Exact observer-based structures for arbitrary compensators

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Abstract

In this paper, some new techniques for determining the observer-based or LQG form of any compensator with arbitrary order are discussed. The practical appeal of such techniques is that they allow for a simplified implementation and reduced memory storage of general controllers and offer additional flexibility for handling gain-scheduling and input saturation constraints as compensator states become meaningful variables. The derived observer-based controllers are input-output equivalent to the original controller but with an explicit separated estimation/control structure. Such structures involve both static control and estimation gains with an extra Youla parameter that can be either static or dynamic. The proposed techniques are applicable both in continuous- and discrete-time, to full-order controllers, that is, controllers whose order is the same as the plant's order but also to augmented- and reduced-order controllers whose orders are greater or smaller, respectively. Necessary conditions to apply this general controller equivalence principle are derived.

The interest and practicality of such techniques are then investigated with regards to the LQG implementation of H_∞ and μ controllers, classes of controllers that does not generally enjoy ease of implementation.

Keywords : LQG compensator - observer-based controller, Luenberger observer, estimator-controller form, Youla parameterization, H_∞ control

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1 Introduction

LQG compensators are interesting for different practical reasons. Probably the key advantage of these controller structures lie in the fact that the controller states are meaningful variables as estimates of the physical plant states. It follows that the controller states can be used to monitor (on-line or off-line) the performance of the system. Note that this simple property does not hold for general controllers with state-space description ;

$$\left\{ \begin{array}{lcl} \dot{x}_K & = & A_K x_K + B_K y \\ u & = & C_K x_K + D_K y \,. \end{array} \right. \tag{1}$$

Another well-appreciated advantage comes from the ease of implementation of observer-based controllers. In addition to the plant data, only two static gains define the entire controller dynamics. In return, this facilitates the construction of gain-scheduled or interpolated controllers. Indeed, assuming the plant model is available in real-time, observer-based controllers will only require the storage of two static gains of lower dimensions instead of the huge set of numerical data in (1) to update the controller dynamics at each sample of time. Note that if we are using an interpolating procedure to update the controller dynamics, the general representation in (1) is highly questionable from an implementation viewpoint and in many cases will lead to an insuperable computational effort. This was in our opinion a major impediment for a widespread use of modern control techniques such as H_∞ and μ syntheses in realistic applications and particularly for problems necessitating real-time adjustment of the controller gains. An in-depth discussion of these questions is provided in [1] and references therein, together with the introduction of various techniques to overcome such difficulties. More theoretical discussions on the implementation of gain-scheduled controllers which exploit informations on plant non-linearities are given in [20] and [21].

Among other potential advantages of our method, we would like to point out the possibility to handle actuator saturation constraints by exploiting this information into the prediction equation. Since we do not cover this matter in this paper, the reader is referred to [19] and references therein for more details.

Two important contributions investigating the estimator/controller structure of any compensator are those of Schumacher in [5] and Bender and Fowell in [2, 3, 4]. In [5], Schumacher introduced the *deterministic separation principle* and showed in the abstract setting of the geometric theory that generically any compensator can be given an observer-based form. As a consequence, the differences between the general controllers (1) are essentially an artifact of the theory. In [2], Bender and Fowell developed simple and practical techniques for computing the estimator/controller form of arbitrary full-order compensators. Their technique consists in mapping the states of the compensator to those of the observer-based controller via a linear state-space transformation. The computation of the transformation matrix involves solving a generalized non-symmetric algebraic Riccati equation. Such Riccati equations have been previously introduced by Kokotovic in [9] in the context of singularly perturbed systems where the existence and uniqueness of the solution can be proved in some special situations. The reader is referred to [10] for more on this subject. In the general case, however, the solutions are not unique but correspond to a finite combinatoric associated with the different possible choices of n closed-loop eigenvalues among the set of $2n$ closed-loop eigenvalues, where n stands for the plant's order.

The contribution in this paper is as follows. In section 2, the results of Bender and Fowell are generalized to augmented order compensators and, under some conditions, to reduced order compensators. It is shown that the Q-parameterization of controllers and Luenberger observers formulation can be exploited to derive equivalent observer-based state-space representations with an explicitly separated structure. Again we would like to emphasize that such techniques are very general and encompass proper or non-strictly proper controllers and plants and also the case of discrete-time controllers. The latter case has been rejected in an appendix for simplicity of the presentation. Necessary conditions of intrinsic nature for the technique to be applicable are also discussed. In Section 3 the implications of these results for the implementation of H_∞ or μ controllers are discussed and illustrated by examples.

The notation used in the paper is standard. The notation $\mathcal{R}$ stands for the set of real number. Variables wearing a hat designate estimates. For instance $\widehat{x}$ denotes an estimate of the variable x. The notation $\text{spec}(M)$ is used to denote the spectrum of matrix M. The notation $A|S$ denotes the restriction of some map A to the linear subspace S. M^* is the conjugate-transpose of M. With a slight abuse, we shall sometimes use the term LQG structure in place of observer-based structure.

2 LQG form computation

In this section, we briefly recall the central ideas behind the Youla parameterization and show how it can be used to find the state estimator-state feedback structure of an arbitrary compensator associated with a given plant.

The plant assumed strictly proper without loss of generality is defined as :

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu, \\ y & = & Cx \end{array} \right. \tag{2}$$

where $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, and $C \in \mathcal{R}^{p \times n}$. The so-called Youla parameterization of all stabilizing compensators built on the LQG form associated with the plant is depicted in figure 1, where K_c, K_f and $Q(s)$ are respectively the state feedback gain, the state estimator gain and the Youla parameter. The compensator associated with this structure is easily shown to have the following state-space description :

$$\left\{ \begin{array}{rcl} \dot{\widehat{x}} & = & A\widehat{x} + Bu + K_f(y - C\widehat{x}) \\ \dot{x_Q} & = & A_Q x_Q + B_Q(y - C\widehat{x}) \\ u & = & -K_c\widehat{x} + C_Q x_Q + D_Q(y - C\widehat{x}) \end{array} \right. \tag{3}$$

where A_Q, B_Q, C_Q and D_Q are the 4 matrices of the state-space representation of $Q(s)$ associated with the state variable x_Q. Hereafter, $\widehat{x}$ denotes an estimate of the plant state x. The Youla parameterization principle is based on the fact that the closed-loop transfer function between the input e and the innovation $\varepsilon_y = y - C\widehat{x}$ is null (see [6] for instance). As a consequence, changing $Q(s)$ leads to various compensators but the closed-loop transfer function remains unaffected. It is readily shown that this closed-loop transfer function can

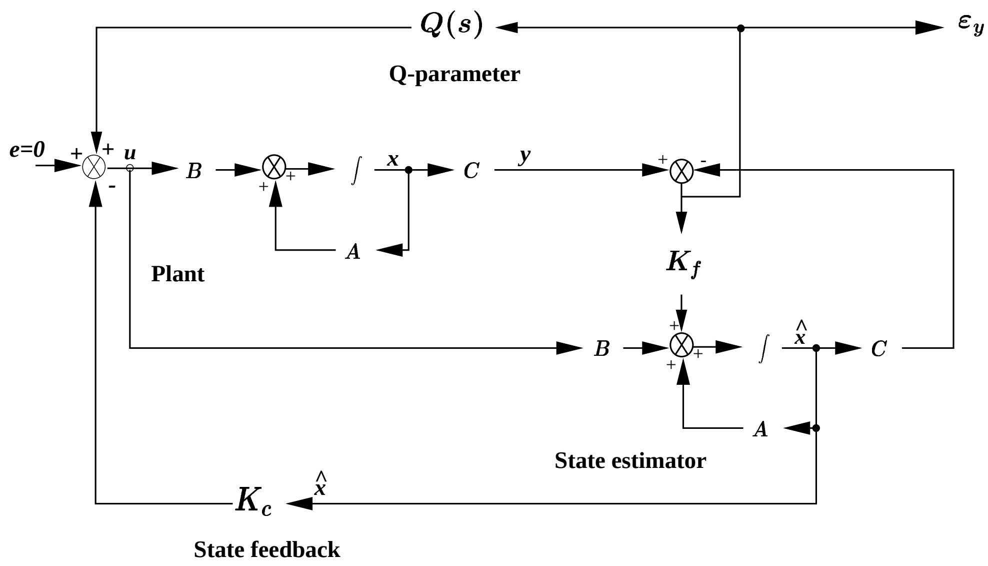

Figure 1: LQG structure and Youla parameterization

be represented by the state-space form (4) involving the estimation error $\varepsilon_x = x - \widehat{x}$:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x_Q} \\ \dot{\varepsilon_x} \end{bmatrix} &= \begin{bmatrix} A - BK_c & BC_Q & BK_c + BD_QC \\ 0 & A_Q & B_QC \\ 0 & 0 & A - K_fC \end{bmatrix} \begin{bmatrix} x \\ x_Q \\ \varepsilon_x \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} e \\ \varepsilon_y &= \begin{bmatrix} 0 & 0 & C \end{bmatrix} \begin{bmatrix} x \\ x_Q \\ \varepsilon_x \end{bmatrix} \end{aligned} \tag{4}$$

From this representation, the separation principle appears clearly and can be stated in the following terms :

- the closed-loop eigenvalues can be separated into n closed-loop state-feedback poles $(\mathrm{spec}(A - BK_c))$, n closed-loop state-estimator poles $(\mathrm{spec}(A - K_fC))$ and the Youla parameter poles $(\mathrm{spec}(A_Q))$,
- the closed-loop state-estimator poles and the Youla parameter poles are uncontrollable by e,
- the closed-loop state-feedback poles and the Youla parameter poles are unobservable from ε_y. The transfer function from e to ε_y always vanishes.

Now let us consider a given n_Kth-order compensator defined by the following state space representation :

$$\left\{ \begin{array}{rcll} \dot{x_K} &=& A_Kx_K + B_Ky & \text{(a)} \\ u &=& C_Kx_K + D_Ky & \text{(b)} \end{array} \right. \tag{5}$$

We are first going to express the compensator state equation (5.a) as an Luenberger observer of the variable $z = Tx$. So, we will denote:

$$x_K = \widehat{z} \tag{6}$$

According to Luenberger's formulation [6], this problem can be stated as the search of

$$T \in \mathcal{R}^{n_K \times n},\ F \in \mathcal{R}^{n_K \times n_K},\ G \in \mathcal{R}^{n_K \times p}$$

such that

$$\dot{\widehat{z}} = F\widehat{z} + Gy + TBu \tag{7}$$

is an (asymptotic) observer of the variable z, that is $z - \widehat{z}$ vanishes as t goes to infinity. Luenberger has shown that the constraints :

$$TA - FT = GC, \quad \text{and } F \text{ stable }, \tag{8}$$

ensure that this holds true. Then, with the output equation (5.b), the state space representation of the compensator reads :

$$\left\{ \begin{array}{lcl} \dot{\widehat{z}} & = & (F + TBC_K)\widehat{z} + (G + TBD_K)y \\ u & = & C_K\widehat{z} + D_K y \end{array} \right. \tag{9}$$

The separation principle is still true and one can easily shown from (2) and (9) that the closed-loop dynamic can be expressed as :

$$\left[\begin{array}{c} \dot{x} \\ \dot{\varepsilon}_z \end{array} \right] = \left[\begin{array}{cc} A + B(D_K C + C_K T) & -BC_K \\ 0 & F \end{array} \right] \left[\begin{array}{c} x \\ \varepsilon_z \end{array} \right] \tag{10}$$

where $\varepsilon_z = Tx - \widehat{z}$. Note that the stability of F is secured whenever the original controller (5) is stabilizing.

With (6), the identification of (9) and (5) leads to the algebraic relations:

$$F = A_K - TBC_K \tag{11}$$

$$G = B_K - TBD_K \tag{12}$$

These equations with (8) guarantee that we are dealing with an observer-based controller. Substituting (11) and (12) in the first relation in (8), we get :

$$A_K T - T(A + BD_K C) - TBC_K T + B_K C = 0 \tag{13}$$

So, the problem is reduced to solving in T the generalized non-symmetric and rectangular Riccati equation (13) and next to compute F and G using (11) and (12) respectively.
Equation (13) can also be reformulated as :

$$[-T \ \ I] \left[\begin{array}{cc} A + BD_K C & BC_K \\ B_K C & A_K \end{array} \right] \left[\begin{array}{c} I \\ T \end{array} \right] = 0 \tag{14}$$

Therefore, the Hamiltonian matrix associated with the Riccati equation (13) is nothing else than the closed-loop system matrix :

$$A_{cl} := \left[\begin{array}{cc} A + BD_K C & BC_K \\ B_K C & A_K \end{array} \right] . \tag{15}$$

The Riccati equation (13) can then be solved by standard invariant subspace techniques which consist in :

- finding a n-dimensional invariant subspace $\mathcal{S} := Range(U)$ of the closed-loop system matrix A_{cl}, that is,
$$A_{cl}U = U\Lambda \tag{16}$$
This subspace is associated with a set of n eigenvalues, spec(Λ), among the $n+n_K$ eigenvalues of A_{cl}. Such subspaces are easily computed using Schur factorizations of the matrix A_{cl}. See [13] for more details.

- partitioning the vectors U which span this subspace conformably to the partitioning in (15).
$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad U_1 \in \mathcal{R}^{n\times n}. \tag{17}$$

- computing the solution : $T = U_2 U_1^{-1}$.

Narasimhamurthi and Wu have shown in [10] that the existence of a solution T satisfying (13) is guaranteed whenever the eigenvalues of the Hamiltonian matrix A_{cl} are distinct. In proposition 2.2, a necessary condition is given for the existence of a solution T. In the general case, however, there are finitely many admissible subspaces $\mathcal{S}$ and thus many solutions. Each solution corresponds to a particular choice of n eigenvalues among the set of closed-loop eigenvalues of A_{cl}.

Then, given a nth-order plant and a n_Kth-order compensator, one can compute the linear combination $T_{n_K\times n}x$ of the plant states which is estimated by the compensator state. An analogous result is also discussed by Fowell and *al* in [3].

2.1 Augmented-order compensators

In this section, we consider the problem where $n_K > n$ and our aim is to find a state-feedback gain K_c, a state-estimator gain K_f and a dynamic Youla parameter $Q(s)$ with order $n_K - n$, such that the observer-based compensator structure in figure 1 is equivalent to the original controller (5). We will assume that T has been computed by the previous technique according to an admissible choice of n poles among the $n+n_K$ closed-loop poles. Next, F and G can be computed from (11) and (12). Now consider a Schur decomposition of the matrix F and a partition of the resulting Schur matrix as a 2 by 2 matrix with block sizes $n_K - n$ and n.

$$F = V\widetilde{F}V^* = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \widetilde{F_{11}} & \widetilde{F_{12}} \\ 0 & \widetilde{F_{22}} \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} \tag{18}$$

with $VV^* = I_{n_K\times n_K}$, $\widetilde{F_{11}} \in \mathcal{R}^{n_K-n\times n_K-n}$ and $\widetilde{F_{22}} \in \mathcal{R}^{n\times n}$.

Let us perform the change of variable :

$$\widehat{z} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{19}$$

in equations (7) and (8) and introduce the notations :

$$\begin{bmatrix} \widetilde{G_1} \\ \widetilde{G_2} \end{bmatrix} = \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} G; \begin{bmatrix} \widetilde{T}_1 \\ \widetilde{T}_2 \end{bmatrix} = \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} T. \tag{20}$$

Equations (7) and (8) then become :

$$\left\{\begin{array}{lcllll} \dot{w}_1 & = & \widetilde{F_{11}}w_1 + \widetilde{F_{12}}w_2 & +\widetilde{G_1}y & +\widetilde{T_1}Bu & \text{(a)} \\ \dot{w}_2 & = & \widetilde{F_{22}}w_2 & +\widetilde{G_2}y & +\widetilde{T_2}Bu & \text{(b)} \end{array}\right. \tag{21}$$

and

$$\left\{\begin{array}{llcll} \widetilde{T_1}A & -\widetilde{F_{11}}\widetilde{T_1} - \widetilde{F_{12}}\widetilde{T_2} & = & \widetilde{G_1}C & \text{(a)} \\ \widetilde{T_2}A & -\widetilde{F_{22}}\widetilde{T_2} & = & \widetilde{G_2}C & \text{(b)} \end{array}\right. \tag{22}$$

Now, we will assume that the Schur decomposition has been performed in such a way that $\widetilde{T_2} = V_2^*T$ is non singular (in proposition 2.3, a necessary condition for T to be full column rank is given) and we perform the second change of variable :

$$w_2 = \widetilde{T_2}\widehat{x} \tag{23}$$

From equations (21.b) and (22.b), one can derive :

$$\dot{\widehat{x}} = A\widehat{x} + Bu + \widetilde{T_2}^{-1}\widetilde{G_2}(y - C\widehat{x}) \tag{24}$$

Using now (22.a) and (23) to substitute $\widetilde{F_{12}}w_2$ into equation (21.a), we get :

$$\dot{w}_1 = \widetilde{F_{11}}(w_1 - \widetilde{T_1}\widehat{x}) + \widetilde{G_1}(y - C\widehat{x}) + \widetilde{T_1}(A\widehat{x} + Bu) \tag{25}$$

Pre-multiplying equation (24) by $\widetilde{T_1}$, subtracting it from equation (25) and using the last change of variable :

$$w_1 - \widetilde{T_1}\widehat{x} = x_Q \tag{26}$$

we obtain :

$$\dot{x_Q} = \widetilde{F_{11}}x_Q + (\widetilde{G_1} - \widetilde{T_1}\widetilde{T_2}^{-1}\widetilde{G_2})(y - C\widehat{x}) \tag{27}$$

From (6), (19), (23) and (26), one can easily derive the global linear transformation between the compensator original state x_K and the new states $\widehat{x}$ and x_Q :

$$x_K = \widehat{z} = [V_1 \; T]\begin{bmatrix} x_Q \\ \widehat{x} \end{bmatrix} \tag{28}$$

Then, the compensator output equation (5.b) can be expressed as ;

$$u = C_KT\widehat{x} + C_KV_1x_Q + D_Ky \tag{29}$$

or :

$$u = (C_KT + D_KC)\widehat{x} + C_KV_1x_Q + D_K(y - C\widehat{x}) \tag{30}$$

The identification of the set of equations (24), (27) and (30) with equation (3) provides all the parameters for the observer-based controller structure shown in figure 1 :

$$K_f = \widetilde{T_2}^{-1}\widetilde{G_2} = (V_2^*T)^{-1}V_2^*G \tag{31}$$

$$K_c = -C_KT - D_KC \tag{32}$$

$$A_Q = \widetilde{F_{11}} = V_1^*FV_1 \tag{33}$$

$$B_Q = \widetilde{G_1} - \widetilde{T_1}\widetilde{T_2}^{-1}\widetilde{G_2} = V_1^*[I_{n_K \times n_K} - T(V_2^*T)^{-1}V_2^*]G \tag{34}$$

$$C_Q = C_KV_1 \tag{35}$$

$$D_Q = D_K \tag{36}$$

In brief, the procedure to compute the observer-based form and the dynamic Youla parameter of a given n_Kth-order compensator associated with a nth-order plant ($n_K > n$) can be summarized as follows:

- compute the closed-loop matrix A_{cl} (equation (15)) and choose n eigenvalues in spec(A_{cl}) which will be assigned to the closed-loop state-feedback poles (see proposition 2.1),
- solve in T the non-symmetric Riccati equation (13) and compute F and G with the help of (11) and (12),
- select a partition of n eigenvalues (which will be assigned to the closed-loop state-estimator poles) and $n_K - n$ eigenvalues (which will be assigned to Q-parameter poles) in spec(F) and compute matrices V_1 and V_2 from a Schur decomposition of F according to this partition (equation (18)),
- compute the sought parameters K_c, K_f, A_Q, B_Q, C_Q and D_Q using (31)-(36).

2.2 Discussion

There is a combinatoric of solutions according to the choice of the partition of the closed-loop eigenvalues, first, in the computation of matrix T, and secondly, in the Schur decomposition of matrix F. Hereafter some rules are proposed to reduce the number of admissible choices. These rules are the extension of remarks previously stated by Bender and Fowell [2], in the full-order case but we provide here a simple proof as they will be the key for the application of this method to the standard control problem presented in section 3.

Proposition 2.1 *The n eigenvalues chosen for the computation of the solution T of the Riccati equation (13) using the Hamiltonian approach are the n eigenvalues of the closed-loop state feedback associated with the equivalent LQG-compensator, i.e, spec($A - BK_c$).*

Proof : From (15), (16) and (17), we have :

$$\begin{bmatrix} A + BD_KC & BC_K \\ B_KC & A_K \end{bmatrix} \begin{bmatrix} I_{n\times n} \\ T \end{bmatrix} = \begin{bmatrix} I_{n\times n} \\ T \end{bmatrix} U_1 \Lambda U_1^{-1} \tag{37}$$

the first row of this matrix equality reads :

$$A + B(D_KC + C_KT) = U_1 \Lambda U_1^{-1} \tag{38}$$

using (32), we have :

$$A - BK_c = U_1 \Lambda U_1^{-1} \tag{39}$$

So, the eigenvalues of Λ are the eigenvalues of $A - BK_c$. As a consequence, the n_K remaining eigenvalues are the Luenberger observer poles (i.e. spec(F), see also equation (10)), which are shared, in the Schur decomposition (18), between the $n_K - n$ Youla parameter poles (i.e. spec(A_Q)) and the n closed-loop state estimator poles (i.e. spec($A - K_fC$)). □

Hereafter, we are considering the set of equations (from (15), (16) and (17)) :

$$\begin{bmatrix} A+BD_KC & BC_K \\ B_KC & A_K \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \Lambda \tag{40}$$

and we shall give a necessary condition, on the choice of the subspace $\mathcal{S}$, for the existence of a solution T (that is, for U_1 to be invertible).

Proposition 2.2 *Consider U_1 and U_2 associated with some n-dimensional invariant subspace $\mathcal{S}$ of A_{cl}. Assume there is some uncontrollable plant eigenvalue which is not in spec($A_{cl}|S$) then U_1 is singular. In other words,*

$$if\ \exists\, \lambda\ \notin\ spec(\Lambda)\ s.\ t.\ \lambda\ is\ (A,B)\ uncontrollable,\ then\ U_1\ is\ singular \tag{41}$$

Proof : Consider the (A,B)-pair and let λ denote an uncontrollable eigenvalue with associated left-eigenvector u. That is,

$$u^T[A-\lambda I \mid B] = 0 \tag{42}$$

then, pre-multiplying (40) by $[u^T\ 0]$, we get :

$$u^T[(A+BD_KC)U_1 + BC_KU_2] = u^TU_1\Lambda \tag{43}$$

From (42) and (43) it follows that :

$$u^TU_1(\Lambda - \lambda I) = 0 \tag{44}$$

So, if $\lambda \notin$ spec(Λ) then $u^TU_1 = 0$, that is U_1 is singular. □

We also have a dual property which concerns the column rank of T (that is, for U_2 to be full column rank). It can be stated as follows.

Proposition 2.3 *Consider U_1 and U_2 associated with some n-dimensional invariant subspace $\mathcal{S}$ of A_{cl}. Assume there is some unobservable plant eigenvalue in spec($A_{cl}|S$), then U_2 is column rank deficient. In other words,*

$$if\ \exists\, \lambda\ \in\ spec(\Lambda)\ s.\ t.\ \lambda\ is\ (A,C)\ unobservable,\ then\ U_2\ is\ column\ rank\ deficient. \tag{45}$$

Proof : Omitted for brevity. See proposition 2.2. □

Remark 2.4 If $n_K = n$, then T is square and the Schur decomposition (18) of F is such that $V_2 = I_{n\times n}$ and V_1 is empty. Then equations (31)-(36) become :

$$\begin{aligned} K_f &= T^{-1}G = T^{-1}B_K - BD_K &(46)\\ K_c &= -C_KT - D_KC &(47)\\ Q(s) &= D_Q = D_K &(48) \end{aligned}$$

Our results then specialize to those of [2].

Remark 2.5 Among all the admissible choices, the only restriction which can help us is that complex conjugate pairs of poles cannot be separated if we are seeking state-space representations with real coefficients. Note that such a choice is not always possible. For instance, consider the plant $P(s) = 1/s$ and the compensator $2/(s+2)$, then the computation of the state feedback-state estimator form leads to $Q = 0$, $K_c = 1 + i$ (or $1 - i$) and $K_f = 1 - i$ (resp. $1 + i$). Although the gains K_c and K_f are complex, the transfer function of the controller has real coefficients. The following selection rules have proved also useful in practical applications of the method:

- affect the fastest poles to spec(A_Q) in such a way that the Youla parameter acts as a direct feedthrough in the compensator,
- assign to spec$(A - BK_c)$ the n closed-loop poles which are the "nearest" from the n plant poles in order to respect the dynamic behavior of the physical plant and reduce the state-feedback gains,
- assign fast closed-loop poles to spec$(A - K_fC)$ to have an efficient state estimator.

Remark 2.6 If the plant has a direct feedthrough matrix D, then this technique should be applied to the strictly proper plant model (A, B, C) between the input u and the fictitious output $\bar{y} = y - Du$. A valid compensator for the original plant is then easily derived with the representation:

$$\left\{ \begin{array}{rcl} \dot{x_K} & = & (A_K + B_K(I - DD_K)^{-1}DC_K)x_K + B_K(I - DD_K)^{-1}\bar{y} \\ u & = & (I - D_KD)^{-1}C_Kx_K + (I - D_KD)^{-1}D_K\bar{y} \end{array} \right. \tag{49}$$

Then, the parameters of the LQG-form (i.e. : K_c, K_f, A_Q, B_Q, C_Q and D_Q) computed from these new compensator and plant must be applied to the equation (3) with y replaced with $\bar{y} = y - Du$.

2.3 Reduced-order compensators case

In the case $n_K < n$ (i.e. $dim(z) < dim(x)$), the LQG structure shown in figure (1) is no longer valid. But we can find an interesting alternative in building a reduced-order estimator.

It is interesting to point out the case where $[T^T\ C^T]$ is a rank n matrix (i.e. $p + n_K \geq n$) then, we can obtain a reduced observer-based representation involving an estimate $\widehat{x}$ of the plant state x by a linear function of the compensator state $\widehat{z}$ and the plant output y (see Luenberger [6] and Newmann [14]) :

$$\widehat{x} = H_1\widehat{z} + H_2y \tag{50}$$

with the constraint :

$$H_1T + H_2C = I_n \tag{51}$$

Then, the separation principle holds and a Youla parameterization (with a static parameter D_Q) built on such a reduced-order estimator reads :

$$\left\{\begin{array}{lcll} \dot{\widehat{z}} & = & F\widehat{z}+Gy+TBu & \text{(a)} \\ \widehat{x} & = & H_1\widehat{z}+H_2y & \text{(b)} \\ u & = & -K_c\widehat{x}+D_Q(y-C\widehat{x}) & \text{(c)} \end{array}\right. \tag{52}$$

$$\left\{\begin{array}{lcl} TA-FT & = & GC \\ H_1T+H_2C & = & I_n \end{array}\right. \tag{53}$$

As well as in the LQG-form compensator, it can be easily shown that the closed-loop poles, with a compensator defined by equations (52) and (53), are distributed between the closed-loop state-feedback poles (spec($A-BK_c$)) and the estimator poles (spec(F)). Equations (13), (11) and (12) which respectively provide T, F and G are still valid. The problem is therefore reduced to computing K_c, H_1, H_2 and D_Q such that (from the identification of (52.b) and (52.c) with (5.b)) :

$$\left\{\begin{array}{ll} C_K=-(K_c+D_QC)H_1 & (a) \\ D_K=-(K_c+D_QC)H_2+D_Q & (b) \\ H_1T+H_2C=I & (c) \end{array}\right. \tag{54}$$

It is easily deduced that :

$$K_c=-C_KT-D_KC \tag{55}$$

This is the same equation as (32), established in the augmented-order compensator case.

To compute H_1, H_2 and D_Q, we distinguish the following situations :

- if $\begin{bmatrix} T \\ C \end{bmatrix}^{-1}$ exists (which implies that $n_K+p=n$) then :

  $$[H_1\ H_2]=\begin{bmatrix} T \\ C \end{bmatrix}^{-1} \tag{56}$$

  and

  $$\begin{bmatrix} T \\ C \end{bmatrix}[H_1\ H_2]=\begin{bmatrix} TH_1 & TH_2 \\ CH_1 & CH_2 \end{bmatrix}=\begin{bmatrix} I_{n_K} & 0 \\ 0 & I_p \end{bmatrix} \tag{57}$$

  Hence, relationships (54) are satisfied for any D_Q and we can choose $D_Q=0$ without loss of generality.

- if $n_K>n-p$, then there are several solutions (H_1, H_2) satisfying (54.c), one can choose for example the least norm solution (in order to reduce the control gains) using the pseudo-inverse of matrix $[T^T\ C^T]$:

  $$\left\{\begin{array}{lcl} H_1 & = & [T^TT+C^TC]^{-1}T^T \\ H_2 & = & [T^TT+C^TC]^{-1}C^T \end{array}\right. \tag{58}$$

  Then, D_Q must be found by solving equations (54.a) and (54.b).

If $n_K < n - p$, we can only mention that, in open-loop, the compensator state $\widehat{z}$ is an estimate of the linear function T of the plant state x, that is, the estimation error $\varepsilon_z = Tx - \widehat{z}$ tends to 0 with the following dynamic behavior :

$$\dot{\varepsilon_z} = (A_K - TBC_K)\varepsilon_z \tag{59}$$

This property is lost in closed-loop as the separation principle is no more satisfied.

In this case ($n_K < n - p$), the only way round consists in performing a reduction of the plant until the previous technique is applicable. The compensator is then interpreted as an observer-based compensator associated to the reduced plant.

3 Application to the standard control problem

3.1 General standard problem

The results established in the previous sections can be exploited to construct equivalent observer-based state-space representations of controllers of arbitrary order. As a particular application, one can examine the observer-based representations of controllers designed using modern robust control techniques such as H_∞ and μ syntheses. Except in special circumstances [15], such controllers does not enjoy such a structure as this is the case for LQG or H_2 controllers. We show in this section that the distinction between these classes is only formal. We note first that H_∞ and μ syntheses generally provide high-order compensators as compared to the plant's order due to the introduction of frequency weightings or scalings [17, 18]. Now, we know from section 2 how to reformulate such a compensator as an observer-based compensator associated with the augmented plant whose states incorporate the physical plant's states but also additional states resulting from the weights and possibly dynamic scalings.

Consider the general standard form interconnection in figure 2 where :

- $G(s)$ is the nominal plant model,
- $W_{in}(s)$ and $W_{out}(s)$ are respectively the input and the output frequency weightings (or might be scalings),
- e, u, z and y are the exogenous input, the control command, the controlled output and the measurement of the augmented plant $P(s)$, respectively
- $K(s)$ denotes the compensator transfer function.

The state-space representation of the augmented plant $P(s)$ is given as :

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ e \\ u \end{bmatrix} . \tag{60}$$

We assume here that the compensator and the augmented plant have the same order. Then, the technique is directly applicable to the augmented plant (A, B_2, C_2, D_{22}) to obtain

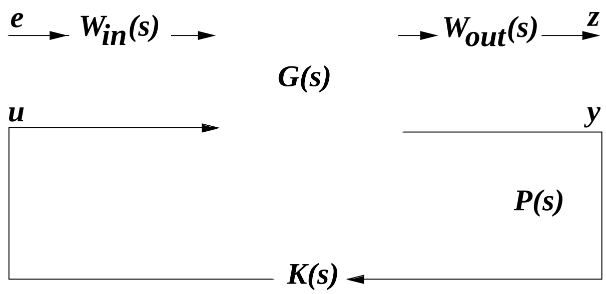

Figure 2: General standard problem

an observer-based representation for $K(s)$ where the state x of the augmented plant is estimated. In other words, the estimator will estimate the states of the model $G(s)$ and the states of the frequency weightings $W_{in}(s)$ and $W_{out}(s)$ and possibly the scalings. As proved earlier, we have to solve a generalized non-symmetric Riccati equation using invariant subspace techniques. This computation must satisfy the constraints derived in propositions 2.2 and 2.3, that is,

- the poles of $W_{in}(s)$ are uncontrollable with respect to the pair $(A,\ B_2)$. Thus, they must be selected in the choice of the invariant subspace,
- the poles of $W_{out}(s)$ are unobservable with respect to the pair $(A,\ C_2)$. Thus, they must be left out in the choice of the invariant subspace.

Alternatively, one can also use the more general results of section 2.1 where the extra dynamic of the augmented plant, which in most cases correspond to weighting functions or scalings, is reflected in a dynamic Youla parameter. In such a case, the order of $Q(s)$ will be the sum of the orders of $W_{in}(s)$ and $W_{out}(s)$ and possibly the scalings. This is illustrated in the example below.

3.2 Example

This example is borrowed from the second demonstration example of the Mu-Analysis and Synthesis Toolbox [16] and can be easily found in the SIMULINK extra library. The problem is the control of an open-loop unstable system $G(s)$ defined as :

$$\left\{ \begin{array}{lcl} \dot{x_G} & = & A_G x_G + B_G u \\ y & = & C_G x_G \end{array} \right. \tag{61}$$

where :

$$A_G = \begin{bmatrix} -36.6 & -18.923 \\ -1.9 & 0.983 \end{bmatrix};\ B_G = \begin{bmatrix} -0.4140 \\ -77.8 \end{bmatrix};\ C_G = \begin{bmatrix} 0 & 57.3 \end{bmatrix}. \tag{62}$$

The synthesis interconnection structure for this problem is depicted in figure 3. Let us denote

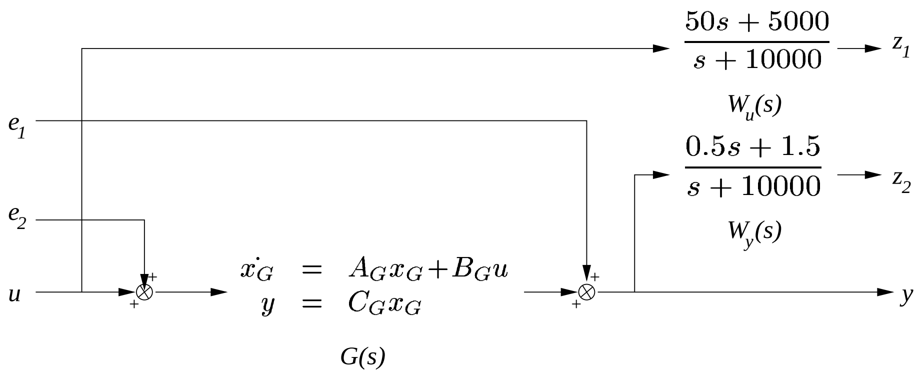

Figure 3: Augmented synthesis model

x_u and x_y, the states associated with the frequency weightings $W_u(s)$ and $W_y(s)$, respectively. The state-space representation (see equation (60)) of the augmented plant, with state vector $x = [x_G,\, x_u,\, x_y]^T$ is thus given as :

$$\begin{array}{l} A = \begin{bmatrix} -36.6 & -18.923 & 0 & 0 \\ -1.9 & 0.983 & 0 & 0 \\ 0 & 0 & -10000 & 0 \\ 0 & -4051.1 & 0 & -10000 \end{bmatrix};\ B_1 = \begin{bmatrix} -0.4140 & 0 \\ -77.8 & 0 \\ 0 & 0 \\ 0 & -70.7 \end{bmatrix}; \\ B_2 = \begin{bmatrix} -0.4140 \\ -77.8 \\ -703.56 \\ 0 \end{bmatrix};\ C_1 = \begin{bmatrix} 0 & 0 & 703.56 & 0 \\ 0 & 28.65 & 0 & 70.7 \end{bmatrix}; \\ C_2 = \begin{bmatrix} 0 & 57.3 & 0 & 0 \end{bmatrix};\ D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}; \\ D_{12} = \begin{bmatrix} 50 \\ 0 \end{bmatrix};\ D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix};\ D_{22} = 0 \end{array} \tag{63}$$

Performing an H_∞ synthesis on this problem yields a compensator described as :

$$\begin{array}{l} A_K = \begin{bmatrix} -3.6600\,10^{1} & -2.3679\,10^{1} & 5.8158 & -3.0532\,10^{-3} \\ -1.8941 & -4.4582\,10^{3} & 1.0929\,10^{3} & -5.7377\,10^{-1} \\ 5.3798\,10^{-2} & -2.7291 & -1.1655\,10^{2} & -5.1887 \\ 0 & 0 & 0 & -1.0000\,10^{4} \end{bmatrix}; \\ B_K = \begin{bmatrix} 2.4703\,10^{-1} & 2.3171\,10^{2} & 0 & -2.1051\,10^{2} \end{bmatrix}^T; \\ C_K = \begin{bmatrix} -2.5680\,10^{-5} & 1.3028\,10^{-3} & -4.7179 & 2.4768\,10^{-3} \end{bmatrix}; \\ D_K = 0 \end{array} \tag{64}$$

with the notations in (5).

3.2.1 Observer-based structure with static Youla parameter

This compensator can be viewed as an observer-based compensator on the whole state of the augmented plant. The transfer between the control input and the control output is then

poles	$\in$ spec$(A - B_2K_c)$	$\in$ spec$(A - K_fC_2)$
-4.3817	*	
-36.590		*
-37.332	*	
-110.76		*
-4457.9	*	
-10000.	*	
-10000.		*
-10000.		*

Table 1: Closed-loop poles distribution - case 1

defined by A, B_2, C_2 and D_{22}. The closed-loop poles are presented on table 1 together with the distribution we have done between the state-feedback poles (spec$(A - B_2K_c)$) and the state-estimator poles (spec$(A-K_fC_2)$). To satisfy the necessary condition in proposition 2.3, the (A, C_2) unobservable poles (-10000 and -10000) associated with the output frequency weightings are affected to the state-estimator poles. As the closed-loop has 3 poles located at $-10000\,rd/s$, it is necessary to examine the eigenvectors associated with these eigenvalues to determine those which are unobservable.

Then, the solution T of the Riccati equation :

$$A_KT - T(A + B_2D_KC_2) - TB_2C_KT + B_KC_2 = 0 \tag{65}$$

is obtained as :

$$T = \begin{bmatrix} 2.9629 & 2.0183 & 2.5178\,10^{1} & 4.6368 \\ -2.7297 & 3.7987\,10^{2} & 4.7017\,10^{3} & 8.6587\,10^{2} \\ -2.0830\,10^{-4} & -1.0923 & 3.0984 & -2.6966 \\ -4.7703\,10^{-1} & -2.5015\,10^{3} & 2.7662\,10^{2} & -6.1725\,10^{3} \end{bmatrix}; \tag{66}$$

and the three parameters K_c, K_f and Q of the observer-based structure (see figure 1) are given by :

$$\begin{array}{l} K_c = [3.8310\,10^{-3},\ \ 5.4749\,10^{-1},\ \ 7.8079,\ \ 1.4379]; \\ K_f = [-3.2331\,10^{-1},\ \ 1.9499,\ \ 3.0543\,10^{-2},\ \ -7.5474\,10^{-1}]^T; \\ Q = 0 \end{array} \tag{67}$$

As discussed before, this new representation gives a clear meaning to the controller states as estimates of the states x_G, x_u and x_y whereas the meaning of the original states in (64) is obscure.

3.2.2 Observer-based structure with dynamic Youla parameter

The compensator (64) can also be interpreted as an observer-based compensator (with a second-order Youla parameter) on the plant nominal model (A_G, B_G, C_G), that is based on the minimal realization of the previous transfer (A, B_2, C_2, D_{22}) from the control input to the measurement. The distribution of the closed-loop poles has been chosen according

poles	$\in$ spec$(A_G - B_G K_c)$	$\in$ spec$(A_G - K_f C_G)$	$\in$ spec(A_Q)
-4.3817	*		
-36.590		*	
-37.332	*		
-110.76		*	
-4457.9			*
-10000.			*

Table 2: Closed-loop poles distribution - case 2

to table 2 in that case. Similar computations yield the following numerical values for the parameters in figure 1.

$$K_c = [8.3640\,10^{-3}, \;\; -7.8403\,10^{-2}], \qquad K_f = [-3.2331\,10^{-1}, \;\; 1.9499]^T$$

and

$$Q(s) = \frac{-3.7514\,10^{-1}\,s \; - \; 3.7514\,10^{3}}{s^2 \; + \; 1.4458\,10^{3}\,s \; + \; 4.4579\,10^{7}}$$

The transformation matrix between the representation (64) and the representation associated with the new state vector $[\widehat{x_G}, \; x_Q]^T$ is given by :

$$M = \begin{bmatrix} 2.9776 & 9.1926\,10^{-14} & 5.3551\,10^{-3} & -1.5243\,10^{-7} \\ 2.2204\,10^{-15} & 2.9776 & 9.9999\,10^{-1} & -2.8464\,10^{-5} \\ 1.7566\,10^{-3} & -1.6430\,10^{-2} & 6.2810\,10^{-4} & 5.2497\,10^{-4} \\ -1.5132\,10^{-4} & -1.2069 & 2.8134\,10^{-5} & 1.0000 \end{bmatrix}. \tag{68}$$

4 Conclusions

In this paper, we have completed some previous work performed by Bender and Fowell on the computation of observer-based structures for general compensators. The Youla parameterization has been used to generalize the technique to arbitrary-order compensators while maintaining the validity of the separation principle. This technique is based upon the resolution of a generalized non-symmetric Riccati equation. Necessary conditions were given for the solvability of this equation in terms of observability and controllability properties of the plant. These results have then been specialized to H_∞ or μ controllers which are issued from an augmented synthesis structure.

Further work is still needed to exploit the multiplicity of choices in the distribution of the closed-loop poles between the closed-loop state-feedback poles, the closed-loop state-estimator poles and the Youla parameter poles. This problem is particularly important to smoothly interpolate or schedule a family of state-feedback gains and state-estimator gains for practical problems requiring some gain-scheduling strategy. The usefulness of these controller structures to handle input saturation constraints is also deserves investigation.

Aknowledgement

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A Discrete-time case

The technique presented in the continuous-time case is now extended to the discrete-time case. Bender and Fowell discussed also this issue using different controller structures in [2]. They have investigated three kinds of sampled data compensators according to the presence of a direct feedthrough term in the plant or the compensator. From our point of view, there is no reason to distinguish theses cases in the context of the Q parameterization considered here. The case of a non-strictly proper plant can be handled as in continuous-time. Therefore, we shall only focus on two classical implementation structures of discrete-time LQG controllers: the predictor and the estimator structures.

A.1 Discrete-time predictor LQG form

The discrete-time plant is defined as :

$$\left\{ \begin{array}{lcl} x(k+1) & = & Ax(k) + Bu(k) \\ y(k) & = & Cx(k) \end{array} \right. \tag{69}$$

The predictor LQG form is described by :

$$\left\{ \begin{array}{lcll} \widehat{x}(k/k) & = & A\widehat{x}(k/k-1) + Bu(k) & \text{Prediction} \\ \widehat{x}(k+1/k) & = & \widehat{x}(k/k) + K_f(y(k) - C\widehat{x}(k/k-1)) & \text{Correction} \\ u(k+1) & = & -K_c\widehat{x}(k+1/k) & \text{Control} \end{array} \right. \tag{70}$$

This case is analogous to the continuous-time one. The construction procedure is therefore the same. It provides the parameters K_c, K_f, A_Q, B_Q, C_Q and D_Q of the Youla-parameterization associated with the predictor LQG form whose state-space representation reads :

$$\left\{\begin{array}{lcl} \widehat{x}(k+1/k) & = & A\widehat{x}(k/k-1) + Bu(k) + K_f(y(k) - C\widehat{x}(k/k-1)) \\ x_Q(k+1) & = & A_Q x_Q(k) + B_Q(y(k) - C\widehat{x}(k/k-1)) \\ u(k) & = & -K_c\widehat{x}(k/k-1) + C_Q x_Q(k) + D_Q(y(k) - C\widehat{x}(k/k-1)) \end{array}\right. \tag{71}$$

A.2 Discrete-time estimator LQG-form

The estimator structure of an LQG controller is described as :

$$\left\{\begin{array}{lcll} \widehat{x}(k+1/k) & = & A\widehat{x}(k/k) + Bu(k) & \text{Prediction} \\ \widehat{x}(k+1/k+1) & = & \widehat{x}(k+1/k) + K_f'(y(k+1) - C\widehat{x}(k+1/k)) & \text{Correction} \\ u(k+1) & = & -K_c'\widehat{x}(k+1/k+1) & \text{Control} \end{array}\right. \tag{72}$$

In contrast to the previous cases, this discrete-time LQG controller exhibits a direct feed-through between $y(k)$ and $u(k)$ but the separation principle still holds : the closed-loop transfer function between the input reference and the innovation $y(k) - C\widehat{x}(k/k-1)$ is zero and the closed-loop poles can be splitted into the closed-loop state-feedback poles (spec$(A - BK_c')$) which are unobservable from the innovation, and the closed-loop state-estimator poles (spec$(A(I - K_f'C))$) which are uncontrollable by the reference input. The Youla-parameterization associated with this structure reads :

$$\left\{\begin{array}{lcl} \widehat{x}(k+1/k) & = & A\widehat{x}(k/k-1) + Bu(k) + AK_f'(y(k) - C\widehat{x}(k/k-1)) \\ x_Q(k+1) & = & A_Q' x_Q(k) + B_Q'(y(k) - C\widehat{x}(k/k-1)) \\ u(k) & = & -K_c'\widehat{x}(k/k-1) + C_Q' x_Q(k) + (D_Q' - K_c'K_f')(y(k) - C\widehat{x}(k/k-1)) \end{array}\right. \tag{73}$$

We know from sections 2 and A.1 how to compute all the parameters (K_c, K_f, A_Q, B_Q, C_Q and D_Q) of the predictor LQG form and the corresponding Youla parameterization, from a given compensator (A_K, B_K, C_K, D_K) and a given plant (A, B, C, D). As a consequence, the parameters (K_c', K_f', A_Q', B_Q', C_Q' and D_Q') of the equivalent estimator LQG form can be obtained by direct identification of the representations (71) and (73). This yields :

$$\begin{array}{c} K_c' = K_c, \;\; K_f' = A^{-1}K_f, \\ A_Q' = A_Q, \;\; B_Q' = B_Q, \;\; C_Q' = C_Q, \;\; D_Q' = D_Q + K_c'K_f' \end{array} \tag{74}$$

Note that this identification is only possible when K_f is in the range of A.