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# Mixed $H_-/H_\infty$ fault detection observer design for multi model systems via nonsmooth optimization approach

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***Abstract*— This paper deals with a fault detection observer design problem for multi models with only a unique observer, using the worst-case fault sensitivity measure, the $H_-$ index, and the worst-case disturbance robustness measure, the $H_\infty$ norm. The fault detection problem with the criteria of $H_-/H_\infty$ can be formulated as a constrained optimization problem, which can be solved by using nonsmooth optimization method. By adding the constraint of the eigenvalues, the proposed method could improve the fast transients of the residual from the faults with the criteria of $H_-/H_\infty$. The proposed method is shown to perform well on two examples: the nonsmooth optimization method will compare with other classical methods with a single model and design a unique observer for a vehicle lateral dynamics switched system with the trade-off between the optimal values of criteria $H_-/H_\infty$ for different models.**

## I. INTRODUCTION

Associated with the increasing demands for the system reliability and dependability, the research of fault detection and isolation (FDI) has received much attention in the last decades, and a great deal of works has been applied on the model-based method to solve FDI problems. Different from the robust control theory, the FDI system should not only be robust to model uncertainty and the disturbances, but also consider the sensitivity of the FDI system to the faults. The perfect decoupling method could be applied to avoid the uncertainty and disturbances [1]. However, in the industrial systems, the possible faults and disturbances are difficult to decouple [2], therefore, the FDI system has to deal with the problems without perfect disturbance decoupling. And it was well recognized that a satisfactory performance of a FDI system should consider the trade-off between the sensitivity to the faults and the robustness to the model uncertainty and the disturbances.

For the robustness of the FDI systems, i.e., insensitiveness to disturbances, noise or uncertainty, a great amount of research has been done by using $H_\infty$ norm optimization techniques to design the robust fault detection observers [3], [4], [5], [6]. To consider the sensitivity to the faults, different definitions have been proposed, especially, the maximum and minimum influences of the faults are investigated in [3], [7], which means the best-case and the worst-case for the sensitivity evaluation separately. Typically, for the worst-case of the influence of faults on residual, the smallest singular value is considered as a suitable sensitivity measure. Hou and Patton [7] proposed a $H_-$ norm by using the minimum singular value of the transfer function from faults to residual at zero frequency, i.e. $\omega = 0$, in which case, the designed observer only considers the worst-case for the faults at zero frequency. Then the literature [3], [6], [8], [1], [9] extended the $H_-$ notation to nonzero singular value over finite frequency ranges. In particular, using the co-inner-outer factorization techniques, [8] got an optimal solution for a multi-objective function, which guarantees the best detectability of faults with the given false alarm rate. To include the possible zero singular values of the transfer function from the faults to the residual, [10], [11] proposed a new minimum sensitivity measure, called $H_-$ index, and the corresponding frequency range could be infinite or finite.

Using the linear matrix inequality (LMI), [12] calculated the $H_-$ norm and designed the fault detection observer with the criterion of $H_-/H_\infty$ , whose condition is sufficient but not necessary. With the defined index, [10], [11] developed an LMI formulation for the multi-objective of the fault detection observer and used the iterative linear matrix inequality (ILMI) to obtain the solutions. Considering the same mixed $H_-/H_\infty$ criterion, some numerical optimization methods such as genetic algorithm [9] proposed to design the robust fault detection observer. In [13], the main idea is to use the pole assignment approach to transform the fault detection problem into an unconstrained optimization problem, and then design a desirable observer gain with the aid of a gradient-based optimization approach for both the infinite and finite cases. But this method makes strong hypotheses for the $H_\infty$ and $H_-$ with the simple singular value and a unique active frequency, whose algorithm leads to nonsmoothness, so the proposed method is not converging or converging slowly. Another point to notice here is that the target poles of observer should be selected at first, which will limit the freedoms to design the observer for the dynamics of the residual.

In the reliable or fault-tolerant control, the fault detection system has to guarantee satisfactory performance in nominal conditions as well as in the case that some system components turn faulty or deviate from nominal conditions. Or a system may have several different normal modes of operation. In [14], a single observer is designed to isolate different faults with the equivalence to design a structurally constrained controller in the standard control problem framework. But it only considers the robustness with respect to the exogenous disturbances and uncertain parameters. [15], [16] considers the LPV model to design the fault detection

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observer with LMI with a varying observer. The idea in this paper is to consider a unique observer gain and residual weighting matrix to detect faults, which will stabilize the observer for different models with the optimal trade-off between the sensitivity to the faults and the robustness to the disturbances. To the best of the authors' knowledge, little work has been done to design a single observer gain and residual weighting matrix with the $H_-/H_\infty$ criteria and simultaneous stabilize the observer for different models. The problem of simultaneously stabilizing the observer for multi models could be formalized as a BMI (Bilinear matrix inequality) problem, however, there are few effective methods to solve the BMI problem. Recently, the developed nonsmooth optimization method is a typical effective method to solve the simultaneous stabilization problem.

In this paper, we focus on the square systems with as many sensors as possible faults. A single fault detection observer is designed for multi models with the performance index of mixed $H_-/H_\infty$ using nonsmooth optimization approach. The cost function in this work includes both disturbance attenuation and fault detection requirements, and the ratio between these objectives is optimized. Comparing with the LMI method, this design method avoids using Lyapunov variables, whose number grows quadratically with the system state size [17]. Thus, the nonsmooth optimization method is suitable for the large size plant. In the optimization, both of the observer gain and the residual weighting matrix are considered. What's more, the constraint of the fast transients of the responses from faults to the residual could be added to the nonsmooth optimization besides the above ratio criterion to improve the transients of the residual from faults. Recently, solvers relying on nonsmooth optimization techniques like Hinfstruct and Systune [17], [18], [19] are well developed. In this contribution we show the applicability of Systune to design a fault detection observer.

The paper is organized as follows. First, Section 2 formulates the problem of fault detection observer design for single model and multi models. Then, in Section 3, the nonsmooth optimization method is presented with two different examples in simulation with the tool of Systune in Matlab. The first example shows the effectiveness of the nonsmooth optimization method to design the fault detection observer for the single model with the performance index of $H_-/H_\infty$, and the results will be compared with the results of other methods from the literature. With the constraint of the eigenvalues of the system, the proposed method improves the rapidity of the responses from the faults to the residual with the optimal value of $H_-/H_\infty$. Considering white Gaussian noise and nonzero mean, deterministic noise, the second practical example will focus on the multi model case to design a single fault detection observer for a vehicle lateral dynamics switched system with 3 subsystems, and the observer is the compromise of the criteria $H_-/H_\infty$ between the different subsystems. Finally, the conclusion is given in Section 4.

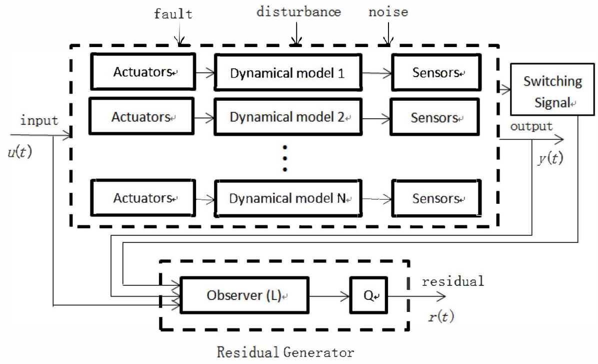


Fig. 1. Residual generator with observer

## II. PROBLEM FORMULATION

### A. Residual generation

Assuming that we have $N \geq 1$ models describing the different normal modes of operation. The linear time invariant (LTI) system for multi models with faults and disturbances is described by

$$\Sigma_0 \begin{cases} \dot{x}(t) = A^i x(t) + B^i u(t) + B_f^i f(t) + B_d^i d(t), \\ y(t) = C^i x(t) + D^i u(t) + D_f^i f(t) + D_d^i d(t), \end{cases} \tag{1}$$

where $i : 1, ..., N$ means the $i^{th}$ model , $x(t) \in R^n$ is the system state vector, $y(t) \in R^m$ represents the output measurement vector, $f(t) \in R^{n_f}$ denotes the fault vector, which can be the process faults, sensor faults, or actuator faults. $d(t) \in R^{n_d}$ is the unknown input vector, including disturbance, modeling error, process and measurement noise or uninterested fault. $u(t) \in R^{n_u}$ is the control input vector. The matrices $A^i$, $B^i$, $C^i$, $D^i$, $B_f^i$, $D_f^i$, $B_d^i$, $D_d^i$ are constant with appropriate dimensions. The single model could be described as the above model $\Sigma_0$ with $i = 1$. Without loss of generality, the following assumptions are used:

- $(A^i, C^i)$ is detectable, $i : 1, ..., N$.
- $f(t)$ and $d(t)$ are $L_2$ norm bounded.

For the generation of the residual, we propose a full-order observer for the multi models in the following form [9], shows as in Fig. 1:

$$\Sigma_1 \begin{cases} \dot{\hat{x}}(t) = A^i \hat{x}(t) + B^i u(t) + L(y(t) - \hat{y}(t)), \\ \hat{y}(t) = C^i \hat{x}(t) + D^i u(t), \\ r(t) = Q[y(t) - \hat{y}(t)]. \end{cases} \tag{2}$$

where $i : 1, ..., N$. $\hat{x}(t) \in R^n$ and $\hat{y}(t) \in R^m$ are the system's state and output estimations, $r(t) \in R^{n_r}$ is the corresponding residual vector, $L \in R^{n \times n_r}$ is the observer gain to design, and $Q \in R^{n_r \times m}$ is the residual weighting matrix, which could be static or dynamic as a $Q(s)$.

Connecting the observer $\sum_1$ (2) with the system $\sum_0$ (1) together as shown in Fig. 1, and considering the state estimation error $e^i(t) = x(t) - \hat{x}(t)$, we can get the residual error dynamic equations:

$$\Sigma_2 \begin{cases} \dot{e}(t) =(A^i - LC^i)e(t) + (B_f^i - LD_f^i)f(t) \\ \qquad + (B_d^i - LD_d^i)d(t), \\ r(t) = QC^i e(t) + QD_f^i f(t) + QD_d^i d(t). \end{cases} \tag{3}$$

The corresponding residual responses from faults and disturbances are:

$$\begin{aligned} r(s) =&Q\{D_f^i + C^i(sI - A^i + LC)^{-1}(B_f^i - LD_f^i)\}f(s) \\ &+ Q\{D_d^i + C^i(sI - A^i + LC)^{-1}(B_d^i - LD_d^i)\}d(s) \\ =&G_{rf}^i(s, L, Q)f(s) + G_{rd}^i(s, L, Q)d(s) \end{aligned} \tag{4}$$

Obviously, the dynamics of the residuals rely on the transfer function from faults and disturbances to the residual, so the multi-objective design of fault detection observer (design the observer gain $L$ and the residual weighting matrix $Q$) contains the following objectives:

i) The $1, ..., N$ residual error dynamics equations (3) with the observer gain $L$ should be stable,
ii) Maximize the effects of faults on the residual,
iii) Minimize the effects of disturbances on the residual.

In order to detect the fault fast, the rapidity of the responses from the fault to the residual is an interesting specification to consider, so it is interesting to introduce the constraint of the fast transients of the responses from the faults to the residual to design a fast fault detection observer.

### B. Criteria for evaluation

Considering the robustness to the disturbances or the unknown signals of the residual, the criterion $H_\infty$ is used in this paper,

$$\|H\|_\infty = \sup_{\omega\in\Phi} \bar{\sigma}(G(j\omega)) \tag{5}$$

where $\bar{\sigma}(G(j\omega))$ denotes the maximum singular value of matrix $G(j\omega)$, and $\Phi$ is the evaluated frequency range, which could be infinite or finite.

For the problem of fault detection observer design, we are more interested in the "worst-case" of the fault detection, so we use $H_-$ index to evaluate the minimum sensitivity of faults to the residual.

According to the concept of structural fault detectability [3], we will follow the definition in [3], which is different from the definition of $H_-$ index in [10][11].

*Definition 2.1:* Given $y(s) = G(s)u(s)$ , the index $H_-$ of $G(s)$ is defined by

$$\|G(s)\|_- = \inf_{u\neq 0} \frac{\|G(s)u(s)\|_2}{\|u(s)\|_2} \tag{6}$$

It is possible that there exists some $u(s) \neq 0$, but $G(s)u(s) = 0$, therefore $\|G(s)\|_- = 0$. Considering the notion of structural fault detectability in [3], it is more interesting to evaluate of the minimum value of $\|G(s)u(s)\|_2$ when $\|u(s)\|_2 = 1$ , which is different from zero. Therefore, the definition in (6) could be rewritten as

$$\|G\|_- = \inf_{\omega\in\Phi} \underline{\sigma}(G(j\omega)) \tag{7}$$

with $\underline{\sigma}(G(j\omega))$ denoting the minimum non-zero singular value of matrix $G(j\omega)$ . And $\Phi$ is the evaluated frequency range where $\underline{\sigma}(G(j\omega)) \neq 0$ , which can be either infinite or finite.

One way to evaluate the rapidity of the responses in the frequency domain is to make all the eigenvalues of $A^i - LC^i$ far from the imaginary axis as much as possible in the negative real part of the complex plane:

$$\begin{gathered} \min_L \ \alpha \\ \text{real}(\text{eig}(A - LC)) < \alpha \end{gathered} \tag{8}$$

where $\alpha$ is a negative value.

### C. Transformation for calculation

Applying the criteria $H_\infty$ and $H_-$ into the residual model (4), the problem of fault detection observer design can be formulated as follows:

i) $A_0^i = A^i - LC^i$ is asymptotically stable;
ii) $\max\limits_{L,Q} \parallel G_{rf}^i \parallel_- = \max\limits_{L,Q} \inf\limits_{\omega\in[\omega_1,\omega_2]} \underline{\sigma}(G_{rf}^i)$ ,
$= \max\limits_{L,Q} \inf\limits_{\omega\in[\omega_1,\omega_2]} \underline{\sigma}(QD_f^i + QC^i(sI - A^i + LC^i)^{-1} \times(B_f^i - LD_f^i))$,
iii) $\min\limits_{L,Q} \parallel G_{rd}^i \parallel_\infty = \min\limits_{L,Q} \sup\limits_{\omega\in[\omega_1,\omega_2]} \bar{\sigma}(G_{rd}^i)$
$= \min\limits_{L,Q} \sup\limits_{\omega\in[\omega_1,\omega_2]} \bar{\sigma}(QD_d^i + QC^i(sI - A^i + LC^i)^{-1} \times(B_d^i - LD_d^i))$,
iv) $\min\limits_L \ \alpha$, where $\text{real}(\text{eig}(A^i - LC^i)) < \alpha$.

[9] proposed that the $\max\limits_{L,Q} \left\|G_{rf}^i\right\|_-$ can be calculated by the following formulation:

$$\begin{aligned} &\min_{L,Q} \sup_{\omega\in[\omega_1,\omega_2]} \bar{\sigma}((G_{rf}^i)^{-1}) \\ &\quad = \min_{L,Q} \sup_{\omega\in[\omega_1,\omega_2]} \bar{\sigma}\left((QD_f^i\right. \\ &\quad \left. +QC^i(sI - A^i + LD_f^i)^{-1}(B_f^i - LD_f^i))^{-1}\right) \end{aligned} \tag{9}$$

with the condition that the matrix of $G_{rf}^i$ is invertible.

### D. Formulation for nonsmooth optimization

There is a number of formulations for the problem of fault detection observer design in [3], and one of which is proposed to solve by LMI method:

$$\begin{gathered} \|G_{rf}^i(L, Q)\|_- > \beta_i \\ \|G_{rd}^i(L, Q)\|_\infty < \gamma_i \\ A^i - LC^i \text{ is asymptotically stable} \end{gathered} \tag{10}$$

where the $\beta_i$ and $\gamma_i$ are the parameters to optimize. And for the LMI method, another important work is to use iterative method to find the maximum $\beta_i$ and the minimum $\gamma_i$, whose convergence rate is slow and the results are always conservative. From the optimization points of view, the above problem could be:

$$\underset{L,Q}{\text{minimize}} \max_{i=1,\ldots,N} \left( \lambda_i \frac{\|G_{rd}^i(L,Q)\|_\infty}{\|G_{rf}^i(L,Q)\|_-} \right), \tag{11}$$

$$A^i - LC^i \text{ is asymptotically stable} \tag{12}$$

where $\lambda^i$ are appropriate non-negative weights. The problem (11) is formulated as a minimax way, which always produces Pareto optimal result, and every Pareto optimal result arises as the solution of the minimax problem (11) with some choice of weights $\lambda^i$ [20].

A point to notice is that because the region of the observer gain $L$ to stabilize the observer for the different models are different, in some cases, the optimal observer gain $L$ could stabilize all the models at the same time, which could be interpreted by the Fig. 2. It is also possible that there are no intersections of the feasible region for the different models, and this paper concentrates on the case that the intersections of the different models exist.

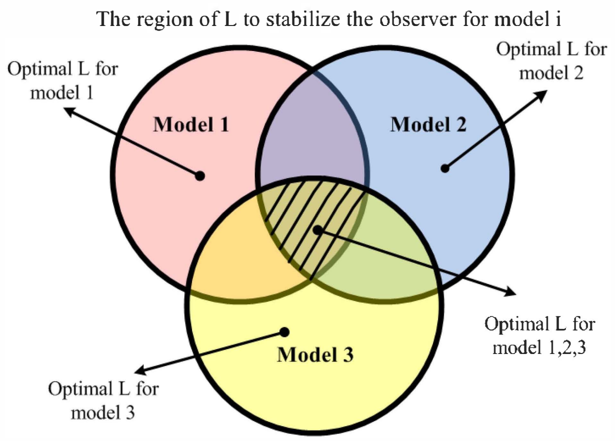


Fig. 2. Interpretation of the formulation (11)

Obviously, with the different selection of the weights $\lambda^i$, the formulation (11) will give the different results. Normally, the weights are meant to express the designer's preferences among different $\|G_{rd}^i(L,Q)\|_\infty / \|G_{rf}^i(L,Q)\|_-$. If the weights $\lambda^i$ are designed as the reciprocal of the best nominal value of $\|G_{rd}^i(L,Q)\|_\infty / \|G_{rf}^i(L,Q)\|_-$, the above observer design optimization for the multi models problem means that there is a trade-off between the criteria $\|G_{rd}^i(L,Q)\|_\infty / \|G_{rf}^i(L,Q)\|_-$ for the different models to design the single observer, so the single balanced optimal observer considers the effects for all the models. This paper will consider this setting for the weights $\lambda^i$ in (11) to design the single observer for multi models. Some other general design procedures to design the weights $\lambda^i$ for the multi-criterion optimization with different preferences are introduced in [20].

## III. NUMERICAL EXAMPLE

Solvers based on nonsmooth optimization techniques Systune and Hinfstruct (Apkarian et al.) have been successfully applied on $H_\infty$ synthesis non-convex problem with structural constraints [21], [22], [23]. With the aid of these solvers, the fault detection observer for multi models could be designed with the criterion of $H_-/H_\infty$ .

The solver Systune uses nonsmooth casts of the form to tuning against multiple requirements:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad g(x) \le c, \ c \in R \end{aligned} \tag{13}$$

where both $f$ and $g$ are $max$-functions

$$f(x) := \max_{i=1,\ldots,N_f} f_i(x), g(x) := \max_{j=1,\ldots,N_g} g_j(x), \tag{14}$$

and $x$ gathers all the parameters to design.

In this section, two different examples are given to illustrate the nonsmooth optimization method to apply to design fault detection observer for the single model with the constraint of the eigenvalues in (8) and for the multi models with a switched system.

### A. Fault detection observer design for single model

To illustrate the effectiveness of the proposed nonsmooth optimization method, here is an example to show the solution for the $H_-/H_\infty$ fault detection problem in infinite frequency range. To compare with the existing methods, the example's model is from [10]. Consider a single MIMO model of the form of (1) with $B$ and $D$ are random to select, which does not affect the optimization result.

$$A = \begin{bmatrix} -0.1210 & -0.5624 & -0.2651 & -0.2500 \\ 4.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0.2500 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -1.4140 & -0.4373 & -0.1768 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}^T \quad D_f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0.02 & -0.02 & 0 \\ 0.02 & 0.1 & 0 \\ 0.02 & -0.02 & 0 \\ 0.02 & 0.1 & 0 \end{bmatrix} \quad D_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the above state space model, [10] included three methods to design the $H_-/H_\infty$ optimal observer, and the solution used by the nonsmooth optimization method will be compared with these three methods to validate the effectiveness of the proposed method.

The observer gain $L$ and the residual weighting matrix $Q$ by Ding's method [8] are:

$$L_{Ding} = \begin{bmatrix} -0.0201 & 0.0981 & -0.0202 & 0.1016 \\ -0.0001 & -0.0034 & 0 & 0.0089 \end{bmatrix}^T$$

$$Q_{Ding} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The method of Nobrega [24] gets the following observer gain:

$$L_{Nobrega} = \begin{bmatrix} 0.2665 & -0.6415 & 0.1534 & 0.2362 \\ -0.0505 & -0.1268 & 0.7282 & 0.6441 \end{bmatrix}^T$$

For the LMI method from [10], the observer gain $L$ and the residual weighting matrix are:

$$L_{Wang} = \begin{bmatrix} -0.0209 & 0.0916 & -0.0161 & 0.1036 \\ -0.0045 & 0.0227 & -0.0406 & 0.0520 \end{bmatrix}^T$$

$$Q_{Wang} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In order to compare the performance of the result by the proposed nonsmooth optimization method with the above three method in [10], the residual weighting matrix $Q$ is designed as a static matrix, just as discussed in [9], which also could be dynamic as $Q(s)$.

$$L_{nonsmooth} = \begin{bmatrix} 0.0513 & 0.2324 & 0.1541 & -0.0586 \\ -0.1532 & 0.2302 & 0.7581 & 0.0988 \end{bmatrix}^T$$

$$Q_{nonsmooth} = \begin{bmatrix} -1.4814 & -0.7006 \\ -1.2189 & 1.0062 \end{bmatrix}$$

Several random start points are applied in the nonsmooth optimization simulation, which does not improve the result. In other words, the local optimal point seems to be the global optimal point.

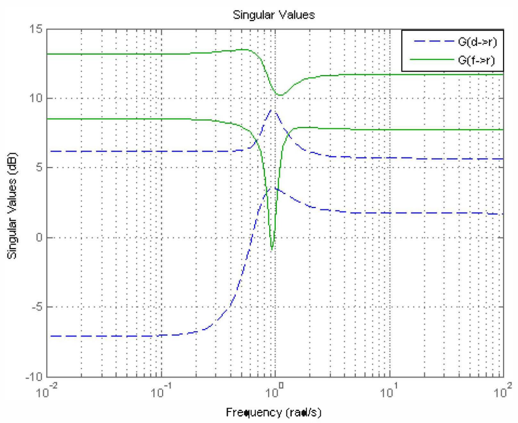


Fig. 3. Singular values of $G_{rf}$ and $G_{rd}$ for the result of nonsmooth method

From Fig. 3, the worst case of the fault detection is at $\omega = 0.928 rad/s$ , in which case, the smallest singular value of $G_{rf}$ is -0.881$db$, which is equal to 0.9016, and the biggest singular value of $G_{rd}$ is 9.0795$db$, which is 2.8443. The corresponding ratio of $\|G_{rd}\|_\infty$ to $\|G_{rf}\|_-$ is 3.1545.

The effects of different methods are listed in Table I, from which, we can find the proposed nonsmooth optimization method works well for the observer design like the classical methods.

From Table I, we can also find that with the different observer gain $L$ and residual weighting matrix $Q$ from different methods, the value of $\|G_{rd}\|_\infty / \|G_{rf}\|_-$ doesn't improve a lot, which means that the effects of the parameters to design on the minimum value of $\|G_{rd}\|_\infty / \|G_{rf}\|_-$ are weak. Thus, it is interesting to add the criterion (8) as an another constraint to the optimization for the above example, which could be solved by the proposed nonsmooth optimization:

TABLE I

COMPARISON FOR DIFFERENT METHODS

| Method | $\|G_{rf}\|_-$ | $\|G_{rd}\|_\infty$ | $\|G_{rd}\|_\infty / \|G_{rf}\|_-$ |
|---|---|---|---|
| Ding | 0.3154 | 1.0002 | 3.1712 |
| Nobrega | 0.3300 | 1.2279 | 3.7209 |
| Wang | 0.3153 | 1 | 3.1716 |
| Nonsmooth | 0.9016 | 2.8443 | 3.1545 |

$$L_{fast} = \begin{bmatrix} -0.2896 & 0.9076 & 0.6822 & -1.1426 \\ 0.1885 & -0.6323 & -0.5896 & 0.9882 \end{bmatrix}^T$$

$$Q_{fast} = \begin{bmatrix} 0.4913 & 0.4282 \\ -0.2321 & 1.5970 \end{bmatrix}$$

With $L_{fast}$ and $Q_{fast}$, the $\|G_{rf}\|_-$ is 0.5521, and the $\|G_{rd}\|_\infty$ is 1.7513. And the corresponding value of $\|G_{rd}\|_\infty / \|G_{rf}\|_-$ is 3.1721, which means that the above solution under the constraint of eigenvalues has almost the same property of sensitivity to the faults and robustness to the disturbances in the worst case.

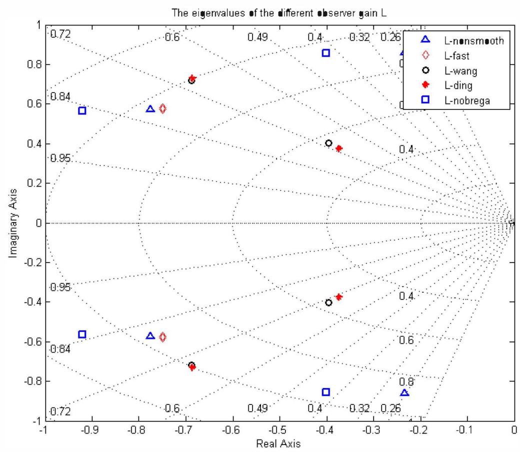


Fig. 4. The diagram of the eigenvalues of the different observer gain $L$

Fig. 4 shows the locations of all the eigenvalues of $A - LC$ with the solutions of above introduced methods and the designed $L_{fast}$. The eigenvalues of $A - L_{fast}C$ are $-0.7503 \pm 0.5810i$ and $-0.7503 \pm 0.5735i$, and the corresponding smallest real part are farthest from the imaginary axis comparing with the other observer gain $L$.

### B. Fault detection observer design for multi models

For the multi model case, the proposed methodology is applied on a vehicle lateral dynamics system. In this application, the different normal modes of operation of the vehicle system are depend on the working speed, which is the switching signal to switch the parameters in the observer, but the observer gain $L$ is unchanged. The model under consideration is the well know bicycle model [25], [26]:

$$\begin{bmatrix} \dot{\beta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} \frac{Y_1}{mv} & \frac{Y_2}{mv^2} \\ \frac{Y_3}{I_z} & \frac{Y_4}{I_z v} \end{bmatrix} \begin{bmatrix} \beta \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} \frac{C_{\alpha V}}{mv} \\ \frac{l_V C_{\alpha V}}{I_z} \end{bmatrix} \delta - \begin{bmatrix} \frac{g}{v} \\ 0 \end{bmatrix} d$$

$$a_y = \begin{bmatrix} \frac{Y_1}{m} & \frac{Y_4}{mv} \end{bmatrix} \begin{bmatrix} \beta \\ \dot{\psi} \end{bmatrix} + \frac{C_{\alpha V}}{m} * \delta - g * d$$

with $Y_1 = -(C_{\alpha V} + C_{\alpha H})$, $Y_2 = l_H C_{\alpha H} - l_V C_{\alpha V} - mv^2$, $Y_3 = l_H C_{\alpha H} - l_V C_{\alpha V}$ and $Y_4 = -(l_V^2 C_{\alpha V} + l_H^2 C_{\alpha H})$, where $\beta$ is the side slip angle, $\psi$ denotes the yaw rate, $a_y$ is the lateral acceleration, $\delta$ is the relative steering wheel angle, $d$ means the disturbance (road bank angle), and $v$ represents the speed of the vehicle .

In the simulation, three subsystems are selected at the speed $v = 7m/s$, $14m/s$ and $20m/s$. In this example, the additive fault in above system is the fault in steering angle $\delta$ measurement, so we set $B_f = B_\delta = \begin{bmatrix} \frac{C_{\alpha V}}{mv} & \frac{l_V C_{\alpha V}}{I_z} \end{bmatrix}^T$ and $D_f = D_\delta = \frac{C_{\alpha V}}{m}$ :

$$\left[\begin{array}{c|c|c} A^1 & B_f^1 & B_d^1 \\ \hline C^1 & D_f^1 & D_d^1 \end{array}\right] = \left[\begin{array}{cc|c|c} -20.7 & -0.46 & 10.1 & -1.4 \\ 21.2 & -27.3 & 63.7 & 0 \\ \hline -145 & 3.74 & 71 & -9.8 \end{array}\right]$$

$$\left[\begin{array}{c|c|c} A^2 & B_f^{12} & B_d^2 \\ \hline C^2 & D_f^2 & D_d^2 \end{array}\right] = \left[\begin{array}{cc|c|c} -9.66 & -0.88 & 4.73 & -0.7 \\ 21.2 & -12.7 & 63.7 & 0 \\ \hline -145 & 1.74 & 71 & -9.8 \end{array}\right]$$

$$\left[\begin{array}{c|c|c} A^3 & B_f^3 & B_d^3 \\ \hline C^3 & D_f^3 & D_d^3 \end{array}\right] = \left[\begin{array}{cc|c|c} -7.24 & -0.93 & 3.55 & -0.5 \\ 21.2 & -9.57 & 63.7 & 0 \\ \hline -145 & 1.31 & 71 & -9.8 \end{array}\right]$$

Because the dimension of the residual in above example is 1, there is no effect of the residual weighting matrix $Q$ on the value of $\|G_{rd}\|_\infty / \|G_{rf}\|_-$. In this example, the observer design will just design the observer gain $L$ by nonsmooth optimization method.

The corresponding region of observer gain $L$ to stabilize the observer for above three subsystems is shown in the Fig. 5, and the relationships between the region of observer gain $L$ to stabilize the observer of the three different subsystems are:

$$\Psi(\text{Subsystem 3}) \subseteq \Psi(\text{Subsystem 2}) \subseteq \Psi(\text{Subsystem 1})$$

Therefore, in this example, it is possible to design a unique observer gain $L$ for the three subsystems with some typical preferences, which should simultaneously stabilizes the observer for the three subsystems. Here, the following design takes the case that the bicycle model evaluates the above three subsystems equally.

First, design the optimal observer for the subsystem separately as in the first example, and we can get:

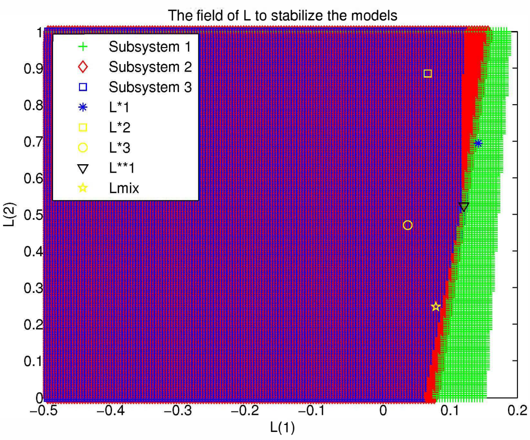


Fig. 5. The region of observer gain $L$ to stabilize the three subsystems

$$L_1^* = [0.1417, 0.6943]^T; \quad \left\|G_{rd}^1\right\|_\infty / \left\|G_{rf}^1\right\|_- = 0.1393$$
$$L_2^* = [0.0663, 0.8857]^T; \quad \left\|G_{rd}^2\right\|_\infty / \left\|G_{rf}^2\right\|_- = 0.1696$$
$$L_3^* = [0.0367, 0.4706]^T; \quad \left\|G_{rd}^3\right\|_\infty / \left\|G_{rf}^3\right\|_- = 0.1940$$

where $L_1^*$ is designed only for Subsystem-1, $L_2^*$ is designed only for Subsystem-2 and $L_3^*$ is only designed for Subsystem-3.

A point we should notice is that in the above example the obtained observer gain $L_1^*$ is in the feasible region of Subsystem-1 but not in the stable region of Subsystem-2 or Subsystem-3 in Fig. 5, which can not stabilize the observer for Subsystem-2 or Subsystem-3. And the eigenvalues of Subsystem-2 and Subsystem-3 with $L_1^*$ are 3.7739, -0.94 and 2.5514, -5.566, which validates the fact that the observer gain $L_1^*$ can not stabilize Subsystem-2 and Subsystem-3. Thus, the constraint of the stability for subsystems may affect other subsystems' optimal value of $\|G_{rd}\|_\infty / \|G_{rf}\|_-$. The selection of weights $\lambda_i$ in (11) to optimize for the three subsystems of the switched system should consider the effects of the stability for these different subsystems. Therefore, solve following criteria with $i = 1,\ 2,\ 3$ separately:

$$\underset{L}{\text{minimize}} \max_{i=1,\dots,N} \left( \lambda_i \frac{\|G_{rd}^i(L,Q)\|_\infty}{\|G_{rf}^i(L,Q)\|_-} \right),$$
$$A^i - LC^i \text{ is asymptotically stable}$$
$$\lambda_i = 1,\ \lambda_j = 0,\ \text{where } i \neq j,\ and\ i,j \in \{1,2,3\}$$

to get $L_1^{**}$ with the optimal value of $\left\|G_{rd}^1\right\|_\infty / \left\|G_{rf}^1\right\|_-$, $L_2^{**}$ with the optimal value of $\left\|G_{rd}^2\right\|_\infty / \left\|G_{rf}^2\right\|_-$ and $L_3^{**}$ with the optimal value of $\left\|G_{rd}^3\right\|_\infty / \left\|G_{rf}^3\right\|_-$ under the constraints of stabilizing the three subsystems simultaneously:

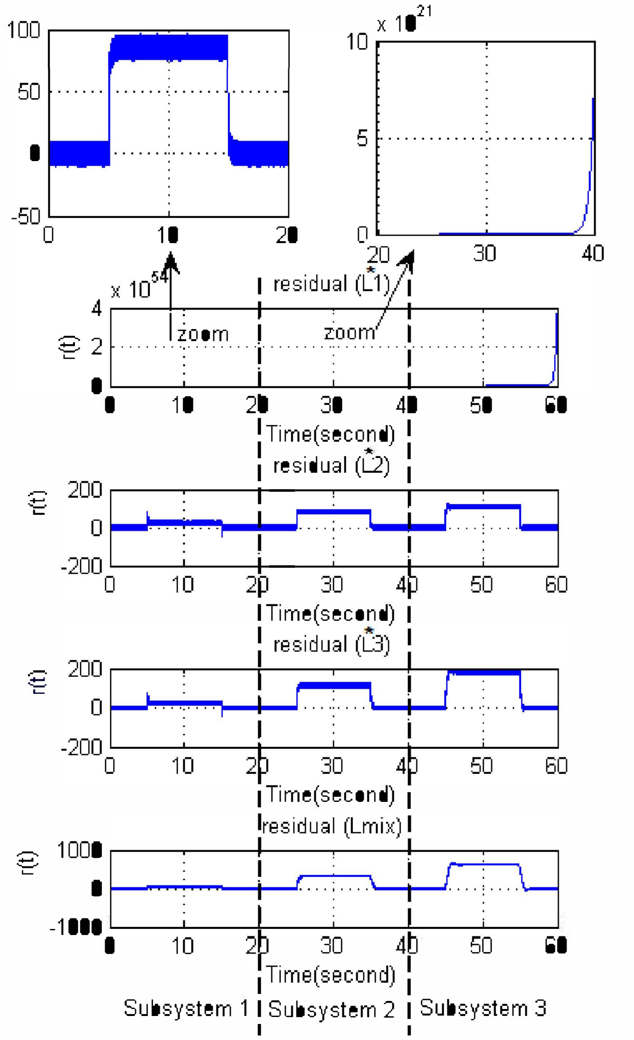


Fig. 6. The residual responses with $L_1^*$, $L_2^*$, $L_3^*$ and $L_{mix}$ for the white Gaussian noise case

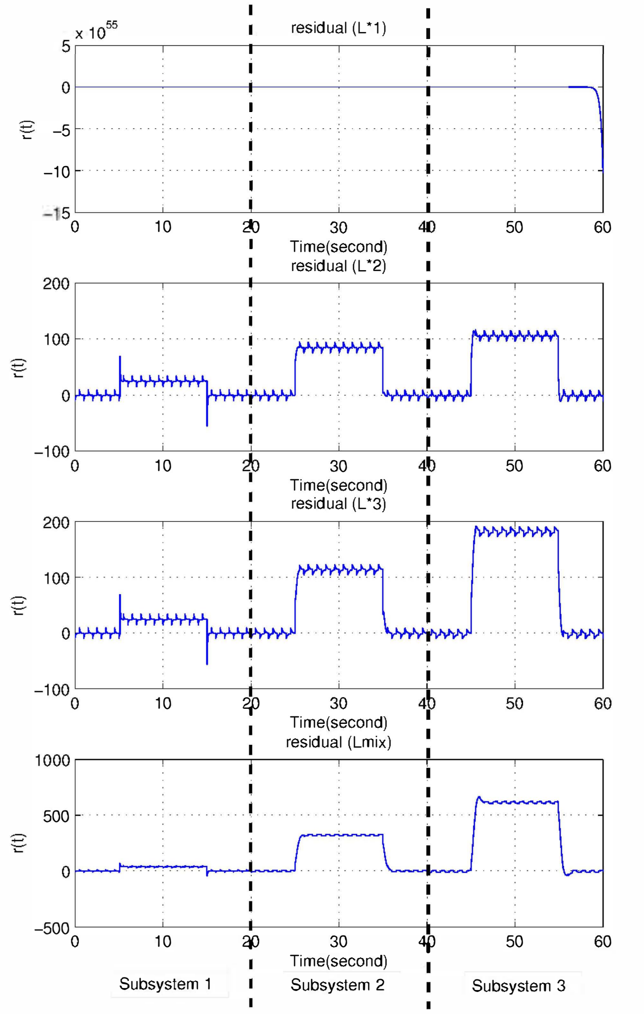


Fig. 7. The residual responses with $L_1^*$, $L_2^*$, $L_3^*$ and $L_{mix}$ for the nonzero mean, deterministic case

$$L_1^{**} = [0.1202, 0.5232]^T; \left\|G_{rd}^1\right\|_\infty / \left\|G_{rf}^1\right\|_- = 0.1658$$
$$L_2^{**} = [0.0663, 0.8857]^T; \left\|G_{rd}^2\right\|_\infty / \left\|G_{rf}^2\right\|_- = 0.1696$$
$$L_3^{**} = [0.0367, 0.4706]^T; \left\|G_{rd}^3\right\|_\infty / \left\|G_{rf}^3\right\|_- = 0.1940$$

As introduced before, the weights are selected as: $\lambda_1 = 1/0.1658$, $\lambda_2 = 1/0.1696$, $\lambda_3 = 1/0.1940$ to consider the three subsystems together. We can get the optimal unique observer design by the nonsmooth optimization method:

$$L_{mix} = [0.0786, 0.2475]^T$$

TABLE II
COMPARISON FOR DIFFERENT OBSERVERS

| | Subsystem-1 | Subsystem-2 | Subsystem-3 |
|---|---|---|---|
| $L_1^{**}$ | 0.1658 | 29.3962 | 47.2731 |
| $L_2^{**}$ | 0.3959 | 0.1696 | 0.2291 |
| $L_3^{**}$ | 0.4440 | 0.1783 | 0.1940 |
| $L_{mix}$ | 0.2727 | 0.1800 | 0.2869 |

Fig. 5 also shows the locations of the different observer gain $L$. And the effects of $L_{mix}$ on the criteria of $\|G_{rd}\|_\infty / \|G_{rf}\|_-$ for the three subsystems comparing with the other three designed observer gain are listed in the table II.

The table II shows that the mixed performance index $\|G_{rd}\|_\infty / \|G_{rf}\|_-$ of the observer gain $L_1^{**}$ for Subsystem-1, $L_2^{**}$ for Subsystem-2 and $L_3^{**}$ for Subsystem-3 are smallest, which means that they are the optimal design for the separate subsystem, however, these designs of course are not good for the other subsystems. Considering the three subsystems equally, the design of the mixed case, $L_{mix}$, gives the Pareto optimal solution for the three different subsystems, whose effects are the trade-off between the three subsystems for the value of $\left\|G_{rd}^i\right\|_\infty / \left\|G_{rf}^i\right\|_-$, which also stabilize all the subsystems at the same time.

Furthermore, these observers are simulated with switched system by taking the disturbances as a white Gaussian noise and the faults as

$$f(t) = \begin{cases} 1, & 5 \leq t \leq 15 \\ 1, & 25 \leq t \leq 35 \\ 1, & 45 \leq t \leq 55 \\ 0, & \text{elsewhere} \end{cases}$$

In this case, there will be a fault in Subsystem-1, Subsystem-2 and Subsystem-3 separately before switching, and the switching time is selected at 20s and 40s. The time responses of the residual signals with the white Gaussian noise case are plotted in Fig. 6.

What's more, from an engineering point of view, a nonzero mean, deterministic noise road band angle is more interesting than the white noise road band angle. So the second simulation with impulse disturbances gives more practical meanings. The disturbances are taken as a kind of impulse disturbances (amplitude=1, period=1s, pulse width=50%) in the simulation. The time responses are shown in Fig. 7. Same as the above analysis, both of the previous white Gaussian noise case and the nonzero mean, deterministic noise case, the observer gain $L_1^*$ cannot stabilize either the Subsystem-2 or the Subsystem-3. And comparing with the observer gains $L_1^*$, $L_2^*$, $L_3^*$, the compromised observer gain, $L_{mix}$, shows a satisfied capability to produce a residual to detect the faults for the switched system.

## IV. CONCLUSIONS

In this paper, we deal with a multi-objective $H_-/H_\infty$ fault detection observer design for multi model problem, where the norm $\|\cdot\|_\infty$ is used to describe robustness to disturbances and $\|\cdot\|_-$ index is used to measure the fault sensitivity. The problem is solved by nonsmooth method with Systune in Matlab. Different from the method of LMI and other methods, the proposed method optimizes the criteria of $H_-/H_\infty$ directly and ensures to converge to an optimal solution. Besides the constraints of sensitivity to the faults and robustness to the disturbances, the proposed method could contain the constraint of the eigenvalues to improve the fast transients of the residual from the faults. What's more, in this formulation, the nonsmooth optimization method could be applied to design an unique observer gain $L$ for multi models with the Pareto optimal $\|\cdot\|_\infty / \|\cdot\|_-$ for the different models and simultaneous stabilize the observer for the different models. The effectiveness of the proposed method is proved by the numerical simulations with a single model and a vehicle lateral dynamics switched system.

## V. ACKNOWLEDGMENTS

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