Parameter-Dependent Lyapunov Functions for Real Parametric Uncertainty

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Submitted to *IEEE Trans. Aut. Contr.*, June 1994

Abstract

A new test of robust stability/performance is proposed for linear systems with uncertain real-valued parameters. This test is an extension of the notion of quadratic stability where the fixed quadratic Lyapunov function is replaced by a Lyapunov function with affine dependence on the uncertain parameters. Admittedly with some conservatism, the construction of such parameter-dependent Lyapunov functions can be reduced to an LMI problem, hence is numerically tractable.

This LMI-based test can be used for both fixed or time-varying uncertain parameters and is always less conservative than the quadratic stability test whenever the parameters cannot vary arbitrarily fast. It is also less conservative than the real μ upper bound for time-varying parameters, and several examples demonstrate that it compares advantageously with this upper bound even for fixed parameters.

Key words: Robust stability, Real parametric uncertainty, Lyapunov stability, Linear matrix inequalities.

1 Introduction

When designing control systems, it is often desirable to assess the robustness of stability and performance against uncertainty on the critical physical parameters of the system. Example of physical parameters include stiffness, inertia, or viscosity coefficients in mechanical systems, aerodynamical coefficients in flight control, the values of resistors and capacitors in electrical circuits, etc. Even though this problem is NP-hard in general, a number of more or less conservative tests are available to estimate stability regions. These include Kharitonov's theorem and related results [26, 5], quadratic stability tests [22, 6, 29, 25], and the real μ or K_m stability margins [14, 15, 33, 34].

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The discussion is restricted to linear systems of the form

$$\dot{x}(t) = A(\theta)\, x(t); \qquad x(0) = x_0 \tag{1.1}$$

where the state matrix $A(\theta)$ is a function of the vector $\theta = (\theta_1, \ldots, \theta_K) \in \mathbb{R}^K$ of real uncertain parameters. For technical reasons, it is important to distinguish between constant and time-varying parameters. Constant uncertain parameters have a fixed value that is known only approximately. In this case, the underlying dynamical system (1.1) is time-invariant. Time-varying parameters $\theta_i(t)$ are parameters whose value varies in some range $[\underline{\theta}_i, \bar{\theta}_i]$ during operation. The resulting dynamical (1.1) system is then time-varying. In general, robust stability against time-varying parameters is more demanding than robust stability against fixed but uncertain parameters (for the same range of uncertainty $[\underline{\theta}_i, \bar{\theta}_i]$).

When θ is time-invariant and each θ_i enters in only one coefficient of the characteristic polynomial

$$p(s) = \det(sI - A(\theta)),$$

Kharitonov's theorem gives a simple necessary and sufficient condition for robust stability. Unfortunately, this decoupling assumption is often restrictive in practice. For time-varying parameters, a numerically appealing test is based on the notion of quadratic stability. It consists of seeking a single matrix $P = P^T > 0$ such that the quadratic Lyapunov function $V(x) = x^T P x$ proves stability of (1.1) for all parameter trajectories $\theta(t)$. When each entry of $A(.)$ is a ratio of multilinear functions of θ, it is shown in [20] that finding an adequate P amounts to solving a system of Lyapunov inequalities, which is a convex program [6].

Quadratic stability guarantees stability against arbitrarily fast parameter variations. As a result, this test can be very conservative for constant or slowly-varying parameters, even in its refined form discussed in [40]. To reduce conservatism in the case of constant parameters, Barmish introduced the notion of parameter-dependent Lyapunov functions [3] . That is, of Lyapunov functions

$$V(x) = x^T P(\theta) x$$

where the Lyapunov matrix $P(\theta)$ is no longer constant, but is now a function of θ. Note that parameter-dependent Lyapunov functions are also discussed in [21] in a more abstract context. The discussion in [3] is restricted to the case of an affine dependence on θ. Even in this simple case, the condition $dV/dt < 0$ leads to a nonconvex optimization problem which does not seem tractable in general. In the special case of rank-one parametric uncertainty, [13] shows that robust stability is equivalent to the existence of an affine parameter-dependent Lyapunov matrix for some augmented system. This important result follows from a simple characterization of strict positive realness [2] for uncertain transfer functions. Unfortunately, This does not seem to extend beyond the rank-one case for which simpler necessary and sufficient conditions are already available.

In this paper, we propose a way of "convexifying" the general affine Lyapunov problem considered in [3]. By imposing additional constraints on the parameter-dependent Lyapunov functions, it is shown that finding a Lyapunov matrix of the form

$$P(\theta) = P_0 + \theta_1 P_1 + \ldots + \theta_K P_K \tag{1.2}$$

can be turned into a linear matrix inequality (LMI) problem of unknown matrices $P_0, P_1, \ldots, P_K$. LMI problems are convex and efficient polynomial-time optimization algorithms are available to solve them [28, 7, 37, 27, 8]. The resulting test is therefore numerically tractable while always less conservative than quadratic stability. Moreover, our test is not restricted to constant parameters, but is also applicable to varying parameters with a well-defined time derivative. In fact, the rate of variation of such parameters can be quantitatively accounted for. This provides a continuous transition between the two extreme cases of constant parameters on the one end, and of arbitrarily fast parameter variations on the other. This feature is valuable and quite unique among available robust stability tests.

The paper is organized as follows. Section 2 recalls the notion of quadratic stability and its LMI formulation when $A(\theta)$ is affine in θ. This notion is then generalized to Lyapunov functions that are affine in θ as well. Section 3 focuses on constant parametric uncertainty and derives LMI-based sufficient conditions for the existence of parameter-dependent Lyapunov matrices of the form (1.2). These LMI conditions are extended to the case of time-varying parameters in Section 4 and refined conditions are proposed in Section 5. Extensions to robust performance assessment are addressed in Section 6 with an emphasis on robust $\mathcal{H}_\infty$ performance. Finally, Section 7 discusses the numerical implementation of this LMI-based robustness test and Section 8 illustrates its performance by a variety of examples. Interestingly, experimental results indicate that the new test is often less conservative than the real μ upper bound, even in the case of constant uncertain parameters. Quantitative insight into the destabilizing effect of time-varying parameters is also provided by this test.

The notation used throughout the paper is fairly standard. For real symmetric matrices M, $M > 0$ stands for "positive definite" and means that all the eigenvalues of M are positive. Similarly, $M < 0$ means "negative definite" (all the eigenvalues of M are negative) and $M \geq 0$ stands for "nonnegative definite" (the smallest eigenvalue of M is nonnegative). The L_2 norm of a causal signal $w(t)$ is defined as

$$\|w\|_2 := \int_0^\infty w(t)^T w(t) dt \ .$$

and the L_2-induced norm of an operator T mapping L_2 into L_2 is denoted by $\|T\|_{L_2}$.

2 Affine Quadratic Stability

Consider the linear system (1.1). Throughout the paper, we assume that

1. each parameter θ_i is real and ranges between known extremal values $\underline{\theta}_i$ and $\bar{\theta}_i$:

$$\theta_i \in [\underline{\theta}_i, \bar{\theta}_i], \tag{2.3}$$

2. the state matrix $A(\theta)$ depends *affinely* on the parameters θ_i. That is,

$$A(\theta) = A_0 + \theta_1 A_1 + \ldots + \theta_K A_K \tag{2.4}$$

where $A_0, A_1, \ldots A_K$ are known fixed matrices.

The first assumption means that the parameter vector θ is valued in an hyperrectangle called the parameter box. In the sequel,

$$\mathcal{V} := \left\{ \omega = (\omega_1, \ldots, \omega_K) \ : \ \omega_i \in \{\underline{\theta}_i, \bar{\theta}_i\} \right\} \tag{2.5}$$

denotes the set of the 2^K vertices or corners of the parameter box. The second assumption is introduced for technical and simplicity reasons. Note that numerous extensions of this approach to more complex parameter dependences are possible. Though somewhat restrictive, the affine model still covers a wide variety of relevant problems. Henceforth, affine parametric uncertainty will refer to the dependence (2.4).

For constant uncertain real parameters, available robust stability tests include Kharitonov's theorem [26, 4, 12] and extensions, and the real $\mu/K_{\rm m}$ upper bound with frequency-dependent D, G scaling matrices [15, 34]. Both tests are numerically tractable, yet conservative except for some special parameter dependence structures $A(\theta)$. For real time-varying parameters, there are essentially two tractable tests of robust stability: quadratic stability and the real μ upper bound with constant D, G scaling matrices, the second being more conservative in the case of affine real parametric uncertainty [39]. Note that both criteria guarantee robustness against *arbitrarily fast* time variations of the parameters [36]. Finally, the quadratic stability test is also applicable to constant uncertain parameters, yet at the expense of overly conservative answers in general (see Section 7).

Since our approach builds upon quadratic stability, we now review the details of the quadratic stability test. This test seeks a matrix $P > 0$ such that $V(x) = x^T P x$ is a Lyapunov function for the time-varying differential inclusion (1.1). That is, such that

$$\frac{dV}{dt}(x(t)) = x^T \left(A(\theta(t))^T P + P A(\theta(t)) \right) x \leq 0 \tag{2.6}$$

along all parameter trajectories $\theta(t)$. Note that this guarantees the asymptotic stability of the dynamical system governed by (1.1). Under the affine dependence assumption (2.4), it can be shown that (2.6) holds if and only if P satisfies the system of LMIs [22, 6]:

$$\begin{aligned} A(\omega)^T P + P A(\omega) &< 0 \qquad \text{for all } \omega \in \mathcal{V} \\ P &> I. \end{aligned} \tag{2.7}$$

In other words, it suffices that P be positive definite and satisfy the Lyapunov inequality at each corner of the parameter box. This reformulation has the merit of reducing a problem with infinitely many constraints to a *finite* set of matrix inequalities. The resulting LMI system (2.7) is then readily solved numerically with existing LMI software [37, 17].

To reduce conservatism in the case of *constant* parameters, Barmish [3] considers Lyapunov functions that depend affinely on the parameters θ_i. The set-up there is slightly more general as A is assumed to range in a polytope of matrices. Specifically,

$$A = q_1 A_1 + \ldots + q_K A_K$$

where $A_1, \ldots, A_K$ are fixed, $q_i \geq 0$, and $\sum_i q_i = 1$. Here the q_i's are not parameter values, but the coefficients of a convex decomposition of A over the set $\{A_1, \ldots, A_K\}$ of vertices of the polytope. In our case, the counterpart of Barmish's affine Lyapunov functions reads

$$V(x) = x^T P(\theta) x \tag{2.8}$$

where the Lyapunov matrix $P(\theta)$ is of the form

$$P(\theta) = P_0 + \theta_1 P_1 + \ldots + \theta_K P_K. \tag{2.9}$$

Such affine parameter-dependent Lyapunov functions are central to our approach. Note that the usual quadratic stability corresponds to the special case when $P_1 = \ldots = P_K = 0$. This suggests the following natural extension of quadratic stability.

Definition 2.1 (Affine Quadratic Stability)
The linear system

$$\dot{x}(t) = A(\theta(t))\, x(t); \qquad x(0) = x_0 \tag{2.10}$$

is affinely quadratically stable (AQS) if there exist $K+1$ symmetric matrices $P_0, \ldots, P_K$ such that

$$P(\theta) \;:= P_0 + \theta_1 P_1 + \ldots + \theta_K P_K \;>\; I \tag{2.11}$$

$$A(\theta)^T P(\theta) + P(\theta) A(\theta) + \frac{dP(\theta)}{dt} \;<\; 0 \tag{2.12}$$

hold for all admissible values and trajectories of the parameter vector $\theta = (\theta_1, \ldots, \theta_K)$.

The function $V(x,\theta) := x^T P(\theta) x$ is then a quadratic Lyapunov function for (2.10). That is, $V(x,\theta) > 0$ for all nonzero x and $\frac{d}{dt} V(x,\theta) \le 0$ for all x_0 and parameter trajectories $\theta(t)$.

■

Note that the negativity of the Lyapunov derivative readily follows from (2.12) and the identity:

$$\frac{dV(x,\theta)}{dt} = x^T \left(A(\theta)^T P(\theta) + P(\theta) A(\theta) + \frac{dP(\theta)}{dt} \right) x.$$

From this definition, Affine Quadratic Stability amounts to finding $K+1$ symmetric matrices $P_0, \ldots, P_K$ that satisfy (2.11)–(2.12). This task is now discussed first in the simpler case of constant uncertain parameters, then in the general case of time-varying parameters.

3 Constant Uncertain Parameters

Throughout this section, the real uncertain parameters θ_i are assumed to be *time-invariant* and valued in the interval $[\underline{\theta}_i, \bar{\theta}_i]$. As a result, the condition (2.12) reduces to

$$L(\theta) \;:= A(\theta)^T P(\theta) + P(\theta) A(\theta) < 0. \tag{3.1}$$

This inequality must hold for a continuum of values of θ. Even when $A(\theta)$ and $P(\theta)$ are affine in θ, assessing whether (1.1) is AQS is not tractable in general, neither analytically nor numerically. In particular, it is no longer sufficient to check (2.11) and (3.1) at the corners of the parameter box as for the quadratic stability test (see [3] for details).

To recover convexity, we must introduce an additional constraint on $P(\theta)$. This constraint restricts the choice of affine Lyapunov matrix $P(\theta)$, hence increases conservatism. Nevertheless, the resulting criterion will always improve on the standard quadratic stability test which corresponds to setting $P_1 = \ldots = P_K = 0$. The next theorem proposes one possible way of enforcing convexity. This approach relies on the concept of *multi-convexity*, that is, convexity with respect to each θ_i when all other θ_j are fixed. Note that this property is less demanding than global convexity in θ.

Theorem 3.1 *Let $\theta = (\theta_1, \ldots, \theta_K)$ be a vector of fixed but uncertain real parameters ranging in the hyperrectangle defined by (2.3), and let $\mathcal{V}$ denote the set of vertices of this hyperrectangle as defined in (1.2). Consider a linear time-invariant system governed by*

$$\dot{x} = A(\theta)\, x \tag{3.2}$$

where $A(\theta)$ depends affinely on θ according to (2.4).

A sufficient condition for AQS of this system is the existence of $K+1$ symmetric matrices $P_0, \ldots, P_K$ such that

$$\begin{aligned} A(\omega)^T P(\omega) + P(\omega) A(\omega) &< 0 \quad \textit{for all } \omega \in \mathcal{V}, && (3.3)\\ P(\omega) &> I \quad \textit{for all } \omega \in \mathcal{V}, && (3.4)\\ A_i^T P_i + P_i A_i &\geq 0 \quad \textit{for i=1,}\ldots\textit{,K} && (3.5)\end{aligned}$$

where

$$P(\theta) \;:= P_0 + \theta_1 P_1 + \ldots + \theta_K P_K. \tag{3.6}$$

When the LMI system (3.3)–(3.5) is feasible, $V(x,\theta) \;:= x^T P(\theta) x$ is a Lyapunov function for (3.2) for all values of θ_i in $[\underline{\theta}_i, \bar{\theta}_i]$.

Proof: First note that the positivity constraint (2.11) is affine in θ_i. Consequently, (2.11) holds for all θ in the parameter box if and only if it holds at all corners [6], which is exactly the condition (3.4). Hence the only difficulty is to enforce of (3.1) over the entire parameter box.

Using the affine expressions (2.4) and (3.6), this condition reads

$$\begin{aligned} L(\theta) = A_0^T P_0 + P_0 A_0 \;&+\; \sum_i \theta_i \,(A_0^T P_i + P_i A_0 + A_i^T P_0 + P_0 A_i)\\ &+\; \sum_{i<j} \theta_i \theta_j \,(A_i^T P_j + P_j A_i + A_j^T P_i + P_i A_j)\\ &+\; \sum_i \theta_i^2 \,(A_i^T P_i + P_i A_i) \;\; < 0. \end{aligned} \tag{3.7}$$

Take any nonzero vector x. Clearly $f(\theta) \;:= x^T L(\theta) x$ is a quadratic scalar function of the form

$$f(\theta_1, \ldots, \theta_K) = \alpha_0 + \sum_i \alpha_i \theta_i + \sum_{i<j} \beta_{ij} \theta_i \theta_j + \sum_i \gamma_i \theta_i^2. \tag{3.8}$$

In general, the negativity of $f(.)$ at all corners in $\mathcal{V}$ does not guarantee its negativity over the entire parameter box. This is the case, however, when $f(.)$ is multi-convex in the θ_i's. That is, when

$$\frac{\partial^2 f}{\partial \theta_i{}^2}(\theta) \geq 0, \qquad i = 1, \ldots, K \tag{3.9}$$

for all θ. Before justifying this claim, note that (3.9) is equivalent to

$$\gamma_i = x^T (A_i^T P_i + P_i A_i) x \geq 0, \qquad i = 1, \ldots, K.$$

Hence the multi-convexity requirement is equivalent to the additional constraint (3.5) in the theorem statement.

Back to our claim, assume that $\gamma_i \geq 0$ for all i and let $\theta^* = (\theta_1^*, \ldots, \theta_K^*)$ be the global maximizer of $f(.)$ over the parameter box. If θ^* is not a corner of the parameter box, we have $\underline{\theta}_i < \theta_i^* < \bar{\theta}_i$ for some i. Then consider the quadratic polynomial

$$g(\theta_i) \ := f(\theta_1^*, \ldots, \theta_{i-1}^*, \theta_i, \theta_{i+1}^*, \ldots, \theta_K^*) = a + b\theta_i + \gamma_i \theta_i^2$$

obtained by setting $\theta_j = \theta_j^*$ for $j \neq i$. This function of θ_i is convex from our multi-convexity constraint $\gamma_i \geq 0$, hence its maximum on $[\underline{\theta}_i, \bar{\theta}_i]$ is attained at the extremities. Consequently,

$$g(\theta_i^*) \leq \max(g(\underline{\theta}_i), g(\bar{\theta}_i))$$

But we also have

$$\max(g(\underline{\theta}_i), g(\bar{\theta}_i)) \leq g(\theta_i^*)$$

since θ^* is the global maximizer of f, and therefore

$$g(\theta_i^*) = \max(g(\underline{\theta}_i), g(\bar{\theta}_i)).$$

It follows that the maximum of $g(.)$ is also attained in the set $\{\underline{\theta}_i, \bar{\theta}_i\}$ of extreme values of θ_i. Repeating the argument for all i, we conclude that f attains its maximum at some corner of the parameter box.

To complete the proof, observe that (3.3) ensures the negativity of f at all corners of the parameter box. Consequently, for nonzero x we have $x^T L(\theta) x < 0$ over the entire parameter box, from which we conclude that $L(\theta) < 0$ for all admissible θ.

■

Summing up, the additional constraint (3.5) reduces the problem of finding affine parameter-dependent Lyapunov matrices to an LMI problem. Though somewhat restrictive, this still provides a significant number of additional degrees of freedom when compared to quadratic stability (case $P_1 = \ldots = P_K = 0$). In light of the examples in Section 8, the resulting test seems all but overly conservative.

Finally, note that in the case of constant parameters, the sufficient conditions of Theorem 3.1 can be combined with a branch-and-bound scheme to reduce conservatism. Specifically, if this test cannot establish AQS over the entire parameter box, the initial hyperrectangle can be divided into smaller hyperrectangles and the test reapplied to each of these hyperrectangles. For a complete discussion of branch-and-bound techniques, see [9, 10].

4 Time-Varying Uncertain Parameters

We now turn to the case of time-varying parameters $\theta_i(t)$ with a bounded rate of variation. As shown below, this more general case can be handled by a minor modification of Theorem 3.1 and the resulting LMI conditions remain less conservative than the quadratic stability test. Throughout the section we assume the following on the parameter variations:

1. the time derivative $\dot{\theta}_i$ is well-defined at all time,
2. the rate of variation $\dot{\theta}_i$ of θ_i satisfies
$$\dot{\theta}_i \in [\underline{\nu}_i, \bar{\nu}_i] \tag{4.1}$$
where $\underline{\nu}_i, \bar{\nu}_i$ are known lower and upper bounds on this rate of variation.

Note that 2. allows for a more accurate modeling of the rate of variation than a mere bound on $|\dot{\theta}_i|$. In particular, sign information is readily included.

To handle the time-varying case, we consider $\dot{\theta}_1, \ldots, \dot{\theta}_K$ as additional time-varying uncertain parameters. As a whole, the vector $\dot{\theta}$ evolves in a K-dimensional hyperrectangle whose vertices are in the set:

$$\mathcal{T} \;:= \{\tau = (\tau_1, \ldots, \tau_K) \;:\; \tau_i \in \{\underline{\nu}_i, \bar{\nu}_i\}\;\} \tag{4.2}$$

With our assumptions and for $P(\theta)$ of the form (3.6), we obtain

$$\frac{dP(\theta)}{dt} = \dot{\theta}_1 P_1 + \ldots + \dot{\theta}_K P_K = P(\dot{\theta}) - P_0. \tag{4.3}$$

The results of Theorem 3.1 can then be generalized to the time-varying case as follows.

Theorem 4.1 *Consider a linear parameter-dependent system governed by (2.10) where*

- *$A(\theta)$ depends affinely on the time-varying parameter vector $\theta(t) = (\theta_1(t), \ldots, \theta_K(t))$ according to (2.4)*
- *$\theta(t)$ satisfies (2.3) and (4.1) ,*

and denote by $\mathcal{V}$ and $\mathcal{T}$ the sets of corners of the parameter box (1.2) and of the rate-of-variation box (4.2), respectively.

With these assumptions, a sufficient condition for the Affine Quadratic Stability of this system is the existence of $K+1$ symmetric matrices $P_0, \ldots, P_K$ satisfying

$$A(\omega)^T P(\omega) + P(\omega)A(\omega) + P(\tau) \;<\; P_0 \quad \text{for all } \omega \in \mathcal{V} \text{ and } \tau \in \mathcal{T}\;, \tag{4.4}$$

$$P(\omega) \;>\; I \quad \text{for all } \omega \in \mathcal{V}, \tag{4.5}$$

$$A_i^T P_i + P_i A_i \;\geq\; 0 \quad \text{for } i=1,\ldots,K \tag{4.6}$$

where

$$P(\theta) \;:= P_0 + \theta_1 P_1 + \ldots + \theta_K P_K. \tag{4.7}$$

When the LMI system (4.4)–(4.6) is feasible, a Lyapunov function for (2.10) and all trajectories $\theta(t)$ satisfying (2.3) and (4.1) is then given by

$$V(x, \theta) \;:= x^T P(\theta) x.$$

Proof: The only difference with Theorem 3.1 is the additional term $dP(\theta)/dt$ in (2.12). Temporarily fix τ so that $P(\tau) - P_0$ can be regarded as a constant term. By an immediate extension of Theorem 3.1, the fact that (4.4)–(4.6) hold for all $\omega \in \mathcal{V}$ ensures

$$
\begin{aligned}
P(\theta) &> 0 && (4.8)\\
A(\theta)^T P(\theta) + P(\theta)A(\theta) + P(\tau) - P_0 &< 0 && (4.9)
\end{aligned}
$$

for any value of θ in the parameter box. Since this holds for any vertex $\tau \in \mathcal{T}$ and since $P(\dot{\theta})$ is affine in the $\dot{\theta}_i$, we deduce by a standard convexity argument that the inequality

$$A(\theta)^T P(\theta) + P(\theta)A(\theta) + P(\dot{\theta}) - P_0 < 0 \tag{4.10}$$

holds over the entire rate-of-variation box (4.1) since it holds at all its corners $\tau \in \mathcal{T}$. Comparing with (4.3), it follows that

$$\frac{dV}{dt}(x(t), \theta(t)) = A(\theta)^T P(\theta) + P(\theta)A(\theta) + \frac{dP(\theta)}{dt} < 0$$

for any parameter trajectory $\theta(t)$ satisfying (2.3) and (4.1), which establishes AQS.

■

Comparing with Theorem 3.1, the sufficient conditions for the time-varying case retain the same LMI structure, the only difference being an increased computational burden. Specifically, the constant parameter case requires solving a system of $2^{K+1} + K$ inequalities, while the fully time-varying case involves

$$2^{2K} + 2^K + K = 4^K + 2^K + K$$

inequalities.

Theorem 4.1 provides a valuable bridge between the two extreme cases addressed by Theorem 3.1 and the quadratic stability test. Specifically, the conditions of Theorem 4.1 reduce to those of Theorem 3.1 in the case of constant uncertain parameters, and to the quadratic stability test in the case of arbitrarily fast parameter variations. Consider the first case for instance. Constant parameters correspond to $\dot{\theta}_1 = \ldots = \dot{\theta}_K = 0$, that is, $\underline{\nu}_i = \bar{\nu}_i = 0$ in (4.1). The set $\mathcal{T}$ then reduces to the single element $(0, \ldots, 0)$. Hence $P(\tau) = P_0$ for all $\tau \in \mathcal{T}$ and we recover the AQS conditions of Theorem 3.1.

Similarly, arbitrarily fast variations of θ_i correspond to setting $\underline{\nu}_i = -\infty$ and $\bar{\nu}_i = +\infty$ in (4.1). Consider the case of a single parameter θ_1 for simplicity. ¿From (2.12), the inequality

$$A(\theta_1)^T P(\theta_1) + P(\theta_1)A(\theta_1) + \dot{\theta}_1 P_1 < 0$$

must hold for all values of $\dot{\theta}_1$ in $[-\infty, +\infty]$, which clearly requires $P_1 = 0$. More generally, we must have $P_1 = \ldots = P_K = 0$ in the vector case. In other words, the only affine Lyapunov functions $P(\theta)$ that can withstand arbitrarily fast parameter variations are the constant functions

$$V(x) = x^T P_0 x,$$

which brings us back to the quadratic stability test.

In light of these comments, our conditions readily apply to mixed-cases where some parameters are constant and other time-varying. To specify that θ_i is constant, simply set $\underline{\nu}_i = \bar{\nu}_i = 0$ in condition (4.4). Similarly, set $P_i = 0$ if θ_i can vary arbitrarily fast (i.e., when $|\dot{\theta}_i|$ can take very large values, or when θ_i can vary discontinuously).

Finally, note that in the face of real varying parameters with bounded rate of variation, the sufficient conditions of Theorem 4.1 are always *less conservative* than the quadratic stability test or the real μ upper bound. This is obvious for quadratic stability which is a special case of Theorem 4.1. As for the real μ upper bound, it happens to be even more conservative than quadratic stability in the context of real varying parameters. Indeed, constant D, G scales must be used for varying parameters regardless of their rate of variation, and quadratic stability is less conservative than the Small Gain condition with constant scalings (see [39] for details).

5 Refinement of the AQS Tests

The affine quadratic stability tests introduced in Sections 3 and 4 can be rendered less conservative by somewhat relaxing the multi-convexity requirement. The idea is as follows. Rather than imposing the multi-convexity of the quadratic form $x^T(dV/dt(\theta))x$, we only require that it be bounded from above by a multi-convex function. For simplicity, details are given only in the case of constant parameters. Consider K symmetric matrices $M_1, \ldots, M_K$ such that

$$A_i^T P_i + P_i A_i + M_i \geq 0 \,, \qquad M_i \geq 0 \tag{5.1}$$

and define

$$L_{\rm ub}(\theta) \;:= L(\theta) + \sum_{i=1}^{K} \theta_i^2 M_i \;. \tag{5.2}$$

where $L(\theta)$ is given by (3.1). Clearly

$$L(\theta) \leq L_{\rm ub}(\theta) \tag{5.3}$$

for all θ since $M_i \geq 0$. Moreover, $x^T L_{\rm ub}(\theta) x$ is a multi-convex function of θ by the same argument as in Theorem 3.1. Indeed, $L_{\rm ub}(\theta)$ has an expansion of the form (3.7) where $A_i^T P_i + P_i A_i + M_i$ replaces $A_i^T P_i + P_i A_i$, and the multi-convexity property follows from $A_i^T P_i + P_i A_i + M_i \geq 0$ for the same reasons.

From (5.3), it is sufficient to enforce $L_{\rm ub}(\theta) < 0$ and $P(\theta) > 0$ to prove AQS, and this is readily expressed as a *finite* number of LMI constraints thanks to the multi-convexity of $L_{\rm ub}(\theta)$. This approach leads to the following refinement of Theorem 3.1.

Theorem 5.1 *Consider a linear parameter-dependent system governed by (2.10) and satisfying the assumptions of Theorem 4.1, and let $\mathcal{V}$ and $\mathcal{T}$ denote the sets of corners of the parameter box (1.2) and of the rate-of-variation box (4.2), respectively.*

A sufficient condition for the affine quadratic stability of this system is the existence of $K+1$ *symmetric matrices* $P_0, \ldots, P_K$ *and* K *symmetric matrices* $M_1, \ldots, M_K$ *such that*

$$L_{\text{ub}}(\omega) + P(\tau) \ < \ P_0 \quad \textit{for all } \omega \in \mathcal{V} \textit{ and } \tau \in \mathcal{T}\ , \tag{5.4}$$

$$P(\omega) \ > \ I \quad \textit{for all } \omega \in \mathcal{V}, \tag{5.5}$$

$$A_i^T P_i + P_i A_i + M_i \ \geq \ 0 \quad \textit{for } i=1,\ldots,K \tag{5.6}$$

$$M_i \ \geq \ 0 \quad \textit{for } i=1,\ldots,K \tag{5.7}$$

where

$$L_{\text{ub}}(\theta) \ := A(\theta)^T P(\theta) + P(\theta) A(\theta) + \sum_{i=1}^{K} \theta_i^2 M_i$$

and $P(.)$ *is defined in (4.7).*

When the LMI system (5.4)–(5.7) is feasible,

$$V(x,\theta) \ := x^T P(\theta) x$$

is a Lyapunov function for (2.10) and all trajectories $\theta(t)$ *satisfying (2.3) and (4.1).*

Proof: The proof is readily adapted from those of Theorem 3.1 and 4.1, and is omitted for brevity. Note that $P(\omega)$, $P(\tau)$, and $L_{\text{ub}}(\omega)$ are all affine expressions in the unknowns $\{P_i\}_{i=0}^K$ and $\{M_i\}_{i=1}^K$. Hence (5.4)–(5.7) is indeed an LMI system.

■

The conditions of Theorem 4.1 correspond to the special case when $M_i = 0$ for $i = 1, \ldots, K$. This corresponds to taking $L_{\text{ub}} = L$, or equivalently to requiring the multiconvexity of L itself. Because of the additional degrees of freedom attached to the M_i variables, these refined conditions should be less conservative than those of Theorem 4.1. However, this improvement is at the expense of the computational overhead since the number of optimization variables is roughly doubled in the new LMI system (5.4)–(5.7). For this reason, the AQS test of Theorem 4.1 may be worth trying first when running times are of primary concern.

6 Affine Quadratic $\mathcal{H}_\infty$ Performance

Since Affine Quadratic Stability is a Lyapunov-based concept, there are numerous immediate extensions to various Lyapunov-based performance measures. These include $\mathcal{H}_2$ performance [30], positivity [2], $\mathcal{H}_\infty$ performance [32, 35, 18], and their extensions using the concept of multipliers [38, 23, 16]. For a fairly complete overview, see [8]. For the sake of illustration, we now discuss in detail the extension to robust H_∞ performance assessment for linear systems subject to affine parametric uncertainty. Specifically, consider a system described in state-space by

$$\begin{cases} \dot{x} &= A(\theta(t))\, x + B(\theta(t))\, w \\ z &= C(\theta(t))\, x + D(\theta(t))\, w \end{cases} \tag{6.1}$$

where (1) x, w, z denote the state vector, the exogenous input, and the performance output, respectively, and (2) the dependence on θ is again assumed affine, that is,

$$\begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} + \theta_1 \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} + \ldots + \theta_K \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} . \tag{6.2}$$

Throughout this section, the assumptions of Section 4 on the parameter vector θ and its time derivative $\dot{\theta}$ remain in force.

Paralleling Definition 2.1, a notion of Affine Quadratic $\mathcal{H}_\infty$ Performance (AQP) can be ironed as follows. The system (6.1) has AQP γ if there exists a quadratic Lyapunov $V(x,\theta) := x^T P(\theta)x$, affine in θ, that simultaneously enforces the two properties:

(i) the system (6.1) is internally stable,

(ii) the L_2-induced norm of the operator T_θ mapping w to z satisfies

$$\|T_\theta\|_{L_2} < \gamma . \tag{6.3}$$

Both properties must hold along all trajectories $\theta(t)$ in the prescribed box of parameter variations as defined by the vertex set $\mathcal{V} \times \mathcal{T}$. If θ is time-invariant, $\|T_\theta\|_{L_2}$ is simply the $\mathcal{H}_\infty$-norm of the transfer function associated with (6.1). The meaning of "enforces" is made more precise by the following formal definition.

Definition 6.1 (Affine Quadratic $\mathcal{H}_\infty$ Performance)
The system (6.1) has Affine Quadratic $\mathcal{H}_\infty$ Performance γ if there exist $K+1$ symmetric matrices $P_0, \ldots, P_K$ such that

$$P(\theta) := P_0 + \theta_1 P_1 + \ldots + \theta_K P_K > 0 \tag{6.4}$$

$$\begin{pmatrix} A(\theta)^T P(\theta) + P(\theta)A(\theta) + P(\dot{\theta}) - P_0 & P(\theta)B(\theta) & C(\theta)^T \\ B(\theta)^T P(\theta) & -\gamma I & D(\theta)^T \\ C(\theta) & D(\theta) & -\gamma I \end{pmatrix} < 0 \tag{6.5}$$

holds for all admissible values of the parameter vector $\theta = (\theta_1, \ldots, \theta_K)$ and of its time derivative $\dot{\theta}$.

In such case, the quadratic function $V(x,\theta) := x^T P(\theta)x$ establishes both the internal stability and the $\mathcal{H}_\infty$ performance bound (6.3).

■

To be convinced of this last claim, first note that (6.4) ensures the positivity of $V(x,\theta) = x^T P(\theta)x$ for all admissible parameter values. Meanwhile, the $(1,1)$-block of (6.5) enforces $dV/dt(x,\theta) < 0$ (see Section 4 and (4.9) for details). Hence the system is asymptotically stable. Finally, it is easily verified from (6.5) that

$$\frac{dV}{dt}(x,\theta) + z^T z - \gamma^2 w^T w < 0 \tag{6.6}$$

for all parameter trajectory $\theta(t)$ and input signal $w(t)$. Integrating (6.6) from $t = 0$ to $t = +\infty$ with initial condition $x(0) = 0$, it follows that

$$\|z\|_2{}^2 - \gamma^2\|w\|_2{}^2 + V(x(\infty)) - V(x(0)) < 0,$$

or equivalently $\|z\|_2{}^2 < \gamma^2\|w\|_2{}^2$ since $x(\infty) = 0$ and $V(0) = 0$. Since this last inequality holds for any $w \in L_2$, the L_2-induced norm of the mapping T_θ is less than γ. Note that (6.5) reduces to the well-known Bounded Real Lemma inequality [1] when θ is time-invariant and $P(\theta) \equiv P_0$.

As for AQS, (6.4)–(6.5) puts an infinite number of constraints on the unknowns $P_0, \ldots, P_K$. To render the problem numerically tractable, we again add a multi-convexity constraint in order to reduce (6.4)–(6.5) to a system of finitely many LMIs. The resulting conditions are similar to those of Theorem 5.1 and presented without proof.

Theorem 6.2 *Consider a linear parameter-dependent system governed by (6.1) where*

(1) *$A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$ depend affinely on the time-varying parameter vector $\theta(t)$ according to (6.2), and*

(2) *$\theta(t)$ satisfies (2.3) and (4.1).*

Let $\mathcal{V}$ and $\mathcal{T}$ denote the sets of corners of the parameter box (1.2) and of the rate-of-variation box (4.2), respectively.

This system has Affine Quadratic $\mathcal{H}_\infty$ Performance γ if there exist $K+1$ symmetric matrices $P_0, \ldots, P_K$ and K symmetric matrices $N_1, \ldots, N_K$ such that

$$\left(\begin{array}{c|c} L_\infty(\omega) + \begin{pmatrix} P(\tau) - P_0 & 0 \\ 0 & -\gamma I \end{pmatrix} & \begin{matrix} C(\omega)^T \\ D(\omega)^T \end{matrix} \\ \hline \begin{matrix} C(\omega) & \quad D(\omega) \end{matrix} & -\gamma I \end{array}\right) < 0 \qquad \forall(\omega,\tau) \in \mathcal{V} \times \mathcal{T} \tag{6.7}$$

$$P(\omega) > 0 \qquad \forall \omega \in \mathcal{V}, \tag{6.8}$$

$$\begin{pmatrix} A_i^T P_i + P_i A_i & P_i B_i \\ B_i^T P_i & 0 \end{pmatrix} + N_i \geq 0 \qquad \text{for } i = 1, \ldots, K \tag{6.9}$$

$$N_i \geq 0 \qquad \text{for } i = 1, \ldots, K \tag{6.10}$$

with the notation

$$\begin{aligned} P(\theta) &:= P_0 + \theta_1 P_1 + \ldots + \theta_K P_K \\ L_\infty(\theta) &:= \begin{pmatrix} A(\theta)^T P(\theta) + P(\theta)A(\theta) & P(\theta)B(\theta) \\ B(\theta)^T P(\theta) & 0 \end{pmatrix} + \sum_{i=1}^{K} \theta_i^2 N_i. \end{aligned}$$

As emphasized earlier, this test provides means of quantifying not only the uncertainty on the parameter values, but also the rate of variation of these parameters. It should therefore allow for finer analysis of the impact of parameter variations on the stability/performance of the system.

The conditions of Theorem 6.2 warrant a few comments from a practical standpoint. First, if B is independent of θ, $P_i B_i = 0$ and there is no loss of generality in taking N_i of the form

$$N_i = \begin{pmatrix} M_i & 0 \\ 0 & 0 \end{pmatrix}.$$

The multi-convexity condition (6.9) then reduces to the same condition as for AQS:

$$A_i^T P_i + P_i A_i + M_i \geq 0.$$

Note that the case of a constant C matrix is handled similarly by duality.

Secondly, we could set $N_i = 0$ to obtain the counterpart of Theorem 4.1. However, (6.9) then reads

$$\begin{pmatrix} A_i^T P_i + P_i A_i & P_i B_i \\ B_i^T P_i & 0 \end{pmatrix} \geq 0$$

which requires $P_i B_i = 0$ and imposes an algebraic constraint on P_i. As a result, the LMI feasibility problem (6.7)–(6.10) can no longer be solved by interior-point methods. This difficulty can be circumvented by treating (6.7)–(6.10) as a linear objective minimization problem as indicated in the next section.

7 Numerical Implementation

Testing the sufficient AQS or AQP conditions of Theorems 4.1 through 6.2 amounts to solving a standard LMI feasibility problem. See [8] for an extensive discussion of LMI problems and their applications to control theory. LMI feasibility problems are convex and can be attacked by a variety of numerical optimization algorithms [28, 7, 37, 27]. This section focuses on details of implementation to optimize the performance of the LMI solvers given the particular structure of these LMIs.

The implementation of the conditions of Theorems 5.1 or 6.2 poses no particular difficulty. In contrast, the simpler conditions (4.4)–(4.6) need some attention. The main difficulty with (4.4)–(4.6) is the fact that the multi-convexity constraints

$$A_i^T P_i + P_i A_i \geq 0 \tag{7.11}$$

cannot be strengthened to strict inequalities. Indeed, the A_i matrices are typically of low rank since the parameter θ_i only appears in a few entries of $A(\theta)$ in general. As a result, the feasibility set, that is, the set of matrices $P_0, \ldots, P_K$ satisfying (4.4)–(4.6), has empty interior. This rules out the direct use of currently available polynomial-time solvers for LMI feasibility problems. Indeed, as interior-point methods such solvers can only handle *strictly* feasible problems [28, 27, 8].

There are various ways of alleviating this difficulty. One way is to observe that (7.11) is equivalent to some strict inequality together with some algebraic constraints on the P_i. These constraints indicate that some optimization variables should be eliminated to recover strict feasibility, and a systematic procedure for carrying out this elimination is discussed in

[8]. This approach has the disadvantage of destroying the specific structure of (7.11). When using LMI solvers that exploit this structure to speed up computations, this will result in a degradation of performance. This is the case, in particular, for the LMI solvers available in the package *LMI-Lab* [17, 27].

A second possibility is to replace the feasibility problem by a linear program under LMI constraints. Specifically, to find solutions $x \in \mathbb{R}^p$ of a non strictly feasible LMI

$$L(x) \leq 0, \tag{7.12}$$

we can solve instead

$$\text{Minimize } t \quad \text{subject to } L(x) \leq tI. \tag{7.13}$$

This is also a standard LMI problem and $L(x) \leq tI$ is always strictly feasible. Moreover, the initial non strictly feasible problem has a solution if and only if the global minimum of (7.13) is $t^* = 0$. This second approach has the merit of preserving the LMI structure, hence the high performance of structure-oriented LMI solvers. In practice however, much effort will be spent approximating the global minimum $t^* = 0$ to a high accuracy when (7.12) is not strictly feasible.

To circumvent this last difficulty, we suggest replacing the feasibility problem (4.4)–(4.6) by the following feasibility problem of matrix variables $P_0, \ldots, P_K$ and scalar variables $\lambda_1, \ldots, \lambda_K$:

$$\begin{aligned}
A(\omega)^T P(\omega) + P(\omega)A(\omega) + P(\tau) + \left(\sum_i \lambda_i \omega_i^2\right) I &< P_0 \quad (\forall(\omega,\tau) \in \mathcal{V} \times \mathcal{T}) && (7.14)\\
P(\omega) &> I \quad (\forall \omega \in \mathcal{V}) && (7.15)\\
A_i^T P_i + P_i A_i + \lambda_i I &\geq 0 \quad (i = 1, \ldots, K) && (7.16)\\
\lambda_i &\geq 0 \quad (i = 1, \ldots, K). && (7.17)
\end{aligned}$$

To understand why this removes the difficulty with non strict feasibility of (7.11), assume that $P_0, \ldots, P_K$ solve (4.4)–(4.6). From the strict nature of (4.4) and the fact that each θ_i is bounded, the condition (7.14) must then hold for small enough $\lambda_i > 0$. It readily follows that (7.14)–(7.17) is strictly feasible. Conversely, (7.14)–(7.17) is clearly sufficient for AQS since this is a special case of Theorem 5.1 corresponding to $M_i = \lambda_i I$.

From a numerical viewpoint, this "trick" will be most effective when (4.4) can be made reasonably negative, the excess of negativity being used to desaturate (4.6). This is the best we can expect from the efficiency standpoint. Indeed, the global minimum t^* of (7.13) is nearly 0 when (4.4) can only be made marginally negative, and proving feasibility then always takes more work for the LMI solvers. Note that it is still desirable to use the reformulation (7.13) of feasibility problems to enhance numerical stability.

8 Numerical Examples

This section illustrates the potential of AQS techniques by a few simple examples. The emphasis is on assessing the sharpness of this new test in comparison to well-established robustness measures such as the μ structured singular value or the quadratic stability test. Judging from the experimental results reported below, the AQS test seems to compare very favorably with other approaches and to exhibit little conservatism. In addition, it brings interesting quantitative insight into the way parameter variations can affect stability.

All LMI-related computations were performed with the software package *LMI-Lab* developed by A. Nemirovskii and the first author [17]. *LMI-Lab* is part of MATLAB's forthcoming *LMI Control Toolbox* [19].

8.1 An illustrative example

Before moving to more sophisticated experiments, we begin with a simple example that illustrates the problem setup and its LMI solution. Consider a mass/spring/damper system with unit mass and state-space representation

$$\dot{x} = A(f,k)x \tag{8.1}$$

where

$$A(f,k) := \begin{pmatrix} 0 & 1 \\ -k & -f \end{pmatrix}.$$

The stiffness and viscosity coefficients k and f are assumed to be time-invariant. Clearly $A(.,.)$ is jointly affine in f, k and can be decomposed as

$$A(f,k) = A_0 + fA_1 + kA_2 \ ,$$

with

$$A_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \qquad .$$

This system is stable for all positive values of f, k. To assess the conservatism of our AQS conditions, we let f, k range in

$$f \in [10^{-4}\ ,\ 10^4], \qquad k \in [10^{-4}\ ,\ 10^4]$$

and seek an affine Lyapunov matrix

$$P(f,k) = P_0 + fP_1 + kP_2$$

that would prove stability over this entire range of parametric uncertainty.

Consider the four corners

$$v_1 = (10^{-4}, 10^{-4}), \quad v_2 = (10^{-4}, 10^4), \quad v_3 = (10^4, 10^{-4}), \quad v_4 = (10^4, 10^4)$$

of the parameter box $[10^{-4}, 10^4] \times [10^{-4}, 10^4]$. Using the weaker results of Theorem 3.1, an adequate Lyapunov matrix $P(f,k)$ exists whenever the LMI system

$$A(v_i)^T P(v_i) + P(v_i)A(v_i) \ < \ 0, \qquad i = 1, \ldots, 4 \tag{8.2}$$

$$P(v_i) \ > \ I, \qquad i = 1, \ldots, 4 \tag{8.3}$$

$$A_j^T P_j + P_j A_j \ \geq \ 0, \qquad j = 1, 2 \tag{8.4}$$

is feasible. Taking into account the remarks of Section 7, a solution to this LMI problem was computed as:

$$P_0 = \begin{pmatrix} 67.077 & 56.680 \\ 56.680 & 6.707 \times 10^5 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 56.605 & -8.829 \times 10^{-6} \\ -8.829 \times 10^{-6} & -0.530 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 6.707 \times 10^5 & -2.300 \times 10^3 \\ -2.300 \times 10^3 & -9.220 \times 10^{-12} \end{pmatrix}.$$

To gain insight into the way the parameter dependence shapes the Lyapunov matrix $P(f,k)$, we plotted the smallest and largest eigenvalues of this matrix as functions of the parameter values f and k. The resulting shapes appear in Figures 8.1 and 8.2, respectively. The largest eigenvalue of

$$L(f,k) \ := A(f,k)^T P(f,k) + P(f,k)A(f,k)$$

is also plotted in Figure 8.3.

These graphs confirm that $P(f,k)$ satisfies the sufficient AQS conditions (8.2)–(8.4). As would be expected, the AQS constraints tend to saturate where the system becomes marginally stable, that is, for small values of f or k. This is clear from Figures 8.1 and 8.3. In contrast, they are amply satisfied in the more stable regions of the parameter space.

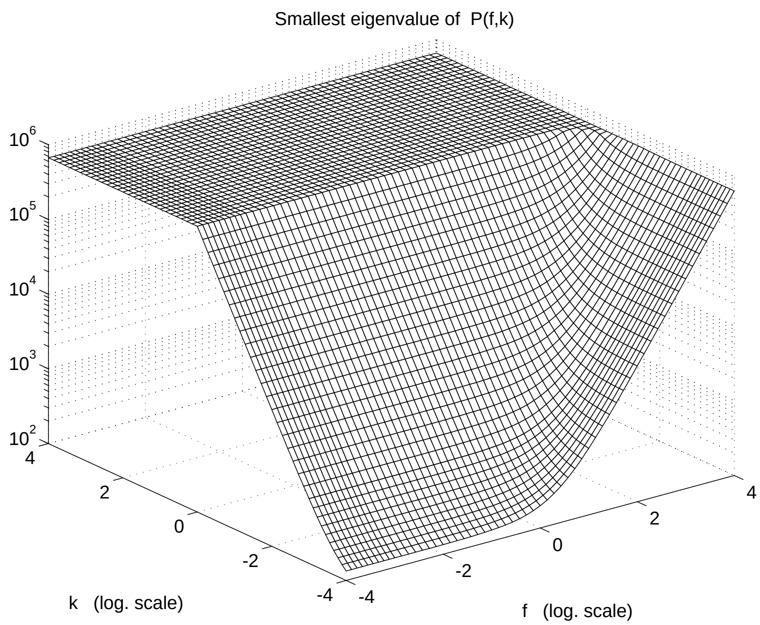

Figure 8.1: Smallest eigenvalue of $P(f,k)$

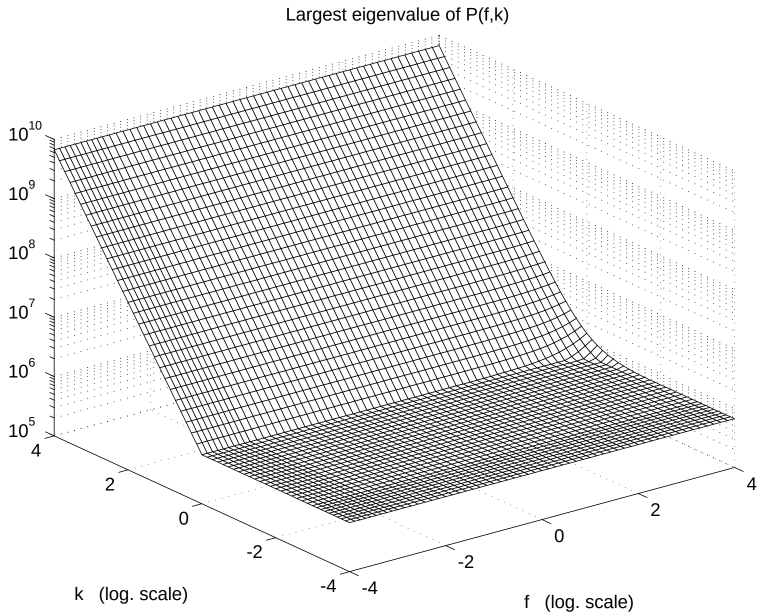

Figure 8.2: Largest eigenvalue of $P(f,k)$

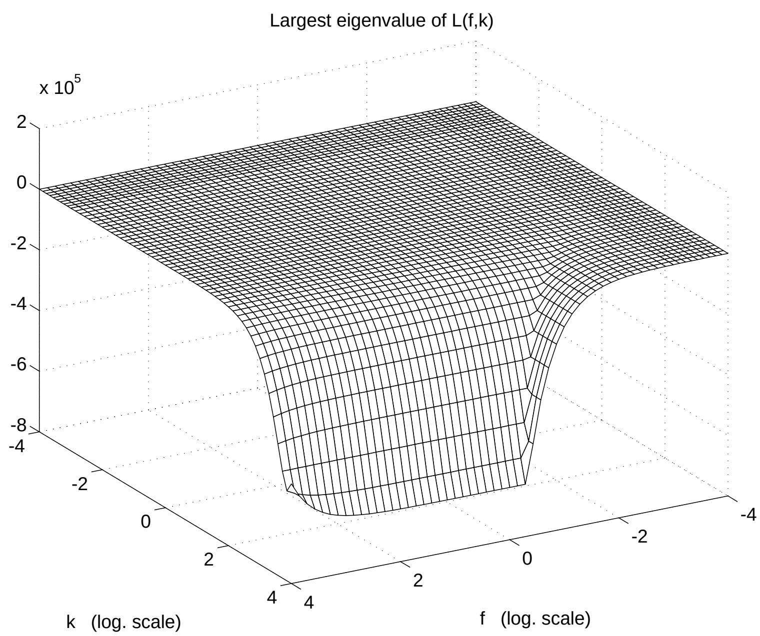

Figure 8.3: Largest eigenvalue of $L(f,k)$

8.2 Comparative study for a mass-spring system

We now investigate the performance of the AQS test of Theorem 3.1 as an alternative tool for parameter margin evaluation in uncertain *time-invariant* systems. By parameter margins we mean the amount of parametric uncertainty that can be tolerated without compromising stability. It must be stressed that no general conclusions should be drawn from the results presented below. Our sole purpose is to demonstrate the validity of this approach and to motivate further study of this new tool for robustness analysis.

Three different techniques are compared in this experiment:

- the *quadratic stability* test which seeks a single Lyapunov quadratic function ensuring stability over the entire uncertainty range,
- *real-μ analysis* based on the upper bound on the structured singular value $\mu_{\mathbb{R}}$ for real parametric uncertainty. The evaluation of this bound involves the computation of optimal frequency-dependent multipliers [15, 34]
- the *affine quadratic stability* test developed in Section 2. As pointed out earlier, this technique is similar in spirit to the quadratic approach except that it exploits the time-invariant nature of the uncertainty.

All three techniques give only sufficient conditions for robust stability and may be more or less conservative. Quadratic stability is typically more conservative since it also allows for time-varying parameters. Interestingly, these tests all reduce to LMI problems. In the experiments described below, the upper bound for the real μ was computed using *μ-Tools* [11], while the AQS test was performed with *LMI-Lab*.

In this example, the system under consideration is the two masses/two springs system depicted in Figure 8.4. A fourth-order state-space model of this system is easily derived as

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{f}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & -\frac{f}{m_2} \end{pmatrix} x \tag{8.5}$$

where f is the motion friction coefficient, k_1, k_2 are the spring stiffnesses, and m_1, m_2 denote the masses of the rigid bodies. In the sequel, we assume that m_1 and m_2 are known exactly with values

$$m_1 = 1, \qquad m_2 = 20$$

while the parameters k_1, k_2, and f are uncertain with nominal values k_{10}, k_{20}, and f_0, respectively. With the multiplicative uncertainty representation

$$k_1 = k_{10}(1+\theta_1), \qquad k_2 = k_{20}(1+\theta_2), \qquad f = f_0(1+\theta_3), \tag{8.6}$$

an affine expression for the uncertain matrix A reads:

$$A(\theta) = A_0 + \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3 \tag{8.7}$$

where

$$A_0 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_{10}+k_{20}}{m_1} & -\frac{f_0}{m_1} & \frac{k_{20}}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_{20}}{m_2} & 0 & -\frac{k_{20}}{m_2} & -\frac{f_0}{m_2} \end{pmatrix}; \qquad A_1 := k_{10} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \tag{8.8}$$

$$A_2 := k_{20} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{m_1} & 0 & \frac{1}{m_1} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m_2} & 0 & -\frac{1}{m_2} & 0 \end{pmatrix}; \qquad A_3 := f_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{m_2} \end{pmatrix}. \tag{8.9}$$

This expression is most appropriate for the quadratic and affine quadratic stability tests. For μ-analysis purposes however, it is more convenient to rewrite (8.7) as

$$A(\theta) = A_0 + B\Theta C \tag{8.10}$$

where $\Theta = \mathrm{Diag}(\theta_1, \theta_2, \theta_3 \times I_2)$ and

$$B := \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{m_1} & -\frac{1}{m_1} & -\frac{1}{m_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 & -\frac{1}{m_2} \end{pmatrix}, \qquad C := \begin{pmatrix} k_{10} & 0 & 0 & 0 \\ k_{20} & 0 & -k_{20} & 0 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & 0 & f_0 \end{pmatrix}.$$

The uncertain system can then be viewed as the interconnection of the uncertainty structure Θ with the LTI system

$$G(s) \ := C(sI - A_0)^{-1}B. \tag{8.11}$$

Note that since the matrix A_3 in (4.4) has rank two, the uncertain parameter θ_3 associated with f is repeated in Θ. As a result, Kharitonov-like techniques may lead to conservative estimates in this problem.

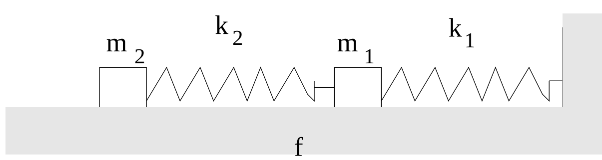

Figure 8.4: Mass-Spring System

8.2.1 Parameter margin estimation

The parameter margin K_m is defined as the smallest relative deviation from nominal parameter values that is needed to destabilize the system [34]. With our uncertainty representation (8.6), this is simply

$$K_\mathrm{m} = \min\{\|\theta\|_\infty \ : \ A(\theta) \text{ is unstable } \}$$

where

$$\|\theta\|_\infty \ := \max(|\theta_1|, |\theta_2|, |\theta_3|).$$

Note that the reciprocal of this number is exactly the real μ for (8.10)–(8.11) and the uncertainty structure $\Theta = \mathrm{Diag}(\theta_1, \theta_2, \theta_3 \times I_2)$. Specifically,

$$\frac{1}{K_{\mathrm{m}}} = \mu_{\mathbb{R}}(G, \Theta) \ := \sup_{\omega} \ \mu_{\mathbb{R},\Theta}(G(j\omega))$$

where $G(s)$ is given by (8.11).

It is well known that the mass-spring system of Figure 8.4 is stable whenever the parameters k_1, k_2, and f are positive. Conversely, $f > 0$ is necessary for stability. Hence

$$\mu_{\mathbb{R}}(G, \Theta) = \frac{1}{K_{\mathrm{m}}} = 1$$

in this example, and the conservatism of our three robust stability tests can be measured up against this exact value. Specifically, the following three lower bounds on K_{m} are computed and compared to the true value $K_{\mathrm{m}} = 1$:

- K_{QS} defined as the smallest $\delta > 0$ such that (8.5) is quadratically stable on the parameter box
$$\mathcal{H}(\delta) = \{ \ (\theta_1, \theta_2, \theta_3) \ : \ \theta_i \in [-\delta, \delta] \} \tag{8.12}$$
- $K_\mu = 1/\mu_{\mathrm{ub}}$ where μ_{ub} is the computable upper bound on $\mu_{\mathbb{R}}(G, \Theta)$ introduced in [15],
- K_{AQS} defined as the smallest $\delta > 0$ for which the sufficient AQS conditions of Theorem 3.1 are feasible on the parameter box $\mathcal{H}(\delta)$ defined above.

Here the parameter box $\mathcal{H}(\delta)$ has $2^3 = 8$ corners, and the AQS test requires computing four 4×4 symmetric matrices $P_0, \ldots, P_4$.

For meaningful comparison, we computed these three quantities for a wide variety of nominal models. Specifically, we set $k_{10} = k_{20} = 20$ and considered values of f_0 ranging in $[10^{-3}, 1]$ with a sampling step of 0.02. Since K_{QS} and K_{AQS} cannot be directly computed from the associated LMI conditions, K_{QS} was determined by dichotomy, and AQS was tested directly for the parameter box $\mathcal{H}(0.999)$.

The results of this experiment appear in Figure 8.5 where the estimated parameter margins K_{QS}, K_μ, and K_{AQS} are compared to the true value $K_{\mathrm{m}} = 1$ for various values of f_0. By stability region, we simply mean the largest box $\mathcal{H}(\delta)$ where stability could be established by the tests under scrutiny. Remarkably, the uncertain system was proved AQS in $\mathcal{H}(0.999)$ for all values of f_0. In other words, the AQS test estimated the correct parameter margin with 0.1% accuracy. In contrast, the quadratic stability test performed very poorly, and even the real-μ upper bound proved relatively inaccurate. For both K_{QS} and K_μ, the largest errors occurred for small values of f_0, that is, when the system is poorly damped. Quantitatively, the AQS estimate outperformed K_μ by a factor 10 in the badly damped region and by a factor 1.25 near $f_0 = 1$.

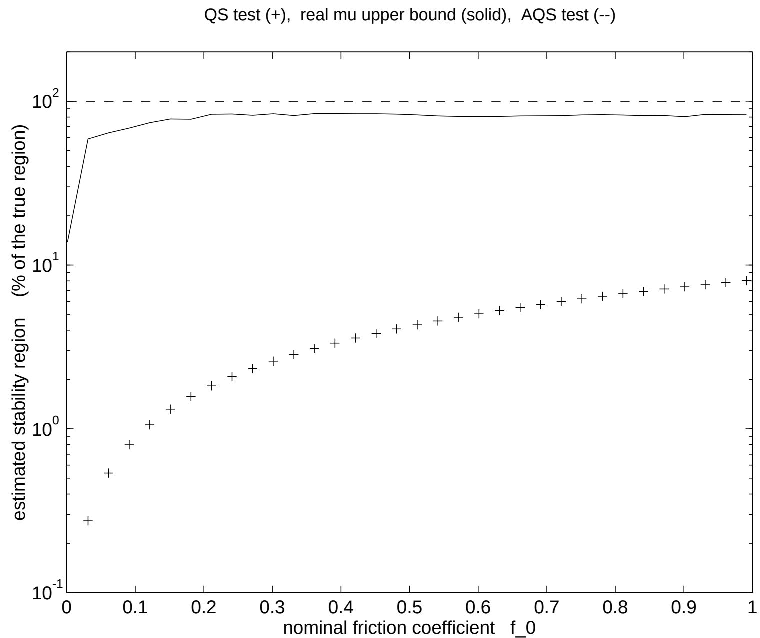

Figure 8.5: Sharpness of the parameter margin estimates K_{QS}, K_{μ}, and K_{AQS}.

8.2.2 Maximal stability box

Instead of computing the parameter margin K_{m} which measures the distance to instability in relative variation terms, we now seek estimates the largest box in the positive parameter orthant where stability is preserved. The comparison is restricted to the AQS test vs. the real-μ upper bound. The values of m_1 and m_2 are as before, but k_1, k_2, and f are now allowed to vary in $(0, +\infty)$. For μ-analysis purposes, this range should first be mapped to a finite interval centered at 0. To this end, express the parameter dependence as

$$A(\theta) := \mathcal{A} + \mathcal{B}\Theta\mathcal{C}$$

where $\Theta = \mathrm{Diag}(k_1, k_2, f \times I_2)$ is the block-diagonal uncertainty structure and

$$\mathcal{A} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B} := B, \qquad \mathcal{C} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Applying the bilinear transformation

$$\tilde{\Theta} := (\Theta - I)(\Theta + I)$$

to the uncertainty structure, the uncertain system can also be viewed as the interconnection of $\tilde{\Theta}$ with the transfer function

$$\tilde{G}(s) = I + 2\,\mathcal{C}(sI - (\mathcal{A} + \mathcal{B}\mathcal{C}))^{-1}\mathcal{B}.$$

Moreover, the diagonal entries of $\tilde{\Theta}$ now range in $[-1, 1]$ when k_1, k_2, f range in $(0, +\infty)$. As in the previous subsection, it is readily seen that

$$\mu_{\mathbb{R}}(\tilde{G}, \tilde{\Theta}) = 1$$

where $\mu_R(\tilde{G}, \tilde{\Theta})$ denotes the real μ for the system $\tilde{G}$ and the uncertainty structure $\tilde{\Theta}$.

The AQS test of Theorem 3.1 was performed on the parameter box

$$(k_1, k_2, f) \;\in\; \prod_1^3 [10^{-3},\; 10^4],$$

and succeeded to prove AQS over this wide range of values of k_1, k_2, f. In contrast, the real-μ upper bound for the transformed problem $\tilde{G}, \tilde{\Theta}$ evaluated to 2.3015, which only guarantees stability over the parameter box

$$(k_1, k_2, f) \;\in\; \prod_1^3 [0.3942,\; 2.5366].$$

Again the AQS test compares very favorably in this stability box maximization problem.

8.3 Random tests

This subsection pursues the comparison between the AQS test and the real μ upper bound, this time on a larger sample of problems. We consider eighth-order dynamical systems of the form

$$M_0\ddot{x} + F_0\dot{x} + K_0 x = 0 \tag{8.13}$$

where M_0, F_0, K_0 are randomly generated *positive definite* matrices in $\mathbb{R}^{4\times 4}$. We subject this nominal model to rank-four perturbations by replacing F_0, K_0 in (8.13) by

$$K \;:= (1 + \theta_2)K_0\;; \qquad\qquad F = (1 + \theta_2)F_0. \tag{8.14}$$

For $\theta_1 > -1$ and $\theta_2 > -1$, the matrices K and F remain positive definite and the system is dissipative [24], hence stable. Observing that $F > 0$ is required for stability, it follows that the parameter margin for the perturbation (8.14) is

$$K_{\mathrm{m}} = 1$$

as in Subsection 8.2.1.

This parameter margin was estimated using the real-μ upper bound and the AQS test for 100 randomly generated nominal models M_0, F_0, K_0. As earlier, the sharpness of each test was measured by the relative gap to the true value 1. The results of these 100 experiments are displayed in Figure 8.6. Again the AQS test performed very well since the LMI conditions of Theorem 3.1 were always feasible in the box

$$\theta_1 \in [-0.999, 0.999], \qquad \theta_2 \in [-0.999, 0.999].$$

Hence the parameter margin was correctly estimates within 0.1% in all 100 experiments. In comparison, the μ upper bound was found conservative in most cases, with more than 50% error in some problems and an average error around 35%. Interestingly, the μ upper bound was least accurate for problems with repeated-scalar uncertainty.

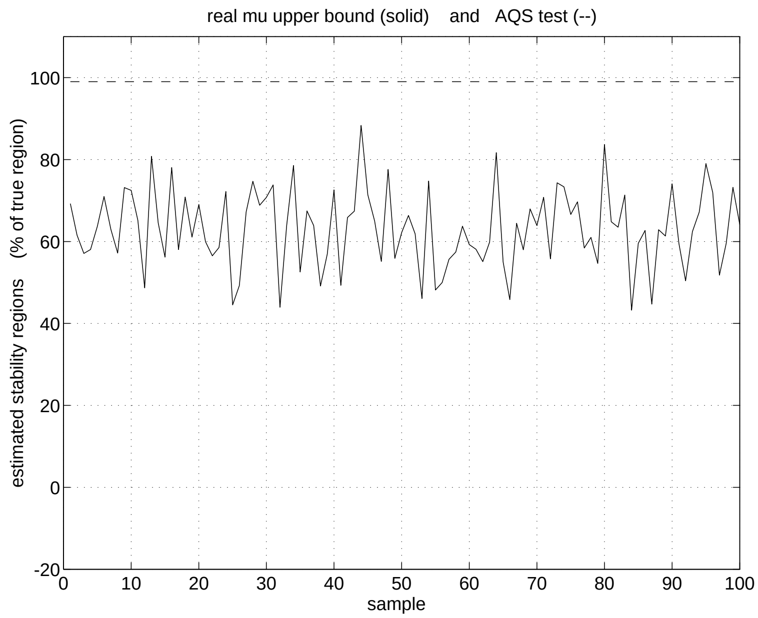

Figure 8.6: Comparative sharpness of the AQS test and of the real μ upper bound

8.4 Time-varying example

The next example illustrates the performance of the AQS test in mixed problems combining constant and time-varying parameters. In particular, it is shown that taking into account available bounds on the rate of parameter variation as suggested by Theorem 4.1 can dramatically reduce the conservatism of the quadratic stability test.

The system under consideration is again the elementary mass-spring system of 8.1, but the stiffness and friction coefficients k and f are now allowed to vary in time. The parameter range is given in multiplicative form as

$$f = f_0(1 + \theta_1), \qquad k = k_0(1 + \theta_2)$$

where f_0 and k_0 are the nominal values of f and k, respectively. With this uncertainty representation, the state matrix of the mass-spring system reads:

$$A(f,k) = \begin{pmatrix} 0 & 1 \\ -k_0 & -f_0 \end{pmatrix} + \theta_1 \begin{pmatrix} 0 & 0 \\ 0 & -f_0 \end{pmatrix} + \theta_2 \begin{pmatrix} 0 & 0 \\ -k_0 & 0 \end{pmatrix}. \tag{8.15}$$

The bounds on the time derivatives of f and k are denoted by $\dot{f}_{max}$ and $\dot{k}_{max}$. Specifically, we impose

$$|\dot{f}(t)| \leq \dot{f}_{max} \qquad\qquad |\dot{k}(t)| \leq \dot{k}_{max} \tag{8.16}$$

where $\dot{f}_{max}$ and $\dot{k}_{max}$ range from zero (constant parameters) to infinity (arbitrarily fast variations).

Again we are interested in evaluating the parameter margin K_{m}. That is, the largest value of

$$\delta \ := \max(|\theta_1|, |\theta_2|)$$

for which the time-varying system

$$\dot{x} = A(f(t), k(t))\, x$$

subject to (8.16) is asymptotically stable. Note that $K_{\mathrm{m}} \leq 1$ since $f > 0$ is required for stability. For this experiment we use the sharper LMI conditions of Theorem 5.1. The LMIs (5.4) are evaluated at the corners of the parameter box

$$(\theta_1, \theta_2) \in [-\delta, \delta] \times [-\delta, \delta]$$

and of the rate-of-variation box

$$(\tau_1, \tau_2) \in [-\dot{f}_{max},\ \dot{f}_{max}] \times [-\dot{k}_{max},\ \dot{k}_{max}].$$

The following three experiments were performed:

(1) for time-invariant k $(\dot{k}_{max} = 0)$ and time-varying f, compute the parameter margin estimate K_{AQS} for several values of $\dot{f}_{max}$ ranging between 10^{-3} to 10^4. This experiment assesses the impact of time variations of f on the overall stability. The results are shown in Figure 8.7.

(2) for time-invariant f $(\dot{f}_{max} = 0)$ and time-varying k, compute K_{AQS} for values of $\dot{k}_{max}$ ranging between 10^{-3} to 10^4. The results are depicted by Figure 8.8.

(3) compute K_{AQS} for f, k both time-varying and for values of their maximum rate of variation ranging between 10^{-3} to 10^4. The purpose is to assess whether combined time variations of f and k can further shrink the estimated stability region. The outcome of this test is presented in the 3D plot of Figure 8.9.

In each experiment, the estimate K_{AQS} was computed by dichotomy.

Figure 8.7 shows that time variations of f alone have no effect on the stability region. This is consistent with the fact that the system remains dissipative regardless of the rate of variation of f. Indeed, the parameter-dependent Lyapunov function $V(x, \dot{x}, k) := \dot{x}^2 + kx^2$ proves stability whenever k is time-invariant and $f > 0$ since then

$$\frac{dV}{dt} = 2\,\dot{x}\,(\ddot{x} + kx) = -2\,f\dot{x}^2 \leq 0.$$

In contrast, Figure 8.8 shows that the estimated stability region is strongly affected by the rate of variation of k. While the parameter margin stays close to its best possible value 1 for slow variations of k, it gradually drops as $\dot{k}_{max}$ increases and finally settles around the quadratic stability value 0.5 near $\dot{k}_{max} = 30$. According to these results, the quadratic stability test is optimal only for rates of variation larger than 30 N/m/s. For moderate rates of variation however, the AQS test is significantly less conservative while offering the same stability guaranties against parameter variations. Finally, Figure 8.5 indicates that simultaneous variations of f and k do not restrict the estimated stability region any further. Hence the stability region seems to be essentially determined by $\dot{k}_{max}$.

Summing up, the AQS approach is a promising alternative for robust stability analysis in the face of time-varying parameters with known maximum rates of variation.

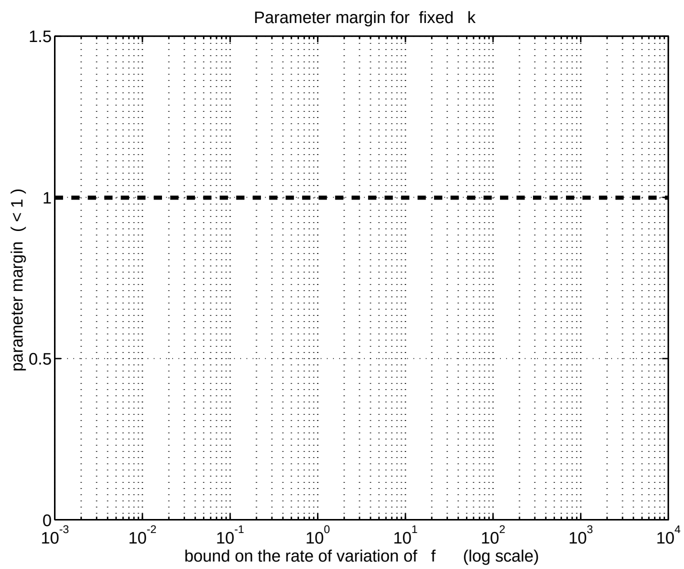

Figure 8.7: AQS parameter margin for time-varying f.

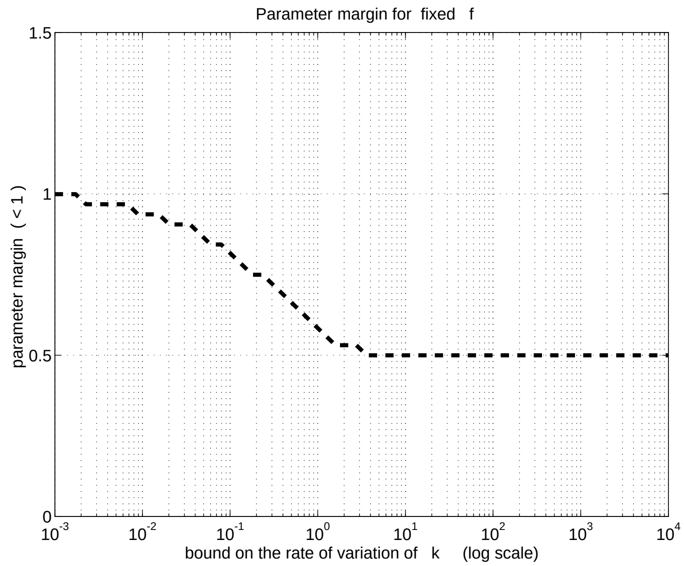

Figure 8.8: AQS parameter margin for time-varying k.

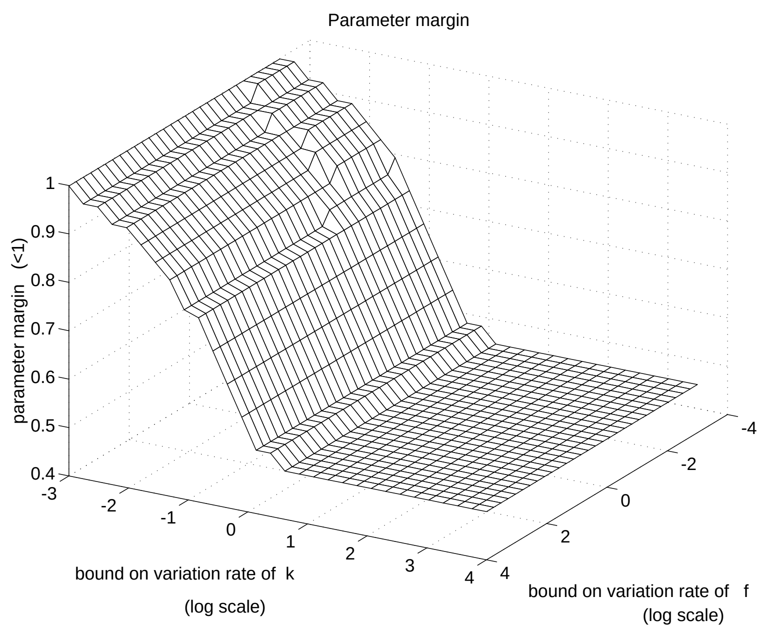

Figure 8.9: AQS parameter margin for time-varying f and k.

8.5 Minimum stability degree

This last example is an application of AQS concepts to the computation of the minimum stability degree (MSD) of a system. For a linear parameter-dependent system

$$\dot{x} = A(\theta)x$$

with θ ranging in a parameter box $\mathcal{H}$, the MSD is defined as

$$\sup\{\alpha : \ A(\theta) + \alpha I \ \text{ is stable for all } \theta \text{ in } \mathcal{H} \ \} \ .$$

In general, computing the MSD of an uncertain system is a NP-hard problem. That is, the required computational effort grows exponentially with the problem size. In special cases however, extensions of Kharitonov's Theorem allow to compute the MSD exactly [31]. This is the case of the example considered in [10] which is now revisited from the AQS perspective.

Consider the system of Figure 8.10 where $\theta_1, \theta_2, \theta_3$ are uncertain time-invariant parameters ranging in:

$$2 \leq \theta_1 \leq 3, \qquad 3 \leq \theta_2 \leq 5, \qquad -1 \leq \theta_3 \leq 1 \ . \tag{8.17}$$

Using a branch and bound technique, the MSD for this system was estimated as -0.2757, which coincides with the true value provided by Kharitonov's theory. For comparison purposes, we computed a lower bound on the MSD using the AQS technique.

In AQS terms, this problem can be formulated as:

$$\text{Maximize } \alpha \ ,$$

subject to the LMI constraints:

$$\begin{aligned} (A(v_i) + \alpha I)^T X(v_i) + X(v_i)\,(A(v_i) + \alpha I) &< 0, \qquad i = 1, \ldots, 8 & (8.18)\\ X(v_i) &> I, \qquad i = 1, \ldots, 8 & (8.19)\\ A_j^T X_j + X_j A_j &\geq 0 \qquad j = 1, \ldots, 3 & (8.20) \end{aligned}$$

where the v_i denote the corners of the parameter box (8.17) and the matrices $A(v_j)$ and A_j are readily derived from the problem description.

Proceeding by dichotomy, these LMI conditions were found feasible for $\alpha = -0.2757$ and unfeasible for $\alpha = -0.2756$. In other words, the AQS approach provides a non-conservative estimate of the MSD in this problem.

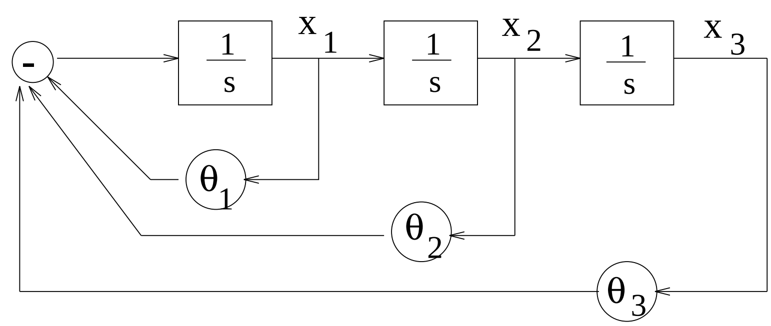

Figure 8.10: System used in the MSD example.

9 Conclusions

We have proposed new LMI-based tests for the robust stability/performance of linear systems with uncertain or time-varying real parameters. These tests rely on the concept of Affine Quadratic Stability (AQS) and involve parameter-dependent Lyapunov functions with an affine dependence on the parametric uncertainty.

Their relative sharpness has been demonstrated on a variety of physically motivated examples. In particular, the AQS test significantly improved on the standard quadratic stability test and often outperformed the real μ upper bound. Interestingly, quantitative information about the rate of parameter variation is readily included in the AQS test to reduce conservatism in the time-varying case.

These preliminary results are encouraging and motivate further investigation of the potential and applications of AQS concepts.

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