defined by

$$\begin{cases} \dot{e} &= (A - LC)\, e + Bd \\ e_0 &= x_0 - \widehat{x}_0 \end{cases} \tag{35}$$

which is not cooperative. Under Hyp. 3, the following system is an interval observer for the observation error dynamics in Eq. (35), where $P_B^+ = \max(PB, 0)$ and $P_B^- = P_B^+ - PB$.

$$\begin{cases} \underline{e}_z &= M\underline{e}_z + P_B^+\underline{d} - P_B^-\overline{d} \\ \overline{e}_z &= M\overline{e}_z + P_B^+\overline{d} - P_B^-\underline{d} \\ \underline{e}_z(0) &= P^+\underline{e}_0 - P^-\overline{e}_0 \\ \overline{e}_z(0) &= P^+\overline{e}_0 - P^-\underline{e}_0 \end{cases} \tag{36}$$

Reverting back to the initial coordinates is done using Eq. (12). The two following methods are considered to determine P and L:

- application of [4, Lemma 1, p. 261];
- application of the nonsmooth control-based method.

6.2.1 Application of the constructive Lemma

In this section, Lemma 1 in [4, p. 261] is used as a practical method to solve the Sylvester equation presented in Sect. 3.2.2. At first, one has to choose the observer gain so that $A - LC$ is Hurwitz, for example $L = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^\top$. Then, a targeted matrix M is selected such that it is Metzler and has the same eigenvalues as $A - LC$:

$$M = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -3 & 2 \\ 2 & 0 & -3 \end{bmatrix} \tag{37}$$

Using [4, Lemma 1, p. 261] with $e_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\top$ and $e_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$, one obtains

$$P = \begin{bmatrix} 0.4085 & 0.8660 & 0.5000 \\ 0.5915 & -0.8660 & 0.5000 \\ -0.0915 & 0.0000 & -1.0000 \end{bmatrix}, \; M = P\,(A - LC)\,P^{-1} = \begin{bmatrix} -3.0000 & 2.0000 & -0.0000 \\ -0.0000 & -3.0000 & 2.0000 \\ 2.0000 & 0.0000 & -3.0000 \end{bmatrix} \tag{38}$$

Table 2: Control-based approach parameters for the example described in Sect. 6.2, $p_{(A-LC)}$ refers to the poles of the observer error dynamics (35)

Syn. model	*Requirement*	*Settings*
(C1–CT)	**(R1–CT)**	-
(C2–CT)	**(R2–CT)**	$m_{ij}^{\max} = 100$
(C3)	**(R3)**-1	$\mathrm{Re}(p_{A-LC}) \in \left[-10, -3.10^{-3}\right]$
	(R3)-2	$\left\Vert \frac{1}{0.8} T_{d \to e}(s) \right\Vert \leq 1$

6.2.2 Control-based approach

In this section, the approach proposed in Sect. 4 and 5 is used. The synthesis models and requirements used, along with the settings values, are presented in Table 2.

Using the control law synthesis algorithm presented in Sect. 5.1, a locally optimal solution is found after 5 restarts requiring less than 400 iterations each. The execution time rises to 15s. The result is given below.

$$P = \begin{bmatrix} 0.4622 & -1.1708 & -1.6588 \\ 0.3043 & 2.0962 & -0.0391 \\ 0.0021 & -0.0165 & 0.0372 \end{bmatrix}, \; L = \begin{bmatrix} 2.9830 \\ 1.2172 \\ -1.2948 \end{bmatrix}, \; M = \begin{bmatrix} -2.8253 & 2.0267 & 0.0000 \\ 0.0000 & -3.1910 & 98.5812 \\ 0.0404 & 0.0000 & -2.9666 \end{bmatrix} \tag{39}$$

where $M = P\,(A - LC)\,P^{-1}$ with no numerical error.

6.2.3 Simulations and comparisons

The system initialized at x_0 is simulated over $10s$ with $d(t) = \sin(2t)$. As expected, the system state diverges. However, considering the observation error e rather than the state x, the bounds $\underline{e}$ and $\overline{e}$ obtained using the interval observer in Eq. (36) are expected to converge. Simulation results are displayed on Fig. 3 and 4.

In this particular example, the results obtained using the proposed control-based approach are better. The obtained matrix M is certified to be Hurwitz Metzler. By adding control requirements and simultaneously synthesizing P and L, the disturbance can be better rejected.

6.3 Discrete-time example

This example is adapted from [18]. The following discrete-time system with sample time $T_s = 1s$ is considered

$$\begin{cases} x_{t+1} &= A_d x_t + B_d d_t \\ y_t &= C_d x_t \end{cases} \tag{40}$$

where $x_t \in \mathbb{R}^5$ and d_t is an unknown disturbance bounded by $[-1, 1]$ and which is simulated by a random number generator on a limited range and with a time sampling of T_s. The numerical values used for the system matrices are

$$A_d = \begin{bmatrix} -0.54 & 0.45 & 0.36 & 0 & 0 \\ 0.63 & 0.45 & 0.18 & 0.36 & 0 \\ 0.09 & 0.45 & 0.27 & 0.09 & 0.18 \\ 0 & 0 & 0.25 & 0.25\sqrt{2} & -0.25\sqrt{2} \\ 0 & 0 & 0 & 0.25\sqrt{2} & 0.25\sqrt{2} \end{bmatrix}, \; B_d = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\top, \; C_d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{41}$$

The system is Schur-stable. It is initialized at $x_0 = \begin{bmatrix} -0.3 & -0.5 & 0.6 & 0.9 & -0.2 \end{bmatrix}^\top$. The Luenberger observer is initialized at $\widehat{x}_0 = 0$ and the initial state is supposed to lie in $[-\overline{x}_0, \overline{x}_0]$ where $\overline{x}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$. The initial observation error e_0 lies in $[\underline{e}_0, \overline{e}_0] = [-\overline{x}_0 - \widehat{x}_0, \overline{x}_0 - \widehat{x}_0]$. The observation error dynamics is defined by

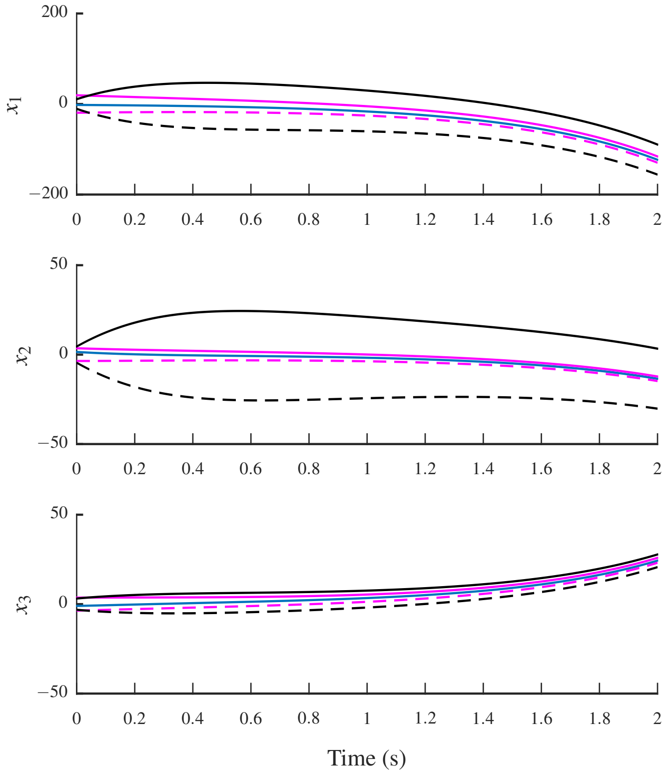

Figure 3: In blue, simulation results over $2s$ for the system in Eq. (33) and in magenta (resp. black), bounds of the interval observer obtained using the control-based approach (resp. Lemma 1 in [4]).

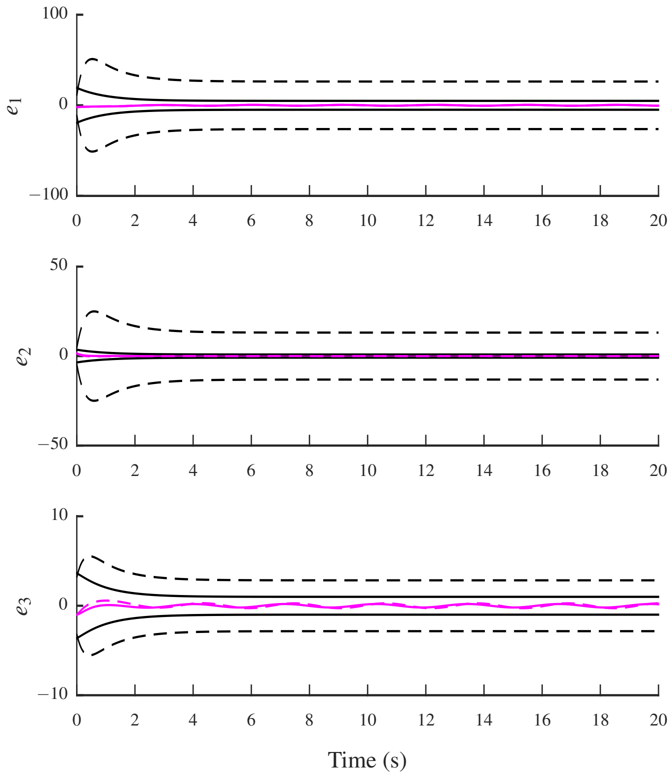

Figure 4: In solid magenta, simulation results over $20s$ for the system in Eq. (35) and in solid black, bounds of the interval observer obtained using the control-based approach. The results obtained using Lemma 1 in [4] are represented in dashed lines.

Table 3: Control-based approach parameters for the example described in Sect. 6.3, $T_{d_t \to e_t}(z)$ refers to the transfer function from e_t to d_t, stability of the observation error dynamics is also ensured

Syn. model	*Requirement*	*Settings*
(C1–DT)	**(R1–DT)**	$m_{ij}^{\max} = 100$
(C3)	**(R3)**-1	Stab. Eq. (42)
	(R3)-2	$\|T_{d_t \to e_t}(z)\|_2 \leq 1$

$$\left\{ \begin{array}{lcl} e_{t+1} & = & (A_d - LC_d)e_t + B_d d_t \\ e_0 & = & x_0 - \widehat{x}_0 \end{array} \right. \tag{42}$$

which is not cooperative for any value of L. Under Hyp. 4, the following system is an interval observer for the observation error dynamics defined in Eq. (42), where $P_B^+ = \max(PB_d, 0)$ and $P_B^- = P_B^+ - PB_d$.

$$\left\{ \begin{array}{lcl} \underline{e}_{z,t+1} & = & M_d \underline{e}_{z,t} + P_B^+ \underline{d}_t - P_B^- \overline{d}_t \\ \overline{e}_{z,t+1} & = & M_d \overline{e}_{z,t} + P_B^+ \overline{d}_t - P_B^- \underline{d}_t \\ \underline{e}_{z,0} & = & P^+ \underline{e}_0 - P^- \overline{e}_0 \\ \overline{e}_{z,0} & = & P^+ \overline{e}_0 - P^- \underline{e}_0 \end{array} \right. \tag{43}$$

Reverting back to the initial coordinates is performed using Eq. (12). In that case, the Sylvester method proposed in Remark 3 yields very poor results with the chosen arbitrary pole placement and non-negative matrix M_d. The two remaining methods: real pole placement as described in Remark 2 and the approach proposed in this article are compared.

6.3.1 Real pole placement and diagonalization

A pole placement is used as described in Remark 2 in Sect. 3.2.1. The desired poles are chosen arbitrarily as real numbers inferior to 1 and positive

$$p_3 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 & 0.9 \end{bmatrix} \tag{44}$$

The following P and L matrices are thus obtained

$$P = \begin{bmatrix} -0.4816 & -1.3788 & -0.8972 & -0.5748 & 0.0763 \\ -0.3375 & -0.5564 & 0.0284 & 0.9356 & -0.9400 \\ -0.1471 & -0.9124 & 0.8567 & 0.2861 & -0.2092 \\ 0.5944 & -1.4806 & -0.7589 & -0.4881 & 0.2457 \\ -0.1690 & 0.0094 & 0.1658 & 0.2565 & 1.1357 \end{bmatrix}, \ L = \begin{bmatrix} -1.1993 & -0.2234 \\ 0.6217 & 0.2992 \\ -0.0161 & 0.2228 \\ -0.0205 & -0.4136 \\ -0.0333 & 0.4069 \end{bmatrix} \tag{45}$$

which leads to the following matrix $M = P(A - LC)P^{-1}$

$$M = \begin{bmatrix} 0.9000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.7000 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.1000 & -0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5000 & -0.0000 \\ -0.0000 & -0.0000 & 0 & -0.0000 & 0.3000 \end{bmatrix} \tag{46}$$

Due to numerical errors, the matrix M is not non-negative up to the machine precision.

6.3.2 Control-based approach

The approach proposed in Sect. 4 and 5 is used, in the case of discrete-time systems. The synthesis models and requirements used are recalled in Table 3.

Using the control law synthesis algorithm presented in Sect. 5.1, a local optimal solution is found after 3 restarts and 400 iterations at maximum. The execution time rises to $41s$. The following matrices are obtained

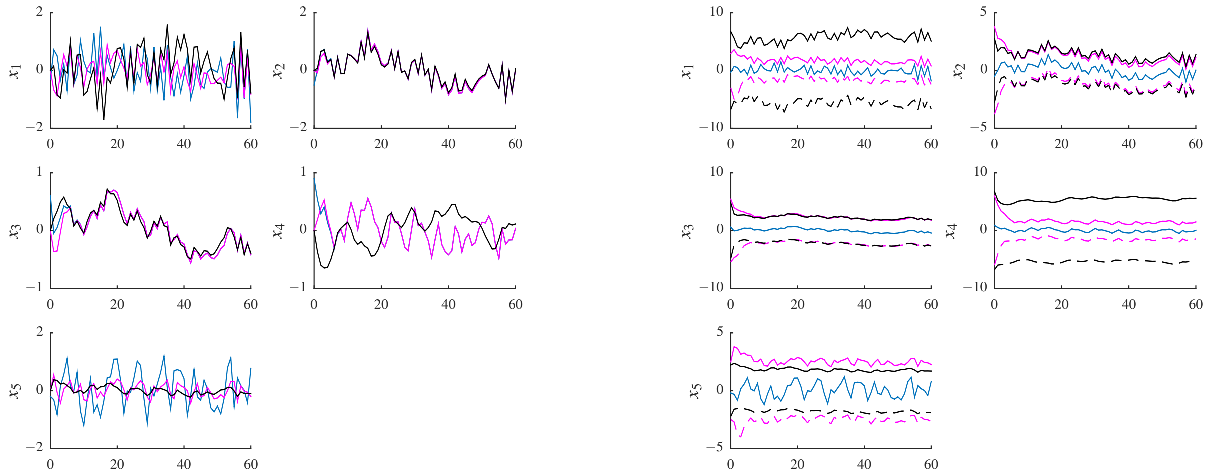

Figure 5: *Left*: in blue, simulation results for system (40) and in magenta (resp. black), for the discrete-time formulation of the classical observer in Eq. (8) obtained using the observer gain L produced by the control-based approach (resp. real pole placement). *Right*: bounds of the interval observer compared to the simulated state.

$$
M_d = \begin{bmatrix} 0.1552 & 0.0166 & 0.0000 & 3.9459 & 0.0000 \\ 0.0108 & 0.3773 & 0.1110 & 0.0035 & 0.0240 \\ 0.0000 & 0.1725 & 0.0529 & 0.0281 & 0.0050 \\ 0.0000 & 0.0761 & 0.0877 & 0.1050 & 0.0000 \\ 0.4027 & 0.0181 & 0.1069 & 0.0066 & 0.4067 \end{bmatrix},
$$

$$
P = \begin{bmatrix} 0.9952 & 0.4521 & 0.0923 & 0.2680 & -0.0638 \\ -0.3055 & 1.0187 & 0.1835 & 0.2143 & -0.3359 \\ -0.5536 & 0.0632 & 0.9009 & 0.1155 & -0.5773 \\ -0.0634 & 0.1535 & 0.1303 & 0.2667 & -0.0216 \\ 0.1970 & 0.2770 & 0.2393 & 0.1984 & 1.2526 \end{bmatrix}, \; L = \begin{bmatrix} -0.5400 & -0.9321 \\ 0.6300 & 0.1244 \\ -0.0900 & -0.4325 \\ 0.3536 & 0.3300 \\ -0.3536 & 0.4908 \end{bmatrix} \quad (47)
$$

where $M_d = P(A_d - LC_d)P^{-1}$ is Schur-stable and non-negative as required.

6.3.3 Simulations and comparisons

The disturbance d_t used for the simulation is randomly generated with $d_t \in [-1, 1]$. The simulation results are shown on Fig. 5.

The results obtained using the control-based approach are very satisfactory. The simultaneous synthesis of P and L leads to a more precise observer. An arbitrary choice of the poles is not required anymore since it is included in the optimization process.

7 Conclusions

In this article, a new approach to compute an appropriate change of coordinates for a system to be cooperative in these new coordinates has been introduced following preliminary results in [5]. Both the continuous- and discrete-time cases have been considered. An overview of the existing techniques to determine such transformation has been performed. The new approach has been applied on three examples and compared with the existing techniques.

The proposed method is based on structured controller synthesis techniques [6, 23]. It is used to enforce specific properties on the state-space matrices of a system. The main addition of the method in comparison with existing ones is the possibility to satisfy additional constraints dealing with the dynamics of the observer. Also, in the case of interval observers, matrices P and L are determined simultaneously while other methods require to choose a targeted Metzler matrix M with the same eigenvalues as $A - LC$, which is not an easy task from the authors point of view.

The current limitation of the method is the computation time which increases with the order of the system. Also, it uses algorithms which were not specifically designed for this task. The consequence is a loss in computational efficiency. As far as the theory of interval observers is concerned, the proposed method currently only applies to determining a time-invariant change of coordinates in the linear time-invariant case. A perspective of this work could be to extend the approach to the time-varying case for which theoretical results have been proposed [8].

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