Mixed H_2/H_∞ Multi-Channel Linear Parameter-Varying Control in Discrete Time

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Abstract

This paper develops a new method for the synthesis of Linear Parameter-Varying (LPV) controllers in discrete time. LPV plants under consideration have a Linear Fractional Transformation (LFT) representation and specifications consist of a set of H_2/H_∞ conditions that can be defined channel-wise. In contrast to earlier results, the proposed method involves a specific transformation of both the Lyapunov and scaling/multiplier variables which renders possible the use of different Lyapunov functions and of different scaling variables for each channel/specification. Appropriate linearizing transformations on the controller data and on the scheduling function are then established to finally recast the problem as an easily tractable LMI program.

Key words. LPV synthesis, mixed H_2/H_∞, multi-channel control, LFT, Linear Matrix Inequalities.

1 Introduction

LPV control techniques have received great attention in the recent years [17, 2, 5, 13, 19]. The main thrust of these techniques is that they provide an elegant and algorithmically attractive setting for addressing the practical needs of gain scheduling or controller interpolation. The most demanding task of these techniques amounts to solving Linear Matrix Inequality (LMI) programs which is relatively easy with currently available Semi-Definite Programming codes. These methods have also been constantly refined and improved in different directions. In [19, 23, 1, 3, 22, 15]. Except from isolated cases [18, 1, 13] which either discuss computationally intensive approaches or propose somewhat conservative schemes, the definition of a genuine mixed H_2/H_∞ and multi-channel LPV methodology is a very challenging issue. Because of the many constraints surrounding most practical designs the development of such a methodology is certainly of crucial importance.

We develop a technique for solving the mixed H_2/H_∞ multi-channel LPV control problem in discrete time which is an extension of previous results in [17, 2]. As a key ingredient, we introduce slack variables in performance characterizations which, to some degree, permits to short-circuit the inherent strong interrelations between Lyapunov and scaling variables on one side and LPV controller variables on the other side. An interesting consequence is that different Lyapunov/scaling variable pairs can be used for each channel and specification, thus improving on earlier results obtained in the context of linear time-invariant multi-objective synthesis [16, 20]. Similar ideas have also been used in [10, 9, 4] for robustness analysis and linear time-invariant synthesis. The work in this paper extends these concepts to LFT systems and the restricted class of full-block symmetric scalings. It also establishes new linearizing transformations of the LPV controller data and of the controller scheduling function to achieve a full LMI program description of the mixed H_2/H_∞ multi-channel LPV synthesis problem.

2 Analysis setup

This section develops analysis tests for robust H_2 and H_∞ performance that will be central in the construction of multi-objective LPV controllers. The LPV plant is described as

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ z_\Delta(k) \\ z(k) \end{bmatrix} &= \begin{bmatrix} A & B_\Delta & B_1 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} \\ C_1 & D_{1\Delta} & D_{11} \end{bmatrix} \begin{bmatrix} x(k) \\ w_\Delta(k) \\ w(k) \end{bmatrix} \\ w_\Delta(k) &= \Delta(k)\, z_\Delta(k), \end{aligned} \tag{1}$$

where $\Delta(k) \in \mathbf{R}^{N\times N}$ is a time-varying matrix-valued parameter evolving in a polytopic set $\mathcal{P}_{\boldsymbol{\Delta}}$, with

$$\mathcal{P}_{\boldsymbol{\Delta}} := \text{co}\,\{\Delta_1, \ldots, \Delta_i, \ldots, \Delta_L\} \ni 0\,, \tag{2}$$

where co stands for the convex envelope and the Δ_i's denote the vertices of $\mathcal{P}_{\boldsymbol{\Delta}}$. The plant with inputs w and outputs z has state-space data entries which are fractional functions of the time-varying parameter $\Delta(k)$. Hereafter, we are using the following notation x for the state vector, w for exogenous inputs and z for controlled or performance variables.

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2.1 Robust H_2 performance

A characterization of robust H_2 performance is provided in the following proposition. For the H_2 performance index to be well defined, we assume that the state-space data are such that the corresponding feedthrough term is zero.

Proposition 2.1 (Robust H_2 performance) *The following statements, involving Lyapunov variables X and Z, scaling pairs (Q_1, R_1), (Q_2, R_2) and general slack matrix variables V, H_1, F_1, H_2, F_2 are equivalent and enforce a bound ν on the variance of the output z for all parameter trajectories $\Delta(k) \in \mathcal{P}_{\mathbf{\Delta}}$:*

$$(i)\ \begin{bmatrix} -X & * & * & * & * \\ 0 & Q_1 & * & * & * \\ 0 & 0 & -\nu I & * & * \\ A & B_\Delta & B_1 & -X^{-1} & 0 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & 0 & R_1^{-1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} -X & * & * & * \\ 0 & Q_2 & * & * \\ C_\Delta & D_{\Delta\Delta} & -R_2^{-1} & * \\ C_1 & D_{1\Delta} & 0 & -Z \end{bmatrix} < 0,\ \mathrm{Tr}\,(Z) < 1, \tag{3}$$

$$\begin{bmatrix} R_1 & \Delta^T \\ \Delta & -Q_1^{-1} \end{bmatrix} > 0,\ \begin{bmatrix} R_2 & \Delta^T \\ \Delta & -Q_2^{-1} \end{bmatrix} > 0,\ \ \forall \Delta = \Delta_i.$$

$$(ii)\ \begin{bmatrix} -X & * & * & * & * \\ 0 & Q_1 & * & * & * \\ 0 & 0 & -\nu I & * & * \\ V^T A & V^T B_\Delta & V^T B_1 & X - (V + V^T) & 0 \\ H_1^T C_\Delta & H_1^T D_{\Delta\Delta} & H_1^T D_{\Delta 1} & 0 & R_1 - (H_1 + H_1^T) \end{bmatrix} < 0,$$

$$\begin{bmatrix} -X & * & * & * \\ 0 & Q_2 & * & * \\ H_2^T C_\Delta & H_2^T D_{\Delta\Delta} & R_2 - (H_2 + H_2^T) & * \\ C_1 & D_{1\Delta} & 0 & -Z \end{bmatrix} < 0,\ \mathrm{Tr}\,(Z) < 1,$$

$$\begin{bmatrix} R_1 & \Delta^T F_1 \\ F_1^T \Delta & Q_1 + F_1 + F_1^T \end{bmatrix} > 0,\ \begin{bmatrix} R_2 & \Delta^T F_2 \\ F_2^T \Delta & Q_2 + F_2 + F_2^T \end{bmatrix} > 0,\ \ \forall \Delta = \Delta_i. \tag{4}$$

Proof: See full version of paper. ■

It is worth mentioning that the conditions in Proposition 2.1 are conservative in two respects. First of all, a fixed Lyapunov function (not depending on parameters) is employed to assess H_2 performance of the uncertain system. This is a well-recognized source of conservatism [8, 12, 11]. Secondly, we are utilizing a subclass of full-block generalized scalings with zero off-diagonal separators in place of the class of generalized scalings or multipliers introduced in [21]. Therefore, these tests should be refined when used for validation purpose. This subclass is, however, more general than the subclass of structured symmetric scalings used in [17, 2]. More importantly, this new characterizations also offer new potentials for deriving tractable characterizations for discrete-time multi-objective LPV control problems which appears delicate using earlier techniques.

There are a few points to have in mind to understand the conditions (3) and (4) and their usefulness.

- In (4), we get rid of the standard Lyapunov terms XA, XB_1, ... and of the scaling terms $R_1 C_\Delta$, $R_1 D_{\Delta 1}$, ... by means of intermediate (slack) variable V, H_1, H_2, F_1 and F_2. These terms generally impose strong limitations in multi-objective control problems since they preclude the use of multiple Lyapunov functions or scalings. Similar ideas have been presented earlier in [10, 9] for Linear Time-Invariant multi-objective synthesis.

- The LMI condition (4) is significantly more costly than its original form (3) because of the additional general matrix variables V and H_1, H_2, F_1 and F_2 . We shall see however that this extra computational overhead is more than offset by new capabilities in multi-objective LPV synthesis. An important consequence is that multiple Lyapunov functions X_j and scalings R_j, Q_j can be employed for each channel and specification.

Finally, the conditions in (3) and (4) guarantee well-posedness of the LFT representation in (1). The property of well-posedness is ensured in most results in this paper and will not be discussed further. See for instance [2, 21] for related texts.

2.2 Robust H_∞ performance

The following result for H_∞ performance parallels those for the H_2 performance in Proposition 2.1.

Proposition 2.2 (Robust H_∞ performance) *The following LMIs involving a Lyapunov variable X, a scaling pair (Q, R) and general slack matrix variables V, H and F enforces a bound γ on the L_2-induced gain of the operator mapping w into z. In different words, H_∞ performance is guaranteed for all parameter trajectories $\Delta(k) \in \mathcal{P}_{\mathbf{\Delta}}$.*

$$\begin{bmatrix} -X & * & * & * & * & * \\ 0 & Q & * & * & * & * \\ 0 & 0 & -\gamma I & * & * & * \\ V^T A & V^T B_\Delta & V^T B_1 & X - (V + V^T) & * & * \\ H^T C_\Delta & H^T D_{\Delta\Delta} & H^T D_{\Delta 1} & 0 & R - (H + H^T) & * \\ C_1 & D_{1\Delta} & D_{11} & 0 & 0 & -\gamma I \end{bmatrix} < 0,$$

$$\begin{bmatrix} R & \Delta^T F \\ F^T \Delta & Q + F + F^T \end{bmatrix} > 0, \qquad \forall \Delta = \Delta_i. \tag{5}$$

Proof: The proof is along the same lines of the proof of proposition 2.1. ■

3 Mixed H_2/H_∞ multi-channel LPV synthesis

3.1 Problem presentation

The general statement of the multi-channel mixed H_2/H_∞ LPV synthesis is detailed below. We are given an LPV plant with LFT structure

$$\begin{bmatrix} x(k+1) \\ z_\Delta(k) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_\Delta & B_1 & B_2 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ C_1 & D_{1\Delta} & D_{11} & D_{12} \\ C_2 & D_{2\Delta} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w_\Delta(k) \\ w(k) \\ u(k) \end{bmatrix},$$
$$w_\Delta(k) = \Delta(k)\, z_\Delta(k), \tag{6}$$

where Δ is defined in (2). Here x, w, w_Δ, z, and z_Δ have the same meaning as in Section 2, u is the control signal, and y is the measurement signal. The pair (w_Δ, z_Δ) is now regarded as the gain-scheduling channel.

For the LPV plant (6) the control problem consists in seeking an LPV controller with LFT structure

$$\begin{bmatrix} x_K(k+1) \\ u(k) \\ z_K(k) \end{bmatrix} = \begin{bmatrix} A_K & B_{K1} & B_{K\Delta} \\ C_{K1} & D_{K11} & D_{K1\Delta} \\ C_{K\Delta} & D_{K\Delta 1} & D_{K\Delta\Delta} \end{bmatrix} \begin{bmatrix} x_K(k) \\ y(k) \\ w_K(k) \end{bmatrix},$$
$$w_K(k) = \Delta_K(k) z_K(k), \tag{7}$$

such that H_2 and H_∞ specifications are achieved for a family of channels (w_1, z_1), (w_2, z_2), etc, where the w_j's and z_j's are sub-vectors of w and z, respectively. The notation Δ_K is used for the controller scheduling function which is a function of the plant parameter Δ, that is, $\Delta_K := \Delta_K(\Delta)$. This scheduling function is part of the design procedure and will be determined in the course of the derivation below.

3.2 LMI characterization

In order to derive closed-loop characterizations of H_2 and H_∞ performance, a standard procedure is to rewrite the LPV plant (6) as an augmented LPV plant with repeated blocks of delay operators $z^{-1}I_n$ and an augmented gain-scheduling block [17, 2]. The resulting closed-loop data are then described as

$$\left[\begin{array}{c|c|c} \mathcal{A} & \mathcal{B}_\Delta & \mathcal{B}_1 \\ \hline \mathcal{C}_\Delta & \mathcal{D}_{\Delta\Delta} & \mathcal{D}_{\Delta 1} \\ \hline \mathcal{C}_1 & \mathcal{D}_{1\Delta} & \mathcal{D}_{11} \end{array}\right] := \left[\begin{array}{cc|cc|c} A & 0 & B_\Delta & 0 & B_1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_\Delta & 0 & D_{\Delta\Delta} & 0 & D_{\Delta 1} \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_1 & 0 & D_{1\Delta} & 0 & D_{11} \end{array}\right]$$
$$+ \left[\begin{array}{ccc} 0 & B_2 & 0 \\ I & 0 & 0 \\ \hline 0 & D_{\Delta 2} & 0 \\ 0 & 0 & I \\ \hline 0 & D_{12} & 0 \end{array}\right] \mathcal{K} \left[\begin{array}{cc|cc|c} 0 & I & 0 & 0 & 0 \\ C_2 & 0 & D_{2\Delta} & 0 & D_{21} \\ 0 & 0 & 0 & I & 0 \end{array}\right], \tag{8}$$

with the definition

$$\mathcal{K} := \begin{bmatrix} A_K & B_{K1} & B_{K\Delta} \\ C_{K1} & D_{K11} & D_{K1\Delta} \\ C_{K\Delta} & D_{K\Delta 1} & D_{K\Delta\Delta} \end{bmatrix}.$$

The new uncertainty or parameter structure associated with the closed-loop data (8) is then given by

$$\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_K(\Delta) \end{bmatrix}.$$

With Each specification/channel is associated an LMI constraint of the form encountered in Propositions 2.1 and 2.2, LMIs (4) and (5). The desired characterization for LPV output-feedback synthesis with multi-objective/channel specifications can be derived in four steps:

1- introduce different Lyapunov variables and scalings (X_j, Z_j) and (Q_j, R_j) for each specification/channel. Also, an H_2 specification requires two pairs of scaling whereas only one is involved in an H_∞ specification. Note that the introduction of different Lyapunov functions and scalings is impractical in earlier developed techniques.

2- introduce slack variables V, H and F common to all channels and specifications.

3- write down expressions characterizing H_2 and H_∞ performance for each channel using Propositions 2.1, 2.2 and the closed-loop data $\mathcal{A}$, $\mathcal{B}_\Delta$, ... in (8).

4- perform adequate congruence transformations for each matrix inequality and use specific linearizing changes of variables to end up with LMI synthesis conditions.

Hereafter, we clarify the proposed procedure. Keeping in mind that all channels (w_1, z_1), (w_2, z_2), etc can be handled in the very same way, we shall only consider the case of an H_2 and H_∞ performance specification for the unique channel (w, z). This greatly simplifies the presentation below. When various channels are under consideration one will simply stack together the corresponding LMI constraints including additional Lyapunov variables and scalings.

In accordance with the partition of $\mathcal{A}$ and $\mathcal{D}_{\Delta\Delta}$ in (8), we introduce a partition of V and of its inverse $W := V^{-1}$, a partition of H and of its dual $G := H^{-1}$ and a partition of F and of its inverse $E := F^{-1}$ in the form

$$V := \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, W := \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, H := \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$
$$G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, F := \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, E := \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

By the strict nature of the LMI constraints involved and a perturbation argument, there is no loss of generality in assuming that V_{21}, W_{21}, H_{21}, G_{21}, F_{21} and E_{21} are invertible. See for instance [7] for a detailed justification. We then introduce the notations

$$\Pi_V := \begin{bmatrix} V_{11} & I \\ V_{21} & 0 \end{bmatrix}, \Pi_W := \begin{bmatrix} I & W_{11} \\ 0 & W_{21} \end{bmatrix}, \Pi_H := \begin{bmatrix} H_{11} & I \\ H_{21} & 0 \end{bmatrix},$$
$$\Pi_G := \begin{bmatrix} I & G_{11} \\ 0 & G_{21} \end{bmatrix}, \Pi_F := \begin{bmatrix} F_{11} & I \\ F_{21} & 0 \end{bmatrix}, \Pi_E := \begin{bmatrix} I & E_{11} \\ 0 & E_{21} \end{bmatrix}.$$

In turn, these matrices are invertible by the assumptions on V_{21}, W_{21}, H_{21}, G_{21}, F_{21} and E_{21}. One can easily verify the identities

$$V\Pi_W = \Pi_V,\ W\Pi_V = \Pi_W,\ H\Pi_G = \Pi_H,\\ G\Pi_H = \Pi_G\ F\Pi_E = \Pi_F,\ E\Pi_F = \Pi_E.$$

For an H_2 specification, we perform the congruence transformations

$$\mathrm{diag}(\Pi_W, \Pi_E, I, \Pi_W, \Pi_G), \qquad \mathrm{diag}(\Pi_W, \Pi_E, \Pi_G, I),$$

on the first and second inequalities (ii) of Proposition 2.1, respectively. For an H_∞ specification, we perform the congruence transformation

$$\mathrm{diag}(\Pi_W, \Pi_E, I, \Pi_W, \Pi_G, I)$$

in (5) of Proposition 2.2. For inequalities involving uncertainty blocks, last inequalities in (4) and (5), we perform the congruence transformation

$$\mathrm{diag}(\Pi_G, \Pi_E)\,.$$

This yields matrix inequalities which solely involves the terms

$$\begin{bmatrix} \Pi_V^T \mathcal{A} \Pi_W & \Pi_V^T \mathcal{B}_\Delta \Pi_E & \Pi_V^T \mathcal{B}_1 \\ \Pi_H^T \mathcal{C}_\Delta \Pi_W & \Pi_H^T \mathcal{D}_{\Delta\Delta} \Pi_E & \Pi_H^T \mathcal{D}_{\Delta 1} \\ \mathcal{C}_1 \Pi_W & \mathcal{D}_{1\Delta} \Pi_E & \mathcal{D}_{11} \end{bmatrix}, \tag{9}$$

and

$$\begin{matrix} \Pi_W^T X_j \Pi_W, & \Pi_E^T Q_j \Pi_E, & \Pi_G^T R_j \Pi_G\,, \\ \Pi_W^T V \Pi_W, & \Pi_G^T H \Pi_G, & \Pi_E^T F \Pi_E\,, \end{matrix} \tag{10}$$

and

$$\Pi_G^T \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_K \end{bmatrix}^T \Pi_F\,. \tag{11}$$

The variables (X_j, Z_j, Q_j, R_j) are attached to a given H_2 or H_∞ specification or channel, while (V, W, H, G, F, E) are slack variables common to all specifications and channels.

Explicit computation and inspection of these terms reveal that by invertibility of V_{21}, W_{21}, H_{21}, G_{21}, F_{21} and E_{21}, one can perform the following linearizing changes of variable:

$$\begin{aligned}
\boldsymbol{D_{K11}} &:= D_{K11}, && (12)\\
\boldsymbol{B_{K1}} &:= V_{21}^T B_{K1} + V_{11}^T B_2 D_{K11}, && (13)\\
\boldsymbol{C_{K1}} &:= D_{K11} C_2 W_{11} + C_{K1} W_{21}, && (14)\\
\boldsymbol{A_K} &:= V_{11}^T A W_{11} + V_{21}^T A_K W_{21} + V_{21}^T B_{K1} C_2 W_{11} \\
&\quad + V_{11}^T B_2 C_{K1} W_{21} + V_{11}^T B_2 D_{K11} C_2 W_{11}, && (15)\\
\boldsymbol{D_{K1\Delta}} &:= D_{K11} D_{2\Delta} E_{11} + D_{K1\Delta} E_{21}, && (16)\\
\boldsymbol{D_{K\Delta 1}} &:= H_{11}^T D_{\Delta 2} D_{K11} + H_{21}^T D_{K\Delta 1}\,, && (17)\\
\boldsymbol{B_{K\Delta}} &:= V_{11}^T B_\Delta E_{11} + V_{21}^T B_{K1} D_{2\Delta} E_{11} + V_{11}^T B_2 D_{K11} D_{2\Delta} E_{11} \\
&\quad + V_{21}^T B_{K\Delta} E_{21} + V_{11}^T B_2 D_{K1\Delta} E_{21}, && (18)\\
\boldsymbol{C_{K\Delta}} &:= H_{11}^T C_\Delta W_{11} + H_{11}^T D_{\Delta 2} D_{K11} C_2 W_{11} + H_{21}^T D_{K\Delta 1} C_2 W_{11} \\
&\quad + \ H_{11}^T D_{\Delta 2} C_{K1} W_{21} + H_{21}^T C_{K\Delta} W_{21}, && (19)\\
\boldsymbol{D_{K\Delta\Delta}} &:= H_{11}^T D_{\Delta\Delta} E_{11} + H_{11}^T D_{\Delta 2} D_{K11} D_{2\Delta} E_{11} \\
&\quad + H_{21}^T D_{K\Delta 1} D_{2\Delta} E_{11} \\
&\quad + H_{11}^T D_{\Delta 2} D_{K1\Delta} E_{21} + H_{21}^T D_{K\Delta\Delta} E_{21}\,, && (20)\\
\boldsymbol{X_j} &:= \Pi_W^T X_j \Pi_W\,, && (21)\\
\boldsymbol{Q_j} &:= \Pi_E^T Q_j \Pi_E, \quad \boldsymbol{R_j} := \Pi_G^T R_j \Pi_G\,, && (22)\\
\boldsymbol{U} &:= V_{11}^T W_{11} + V_{21}^T W_{21},\ \boldsymbol{M} := H_{11}^T G_{11} + H_{21}^T G_{21}, \\
\boldsymbol{N} &:= F_{11}^T E_{11} + F_{21}^T E_{21} && (23)\\
\boldsymbol{\Delta_K} &:= F_{11}^T \Delta G_{11} + F_{21}^T \Delta_K G_{21}\,. && (24)
\end{aligned}$$

Note that these transformations are back and forth because of the invertibility of V_{21}, W_{21}, H_{21}, G_{21}, F_{21} and E_{21}. The matrix inequality terms in (9)-(11) then become

$$\Pi_V^T \mathcal{A} \Pi_W := \begin{bmatrix} V_{11}^T A + \boldsymbol{B_{K1}} C_2 & \boldsymbol{A_K} \\ A + B_2 \boldsymbol{D_{K11}} C_2 & A W_{11} + B_2 \boldsymbol{C_{K1}} \end{bmatrix},$$

$$\Pi_V^T \mathcal{B}_\Delta \Pi_E := \begin{bmatrix} V_{11}^T B_\Delta + \boldsymbol{B_{K1}} D_{2\Delta} & \boldsymbol{B_{K\Delta}} \\ B_\Delta + B_2 \boldsymbol{D_{K11}} D_{2\Delta} & B_\Delta E_{11} + B_2 \boldsymbol{D_{K1\Delta}} \end{bmatrix},$$

$$\Pi_V^T \mathcal{B}_1 := \begin{bmatrix} V_{11}^T B_1 + \boldsymbol{B_{K1}} D_{21} \\ B_1 + B_2 \boldsymbol{D_{K11}} D_{21} \end{bmatrix}$$

$$\Pi_H^T \mathcal{C}_\Delta \Pi_W := \begin{bmatrix} H_{11}^T C_\Delta + \boldsymbol{D_{K\Delta 1}} C_2 & \boldsymbol{C_{K\Delta}} \\ C_\Delta + D_{\Delta 2} \boldsymbol{D_{K11}} C_2 & C_\Delta W_{11} + D_{\Delta 2} \boldsymbol{C_{K1}} \end{bmatrix},$$

$$\Pi_H^T \mathcal{D}_{\Delta 1} := \begin{bmatrix} H_{11}^T D_{\Delta 1} + \boldsymbol{D_{K\Delta 1}} D_{21} \\ D_{\Delta 1} + D_{\Delta 2} \boldsymbol{D_{K11}} D_{21} \end{bmatrix},$$

$$\Pi_H^T \mathcal{D}_{\Delta\Delta} \Pi_E := \begin{bmatrix} H_{11}^T D_{\Delta\Delta} + \boldsymbol{D_{K\Delta 1}} D_{2\Delta} & \boldsymbol{D_{K\Delta\Delta}} \\ D_{\Delta\Delta} + D_{\Delta 2} \boldsymbol{D_{K11}} D_{2\Delta} & D_{\Delta\Delta} E_{11} + D_{\Delta 2} \boldsymbol{D_{K1\Delta}} \end{bmatrix},$$

$$\mathcal{C}_1 \Pi_W := \begin{bmatrix} C_1 + D_{12} \boldsymbol{D_{K11}} C_2 & C_1 W_{11} + D_{12} \boldsymbol{C_{K1}} \end{bmatrix},$$

$$\mathcal{D}_{11} := D_{11} + D_{12} \boldsymbol{D_{K11}} D_{21},$$

$$\mathcal{D}_{1\Delta} \Pi_E := \begin{bmatrix} D_{1\Delta} + D_{12} \boldsymbol{D_{K11}} D_{2\Delta} & D_{1\Delta} E_{11} + D_{12} \boldsymbol{D_{K1\Delta}} \end{bmatrix},$$

$$\Pi_W^T \Pi_V := \begin{bmatrix} V_{11} & I \\ \boldsymbol{U}^T & W_{11}^T \end{bmatrix},\ \Pi_G^T \Pi_H := \begin{bmatrix} H_{11} & I \\ \boldsymbol{M}^T & G_{11}^T \end{bmatrix},$$

$$\Pi_E^T \Pi_F := \begin{bmatrix} F_{11} & I \\ \boldsymbol{N}^T & E_{11}^T \end{bmatrix}.$$

Note that in the H_2 case one must moreover satisfy the zero feedthrough constraints so that the H_2 performance index is well defined.

Thanks to these transformations, the inequalities of Propositions 2.1 and 2.2 which do not involve a parameter block Δ become LMIs as desired. Inequalities associated with the parameter block are rewritten

$$\begin{bmatrix} \boldsymbol{R}_{\mathbf{j},1} & \boldsymbol{R}_{\mathbf{j},2} & \Delta^T F_{11} & \Delta^T \\ \boldsymbol{R}_{\mathbf{j},2}^T & \boldsymbol{R}_{\mathbf{j},3} & \boldsymbol{\Delta_K}^T & G_{11}^T \Delta^T \\ F_{11}^T \Delta & \boldsymbol{\Delta_K} & \boldsymbol{Q}_{\mathbf{j},1} + F_{11} + F_{11}^T & \boldsymbol{Q}_{\mathbf{j},2} + I + \boldsymbol{N} \\ \Delta & \Delta G_{11} & \boldsymbol{Q}_{\mathbf{j},2}^T + \boldsymbol{N}^T + I & \boldsymbol{Q}_{\mathbf{j},3} + E_{11}^T + E_{11} \end{bmatrix} > 0,$$
$$\forall \Delta \in \mathcal{P}_\Delta\,,\ j = 1, \ldots\,. \tag{25}$$

They consist of a set indexed by j of parameterized inequalities with respect to Δ. Recalling that $\Delta(k)$ is evolving in a polytopic set $\mathcal{P}_{\boldsymbol{\Delta}}$, that is,

$$\Delta := \sum_{i=1}^{L} \alpha_i \Delta_i, \quad \sum_{i=1}^{L} \alpha_i = 1, \quad \alpha_i \geq 0,$$

solution candidates can be searched for in the form

$$\boldsymbol{\Delta_K}(\Delta) := \sum_{i=1}^{L} \alpha_i \boldsymbol{\Delta_{K,i}}\,,$$

where the $\boldsymbol{\Delta_{K,i}}$'s are decision variables and the α_i's are the polytopic coordinates of Δ in $\mathcal{P}_{\boldsymbol{\Delta}}$ (see [3] for other potential methods). Under this restriction, the constraints (25) are converted into a finite set of LMIs

$$\begin{bmatrix} \boldsymbol{R_{j,1}} & \boldsymbol{R_{j,2}} & \Delta_i^T & \Delta_i^T F_{11} \\ \boldsymbol{R_{j,2}^T} & \boldsymbol{R_{j,3}} & G_{11}^T \Delta_i^T & \boldsymbol{\Delta_{K,i}^T} \\ \Delta_i & \Delta_i G_{11} & \boldsymbol{Q_{j,1}} + F_{11} + F_{11}^T & \boldsymbol{Q_{j,2}} + I + \boldsymbol{N} \\ F_{11}^T \Delta_i & \boldsymbol{\Delta_{K,i}} & \boldsymbol{Q_{j,2}^T} + \boldsymbol{N}^T + I & \boldsymbol{Q_{j,3}} + E_{11}^T + E_{11} \end{bmatrix} > 0,$$

where i indexes the vertices of $\mathcal{P}_{\boldsymbol{\Delta}}$ and j indexes the channels and specifications.

Since for each channel and H_2 and H_∞ specifications, terms are of the form just derived, we conclude that sufficient existence conditions for the multi-objective/channel LPV control problem can be recast as an LMI program in the variables V_{11}, W_{11}, H_{11}, G_{11}, F_{11}, E_{11}, $\boldsymbol{\Delta_{K,i}}$ and the (bold) variables defined in (12)-(23). See the full version of the paper for detailed LMI descriptions.

3.3 LPV controller construction

Once a feasible solution of the LMI constraints has been computed, the state-space data (7) of the LPV controller are readily obtained as indicated below:

- compute a SVD factorization of $\boldsymbol{U} - V_{11}^T W_{11}$ and deduce invertible matrices V_{21} and W_{21} according to (23). Analogously, compute a SVD factorization of $\boldsymbol{M} - H_{11}^T G_{11}$ and $\boldsymbol{N} - F_{11}^T E_{11}$ and deduce invertible matrices H_{21} G_{21}, F_{21} and E_{21} according to (23).

- compute the LPV controller data by sequentially reverting the changes of variable as specified in (12)-(20) .

- deduce the controller gain-scheduling function as

$$\begin{aligned}\Delta_K(\Delta) &:= F_{21}^{-T}\left(\textstyle\sum_{i=1}^{L} \alpha_i \boldsymbol{\Delta_{K,i}} - F_{11}^T \sum_{i=1}^{L} \alpha_i \Delta_i G_{11}\right) G_{21}^{-1} \\ &:= \textstyle\sum_{i=1}^{L} \alpha_i \left(F_{21}^{-T} \boldsymbol{\Delta_{K,i}} G_{21}^{-1} - F_{21}^{-T} F_{11}^T \Delta_i G_{11} G_{21}^{-1}\right)\,.\end{aligned}$$

Hence, the scheduling function is affine in polytopic coordinates of the parameter block Δ. An enriched class of scheduling functions can be employed which however can play adversely in terms of computational time. If structured symmetric scalings were used, and the controller was forced to replicate the parameter block of the plant, i.e., $\Delta_K := \Delta$, then it can be showed that LMIs involving Δ blocks disappear. This simpler characterization is then equivalent to those in [17, 2] for the single-objective H_∞ control problem. It is, however, more conservative than that proposed in this paper.

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