NONSMOOTH μ SYNTHESIS

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Abstract

We revisit robust complex- and mixed-μ synthesis problems based on upper bounds and show that they can be recast as specially structured controller design programs. The proposed reformulations suggest a streamlined handling of μ synthesis problems using recently developed (local) nonsmooth optimization methods where both scalings or multipliers and a controller of given structure are obtained simultaneously. A first cut of the nonsmooth programming software for structured H_∞ synthesis is made available through the MATLAB R2010b Prerelease, *Robust Control Toolbox Version 3.5* developed by The MathWorks, Inc.

Keywords: μ synthesis, structured controllers, mixed μ, NP-hard problems.

Introduction

Most if not all control engineering problems motivate consideration of plants subject to uncertainties. Uncertainties describe plant variations in the frequency domain and this is often referred to as complex or Linear Time-Invariant uncertainties. They can also capture deviations of some physical quantities from their nominal values and in that case, parametric uncertainties is the appropriate terminology. In a practical situation the nominal plant undergoes variations with respect to both types of uncertainty, this is the so-called mixed uncertainty case. Synthesizing feedback controllers achieving robustness in face of mixed uncertainties remains a very challenging problem at the core of modern robust control theories. When the controller is fixed, assessing the robustness properties of the closed-loop system though apparently much simpler than the synthesis problem is NP-hard in general [45]. A number of potent theories with associated practical tools have been developed over the past 30 to answer the analysis question with mild conservatism. We are pointing here to μ-analysis [22] or to Lyapunov-based analysis techniques where LMIs (Linear Matrix Inequalities) form the central computational component [13]. No such methods and tools with comparable merit and efficiency are available so far for the synthesis problem. The inherent non-convexity of robust synthesis problems is certainly the culprit cause. Among numerous attempts to estimate solutions of robust control problems, μ-synthesis based on upper bounds was probably the most explored avenue. In this approach, the true robustness indicator is replaced by mildly conservative upper bounds. The synthesis problems is then decomposed into analysis and

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synthesis both of which are covered by efficient numerical tools. At this stage it seems important to pinpoint potential sources of weakness in those techniques

- the biconvex nature of the synthesis problem suggests alternating between analysis and synthesis phases till nor further progress is observed. The term $D-K$ or alike is often used in this context. This immediately raises questions about the convergence of such schemes.

- in conventional μ-synthesis, scalings are computed pointwise in frequency and must be fitted by state-space realizable systems [9]. This again questions the numerical stability and accuracy of such procedures.

- in the same approach either scalings are restricted to be stable or stable factorizations are searched for with detrimental impact on conservatism and size inflation in the state dimension of the problem.

- in the alternative passivity approach to the synthesis problem [23, 16], multipliers are sought over the span of a prescribed basis of given length. This again leads to questions about the choice of a rich enough basis to limit the entailed conservatism.

In this work we suggest a different strategy bypassing most of the aforementioned difficulties. The order and structure of the controller is set beforehand as well as the order of scalings and multipliers. Scalings and multipliers are sought in non-necessarily stable state-space realizable form. We do not rely on curve fitting nor on stable factorizations and nor on ad-hoc basis selection. We establish that both complex-μ and mixed-μ syntheses based on upper bounds can be reformulated as synthesis programs with an augmented controller of special structure. The augmented controller encapsulates the original structured controller along with the scalings or multipliers. It follows from the latter cast that the synthesis problem can be solved (locally) with recently available nonsmooth optimization techniques [3]. Synthesis is accomplished in one shot which means the controller together with the scalings or multipliers are obtained simultaneously. The proposed technique is endowed with a sound *local* convergence certificate and thus has potential to overcome premature terminations arising in biconvex $D-K$ iteration schemes. Using our nonsmooth techniques to compute local solutions is certainly less ambitious than global approaches [25, 26] but is more efficient numerically and supported by a wide range of experiments. See [43] and references therein.

The paper is structured as follows. Complex- and mixed-μ synthesis problems are briefly reviewed in section 1. In section 2, we establish that complex-μ synthesis can be formulated as a specially structured controller design problem with nonsmooth objectives and constraints. Similar casts for mixed-μ synthesis are discussed in section 3. Section 4 gives a short account of the key ingredients for solving complex- and mixed-μ programs using the nonsmooth technique introduced in [3]. Finally, experimental results are reported in section 5.

Notation

For complex matrices X^H stands for its conjugate transpose. For Hermitian or symmetric matrices, $X \succ Y$ means that $X-Y$ is positive definite, $X \succeq Y$ that $X-Y$ is positive semi-definite. The maximum eigenvalue of an Hermitian or symmetric matrix X is denoted $\overline{\lambda}(X)$. We use a few

concepts from nonsmooth analysis covered by [17]. For a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, $\partial f(x)$ denotes its Clarke subdifferential at x.

Given matrices N, L and partitioned matrices

$$M := \left[\begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right], \text{ and } J := \left[\begin{array}{cc} J_{11} & J_{12} \\ J_{21} & J_{22} \end{array} \right],$$

of appropriate dimensions, the (assumed well-posed) lower LFT of M and N is defined as

$$\mathcal{F}_l(M, N) := M_{11} + M_{12}N(I - M_{22}N)^{-1}M_{21}$$

while the upper LFT of M and L is obtained as

$$\mathcal{F}_u(L, M) := M_{22} + M_{21}L(I - M_{11}L)^{-1}M_{12}\,.$$

The Redheffer star product [31, 40] denoted '$\star$' of M and J is obtained as

$$M \star J := \left[\begin{array}{cc} \mathcal{F}_l(M, J_{11}) & M_{12}(I - J_{11}M_{22})^{-1}J_{12} \\ J_{21}(I - M_{22}J_{11})^{-1}M_{21} & \mathcal{F}_u(J, M_{22}) \end{array} \right].$$

Defining the bilinear transformation operator

$$\mathcal{B}_q := \left[\begin{array}{cc} I_q & -\sqrt{2}I_q \\ \sqrt{2}I_q & -I_q \end{array} \right]$$

we have

$$\mathcal{B}_q \star \mathcal{B}_q = \left[\begin{array}{cc} 0 & I_q \\ I_q & 0 \end{array} \right]$$

which is the unit element for the Redheffer star product. We will use the fact that for any square complex matrix $X \in \mathbb{C}^{q\times q}$

$$X + X^H \succ 0 \Longleftrightarrow \overline{\sigma}(\mathcal{F}_l(\mathcal{B}_q, X)) < 1$$

and vice-versa. Similarly, we have

$$\overline{\lambda}(X + X^H) < 0 \Longleftrightarrow X + X^H \prec 0 \Longleftrightarrow \overline{\sigma}(\mathcal{F}_l(\widetilde{\mathcal{B}}_q, X)) < 1$$

where

$$\widetilde{\mathcal{B}}_q := \left[\begin{array}{cc} I_q & \sqrt{2}I_q \\ \sqrt{2}I_q & I_q \end{array} \right].$$

To save a space terms denoted $(.)^H$ will be induced by Hermitian symmetry.

Given matrices $X_1, \ldots, X_l$ of arbitrary size, the operator diag constructs the block-diagonal matrix

$$\mathrm{diag}(X_1, \ldots, X_l) := \left[\begin{array}{ccc} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_l \end{array} \right].$$

The notation H_ω is used to designate arbitrary frequency-dependent matrix-valued functions while the notation $H(j\omega)$ implicitly assumes an underlying state-space realization:

$$H(j\omega) := C_H(j\omega I - A_H)^{-1}B_H + D_H\,.$$

For a square matrix $A \in \mathbb{R}^{n\times n}$, $\alpha(A)$ stands for its spectral abscissa, that is,

$$\alpha(A) := \max_{i=1,\ldots,n} \Re\,\lambda_i$$

where the λ_i's are the eigenvalues of A.

1 Complex- and mixed-μ syntheses

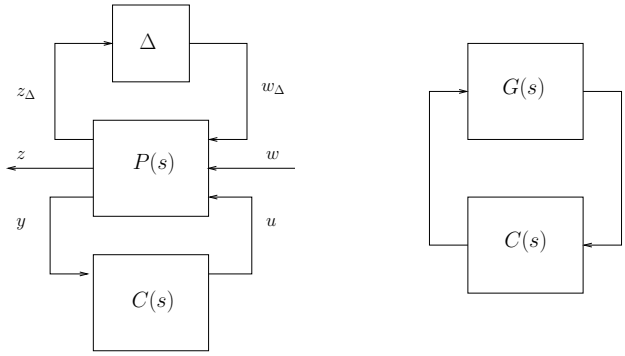

Figure 1: μ synthesis interconnection & nominal stability loop

The general robust synthesis interconnection discussed throughout the paper is shown in figure 1, where $P(s)$ is a linear time invariant plant, $C(s)$ a linear time invariant controller to be designed and Δ refers to uncertainties. The signal pair $(w_\Delta, z_\Delta) \in \mathbb{R}^N \times \mathbb{R}^N$ is the uncertainty channel, $(w, z) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_1}$ is the performance channel, $(u, y) \in \mathbb{R}^{m_2} \times \mathbb{R}^{p_2}$ is the control-measurement channel. We also assume throughout the plant P is state-space realizable in the form

$$P(s) := \left[\begin{array}{c|ccc} A & B_\delta & B_1 & B_2 \\ \hline C_\delta & D_{\delta\delta} & D_{\delta 1} & D_{\delta 2} \\ C_1 & D_{1\delta} & D_{11} & D_{12} \\ C_2 & D_{2\delta} & D_{21} & D_{22} \end{array}\right] . \tag{1}$$

The plant $P(s)$ describes how uncertainties Δ impact the system dynamics, how the controller component $C(s)$ acts on these dynamics, and determines a performance channel (w, z) reflecting practical specifications. It is constructed from an underlying system $G(s)$ with state-space realization

$$G(s) := C_G(sI - A_G)^{-1}B_G + D_G\,,\ D_G \in \mathbb{R}^{p_2 \times m_2}\,.$$

Generally speaking, $P(s)$ is built from $G(s)$ by adding weighting filters which emphasize appropriately chosen frequency ranges [44, 34]. For future use, we shall denote $A(G(s), C(s))$ the state-space A-matrix resulting from the positive feedback loop of $G(s)$ and $C(s)$. We infer that nominal stability of the interconnection in figure 1 can be expressed as $\alpha\{A(G(s), C(s))\} \leq -\epsilon$ for some small enough $\epsilon > 0$. For compactness, we have privileged transfer function notations having in mind that operations on transfer functions have more reliable state-space equivalent.

As usual [49, 22, 19], we consider structural restrictions on the uncertainty block Δ that is $\Delta \in \mathbf{\Delta}$ with the definition

$$\mathbf{\Delta} := \{\Delta = \mathrm{diag}(\delta_1 I_{r_1}, \ldots \delta_q I_{r_q}, \Delta_P) :\ \delta_i \in \mathbb{F},\, \Delta_P \in \mathbb{C}^{m_1 \times m_1}\}\,,$$

with $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ depending on the context and $N = \sum_{i=1}^{q} r_i + m_1$. We will refer to $\delta_i I_{r_i}$ with $\delta_i \in \mathbb{C}$ as repeated scalar LTI uncertainties, while $\delta_i \in \mathbb{R}$ designates repeated parametric uncertainties, and Δ_P will stand for the performance block. To ease the notation, we also introduce the reduced sets

$$\mathbf{\Delta}_\mathbb{F} := \{\Delta = \mathrm{diag}(\delta_1 I_{r_1}, \ldots \delta_q I_{r_q}) :\ \delta_i \in \mathbb{F},\, i = 1, \ldots, q\}\,, \quad \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R}\,.$$

and

$$\mathbf{\Delta_P} := \{\Delta = \Delta_P \in \mathbb{C}^{m_1 \times m_1}\}\,.$$

For $\mathbf{\Delta}$, the notation $\Delta \in \mathbf{B\Delta}$ means $\Delta \in \mathbf{\Delta}$ and $\overline{\sigma}(\Delta) \leq 1$ and similarly for $\mathbf{\Delta_\mathbb{F}}$ and $\mathbf{\Delta_P}$.

A key ingredient in assessing robust performance of the interconnection in figure 1 is the Main Loop Theorem [49, p. 289] which states that assuming nominal internal stability, the interconnection is well-posed, internally stable for all $\Delta \in \mathbf{B\Delta_\mathbb{F}}$ and meets robust performance constraints

$$\|\mathcal{F}_u(\mathcal{F}_l(P(s), C(s)),\, \Delta)\|_\infty < 1, \qquad \forall \Delta \in \mathbf{B\Delta_\mathbb{F}}\,, \tag{2}$$

if and only if

$$\mu_{\mathbf{\Delta}}(\mathcal{F}_l(P(s), C(s))) < 1,\ \forall s = j\omega, \omega \in \mathbb{R} \cup \{\infty\}\,, \tag{3}$$

where here $\mathbf{\Delta}$ is the diagonally augmented structure $\mathbf{\Delta} = \mathrm{diag}(\mathbf{\Delta_\mathbb{F}}, \mathbf{\Delta_P})$.

The quantity $\mu_{\mathbf{\Delta}}$ is the structured singular value computed with respect to the uncertainty structure $\mathbf{\Delta}$. This result means that we can assess robust performance of a closed-loop system by using a frequency evaluation of $\mu_{\mathbf{\Delta}}$. Also, the peak value on the $\mu_{\mathbf{\Delta}}$ plot captures the inverse of the size of the uncertainty against which the loop maintains robust performance.

2 μ SYNTHESIS WITH DYNAMIC D-SCALINGS

In this section, the focus is on the case $\delta_i \in \mathbb{C}$ for $i = 1, \ldots, q$. Except in special instances [35, 38], μ is essentially intractable and is replaced by an easily computable upper bound. That is, the robust performance condition (3) holds whenever there exist for each ω, D-scaling matrices D_ω such that

$$\overline{\sigma}(D_\omega \mathcal{F}_l(P(j\omega), C(j\omega)) D_\omega^{-1}) < 1,\ \forall \omega \in \mathbb{R} \cup \{\infty\}\,, \tag{4}$$

where for all ω

$$D_\omega \in \mathbf{D_\Delta} := \{D :\ D \in \mathbb{C}^{N \times N}, \det(D) \neq 0,\ \Delta D = D\Delta,\ \forall \Delta \in \mathbf{\Delta}\}\,.$$

It follows from this definition that D_ω-scalings enjoy the spatial structure

$$\mathrm{diag}(D_1, \ldots, D_q,\, d_P I_{m_1}),\ D_i \in \mathbb{C}^{r_i \times r_i},\ d_P \in \mathbb{C}\,. \tag{5}$$

The infinite sequence $(D_\omega)_\omega$ neither needs to be state-space realizable nor to correspond to a stable system. Tractability becomes apparent if one reformulates (4)

$$\mathcal{F}_l(P(j\omega), C(j\omega))^H S_\omega \mathcal{F}_l(P(j\omega), C(j\omega)) - S_\omega \prec 0, \qquad \forall \omega \in \mathbb{R} \cup \{\infty\}\,, \tag{6}$$

where $S_\omega := D_\omega^H D_\omega \succ 0$. What we have obtained is an infinite set of LMI constraints indexed by ω. In the classical μ synthesis approach the left-hand side of (4) is minimized by alternating minimizations over $C(s)$ and over $D(s)$ with either one of these terms fixed. This is the so-called $D - K$ iteration. $C(s)$ is obtained as the result of an H_∞ synthesis run for the scaled plant in (4) whereas state-space realizable D-scalings are built by

- computing estimates D_ω of optimal scalings on a discrete grid of frequencies.
- fitting the finite family $(D_\omega)_\omega$ with stable and inversely stable transfer functions $D(s)$.

The rationale in enforcing stable and minimum phase D-scalings originates in the inherent limitations of H_∞ synthesis which by definition must achieve internal stability of the closed-loop scaled plant $D(s)\mathcal{F}_l(P(s), C(s))D(s)^{-1}$. Apart notable exceptions [47, 42, 16], the overall procedure is restricted to a few complex full blocks $\Delta_i \in \mathbb{C}^{f_i \times f_i}$ since D-scalings simplify to $D_i(s) = d_i(s)I_{r_i}$. The outlined scheme would become considerably more complex and costly if a couple of scalar blocks $\delta_i I_{r_i}$ were present as D-scalings are now full unstructured LTI systems $D(s)$. Another hindrance concerns the size inflation in the controller order since H_∞ synthesis is a full-order method both the plant order in (1) and the order of the D-scaling accumulate at the outcome in the order of $C(s)$. Despite apparent difficulties and sources of numerical trouble, the whole procedure has proven to work well in a number of examples [1, 9, 39, 41, 20].

Hereafter, we overcome most of the above-mentioned difficulties by rearranging D-scaling μ synthesis as a nonsmooth program involving a specially structured controller thereby allowing simultaneous construction of both the controller and the scaling in one shot. The resulting program is supported by a sound local optimality certificate established in [6, 3].

For complex uncertainties $\Delta \in \mathrm{diag}(\mathbf{\Delta}_\mathbb{C},\ \mathbf{\Delta_P})$ the suggested rearrangement trick is simple and illustrated in figure 2.

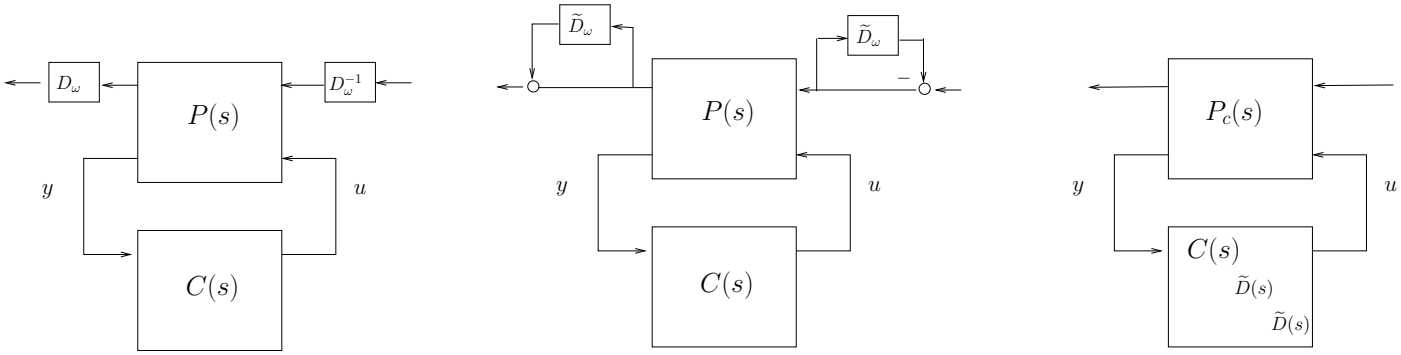

Figure 2: translation on D-scalings

We introduce new scalings $\widetilde{D}_\omega = D_\omega - I_N$ and infer $\widetilde{D}_\omega\Delta = \Delta\widetilde{D}_\omega$ whenever $D_\omega \in D_{\mathbf{\Delta}}$. With the new translated scalings, the left-hand side diagram of figure 2 clearly is an LFT in $C(s)$ and $\widetilde{D}_\omega$ which is readily derived by computing the transfer function around the latter components. We have

$$D_\omega\mathcal{F}_l(P(j\omega), C(j\omega))D_\omega^{-1} = \mathcal{F}_l\left(P_c(s), \begin{bmatrix} C(s) & 0 & 0 \\ 0 & \widetilde{D}_\omega & 0 \\ 0 & 0 & \widetilde{D}_\omega \end{bmatrix}\right), \tag{7}$$

with the definition $P_c(s) := L_c P(s) R_c + Q_c$ with left, right outer factors, and constant

$$P_c(s) = \left[\begin{array}{cc} I_N & 0 \\ \hline 0 & I_{p_2} \\ 0 & 0 \\ I_N & 0 \end{array}\right] P(s) \left[\begin{array}{c|ccc} I_N & 0 & -I_N & 0 \\ 0 & I_{m_2} & 0 & 0 \end{array}\right] + \left[\begin{array}{c|ccc} 0 & 0 & 0 & I \\ \hline 0 & 0 & 0 & 0 \\ I & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Note the partitioning above emphasizes the structure of the LFT in (7).

Accepting for a moment the seemingly mild conservatism induced by searching over state-space realizable D-scalings, the complex μ synthesis problem becomes a design problem with specially structured controllers. Incorporating nominal stability, a nonsmooth cast is obtained as

$$\begin{array}{ll} \underset{K(s)}{\text{minimize}} & \sup_{\omega\in\mathbb{R}^+} \overline{\sigma}\left(\mathcal{F}_l\left(P_c(j\omega), K(j\omega)\right)\right) \\ \text{subject to} & \text{nominal stability} \end{array}$$

$$K(s) := \left[\begin{array}{ccc} C(s) & 0 & 0 \\ 0 & \widetilde{D}(s) & 0 \\ 0 & 0 & \widetilde{D}(s) \end{array}\right], \quad \widetilde{D}(s) \in \widetilde{\mathbf{D}}_{\mathbf{\Delta}}\,,$$

with the shifted scaling set $\widetilde{\mathbf{D}}_{\mathbf{\Delta}}$ defined as

$$\widetilde{\mathbf{D}}_{\mathbf{\Delta}} := \{\widetilde{D}:\ \widetilde{D}\in\mathbb{C}^{N\times N},\ \det(I_N+\widetilde{D})\neq 0,\ \Delta\widetilde{D}=\widetilde{D}\Delta,\ \forall\Delta\in\mathbf{\Delta}\}\,.$$

Nominal stability can be handled in different ways [4]. Adding a constraint on the spectral abscissa of the closed-loop $(G(s), C(s))$ is one possibility:

$$\alpha\{A(G(s),\, C(s))\} \leq -\epsilon\,,$$

Summing up the above, complex μ synthesis is recast as the nonsmooth program with block-diagonal controller $K(s) := \mathrm{diag}(C(s),\, \widetilde{D}(s),\, \widetilde{D}(s)))$:

$$\left|\begin{array}{ll} \underset{K(s)}{\text{minimize}} & \sup_{s=j\omega,\,\omega\in\mathbb{R}^+} \overline{\sigma}\left\{\mathcal{F}_l\left(P_c(s), K(s)\right)\right\} \\ \\ \text{subject to} & \alpha\{A(G(s),\, C(s))\} \leq -\epsilon \\ \\ & K(s) := \left[\begin{array}{ccc} C(s) & 0 & 0 \\ 0 & \widetilde{D}(s) & 0 \\ 0 & 0 & \widetilde{D}(s) \end{array}\right], \quad \widetilde{D}(s) \in \widetilde{\mathbf{D}}_{\mathbf{\Delta}}\,. \end{array}\right. \tag{8}$$

Remarks: A few comments are in order. The supremum in (8) is easily computed using the standard Hamiltonian technique for H_∞ norm computation [14, 12]. It also delivers all the necessary information to construct subdifferential information for program (8).

It is important at this stage to stress that D-scalings $D(s)$ are not necessarily stable hence reducing the conservatism of classical μ-synthesis schemes. The above analysis is easily generalized to multiple complex full uncertainty blocks upon re-defining D-scalings accordingly.

We note that setting $\widetilde{D}(s) = 0$ reduces program (8) to a pure H_∞ synthesis problem while keeping $C(s)$ fixed leads to a μ-analysis problem.

Finally, the reader is pointed to [2] for some hints on how the technique could be generalized to the more delicate problem of arbitrary D-scalings D_ω.

Last but not least, the suggested scheme can be used to compute finely structured controllers $C(s)$ including PIDs, decentralized or lead-lag compensators, etc. This attractive feature of the nonsmooth approach is not accessible with conventional $D-K$ iteration procedures since synthesis phases rely on full-order controllers.

3 MIXED-μ SYNTHESIS WITH MULTIPLIERS

This section is concerned with developing an analogue of section 2 when parametric uncertainties are present. To this aim, we assume $\mathbb{F} = \mathbb{R}$, so that $\delta_i \in \mathbb{R}$ for $i = 1, \ldots, q$. The robust performance problem stated in (3) becomes a mixed-μ synthesis problem. Part of the material in this section is borrowed from [49, 28, 32].

The extension of (6) to parametric uncertainties is the infinite set indexed by ω of LMI constraints

$$F(j\omega)^H S_\omega F(j\omega) + j(G_\omega F(j\omega) - F(j\omega)^H G_\omega) - S_\omega \prec 0, \forall \omega \in \mathbb{R} \cup \{\infty\}\,, \tag{9}$$

with the shorthand $F(j\omega) := \mathcal{F}_l(P(j\omega), C(j\omega))$ and where G_ω scalings are defined as

$$G_\omega \in \mathbf{G}_{\mathbf{\Delta}} := \left\{G :\, G = G^H \in \mathbb{C}^{N\times N},\, G\Delta = \Delta^H G,\, \forall \Delta \in \mathbf{\Delta}\right\}\,.$$

It follows from this definition that G_ω-scalings exhibit for all ω the spatial structure

$$\mathrm{diag}(G_1, \ldots, G_q, 0_{m_1\times m_1}),\; G_i = G_i^H,\; i = 1, \ldots, q\,. \tag{10}$$

Characterization (9) again seems to advocate for a resolution technique where S_ω, G_ω scalings are computed pointwise, curve fitting is used to built rational representations, and stable factors of scalings are derived, followed by an H_∞ synthesis, and so forth, till no further progress is observed. Unfortunately as the author himself admits, this scheme is not practical and prone to failure when there are repeated scalar blocks [48].

Invoking a bilinear transformation applied to $\mathcal{F}_l(P(j\omega), C(j\omega))$, and denoting $M_\omega = -(S_\omega + jG_\omega)$, inequality (9) can be turned into the negative real condition

$$M_\omega \mathcal{B}_N \star \mathcal{F}_l(P(j\omega), C(j\omega)) + (M_\omega \mathcal{B}_N \star \mathcal{F}_l(P(j\omega), C(j\omega)))^H \prec 0,\; \forall \omega \in \mathbb{R} \cup \{\infty\}\,,$$

which by associativity of the star product is easily rewritten as

$$M_\omega \mathcal{F}_l(\mathcal{B}_N \star P(j\omega), C(j\omega)) + (M_\omega \mathcal{F}_l(\mathcal{B}_N \star P(j\omega), C(j\omega)))^H \prec 0,\; \forall \omega \in \mathbb{R} \cup \{\infty\}\,. \tag{11}$$

Note $\mathcal{B}_N \star P$ is well-posed as soon as the LFT model defined in (1) is well-defined on $\mathbf{B}\mathbf{\Delta}$ what we assume throughout. By construction M_ω enjoys the same spatial structure as D-scalings in (5) with the negative real property $M_\omega + M_\omega^H \prec 0$, for all ω. For future use, we denote $\mathbf{M}_{\mathbf{\Delta}}$ the set of all such multipliers.

All put together, and proviso nominal stability of the closed-loop underlying plant, the robust performance condition (2) holds whenever

$$\begin{bmatrix} M_\omega \mathcal{F}_l(\mathcal{B}_N \star P(j\omega), C(j\omega)) & 0 \\ 0 & M_\omega \end{bmatrix} + \begin{bmatrix} M_\omega \mathcal{F}_l(\mathcal{B}_N \star P(j\omega), C(j\omega)) & 0 \\ 0 & M_\omega \end{bmatrix}^H \prec 0,\; \forall \omega \in \mathbb{R} \cup \{\infty\}\,. \tag{12}$$

Introducing left and right outer factors L_r, R_r, and constant Q_r defined as

$$L_r := \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 0 & I_{p_2} \\ I_N & 0 \\ 0 & 0 \end{array}\right],\; R_r = \left[\begin{array}{cc} I_N & 0 \\ 0 & 0 \\ \hline 0 & I_{m_2} \\ 0 & 0 \\ 0 & 0 \end{array}\right]^T,\; Q_r = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{array}\right],$$

inequality in (12) is re-expressed as

$$\mathcal{F}_l\left(L_r(\mathcal{B}_N \star P(j\omega))R_r + Q_r, \begin{bmatrix} C(j\omega) & 0 & 0 \\ 0 & M_\omega & 0 \\ 0 & 0 & M_\omega \end{bmatrix}\right) + (.)^H \prec 0\,. \tag{13}$$

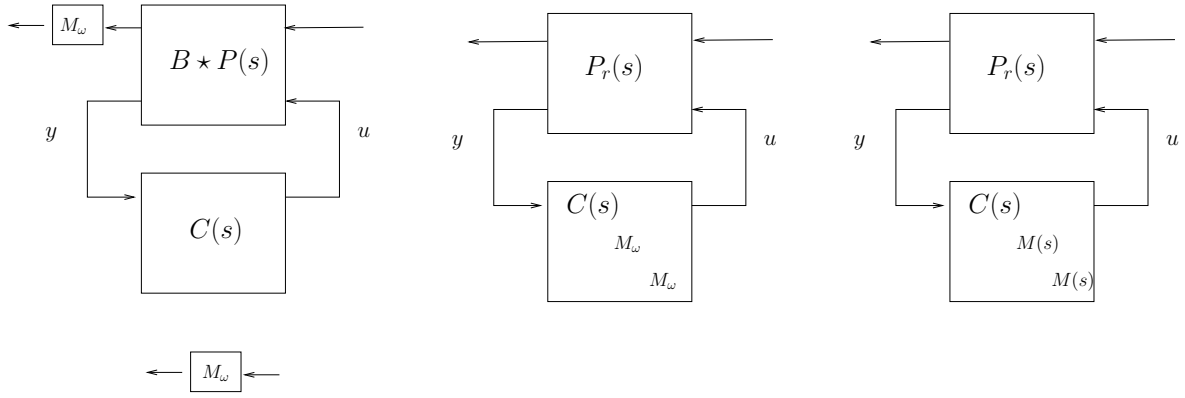

Figure 3: mixed-μ synthesis as structured controller design

Restricting the search to state-space realizable multipliers $M_\omega := M(j\omega)$, and denoting $P_r(s) := L_r(\mathcal{B}_N \star P(s))R_r + Q_r$ leaves us with the nonsmooth cast:

$$\left|\begin{array}{ll} \underset{K(s)}{\text{minimize}} & \sup_{s=j\omega,\,\omega\in\mathbb{R}^+} \overline{\lambda}\left\{\mathcal{F}_l\left(P_r(s), K(s)\right) + \mathcal{F}_l\left(P_r(s), K(s)\right)^H\right\} \\ \text{subject to} & \alpha\{A(G(s),\, C(s))\} \le -\epsilon \\ & K(s) := \begin{bmatrix} C(s) & 0 & 0 \\ 0 & M(s) & 0 \\ 0 & 0 & M(s) \end{bmatrix}, \quad M(s) \in \mathbf{M}_{\mathbf{\Delta}}\,. \end{array}\right. \tag{14}$$

This establishes that mixed-μ upper bound synthesis may be viewed as synthesizing a controller with special (repeated) block-diagonal structure and that the conventional distinction between analysis and synthesis is merely an artefact due to the lack of resolution methods. The case of time-varying parameters with arbitrary rates of variation is easily recovered by restricting the search for multipliers $M(s)$ to real-valued static multipliers with complying structure, $M(j\omega) = M$ for all ω with $M + M^T \prec 0$. Although not developed further here this result has also obvious consequences for the synthesis of robust controllers in the presence of slowly varying parameters and for a number of variants involving multiplier-based characterizations. In the proposed nonsmooth optimization approach the orders of the multipliers as well as the order and structure of the controller is set beforehand. A very favorable feature is therefore that complexity in scalings and multipliers does not meddle in the controller complexity. Multipliers are sought in non-necessarily stable state-space realizable form and do not rely on expansion on finite basis as is often the case with current techniques. As in section 2, the supremum in (14) as well as the accompanying subdifferential information are easily computed using standard Hamiltonian techniques.

If conservatism is an issue in a specific application the outlined approach is easily generalized to synthesis with multiple frequency bands. By this we mean that a single structured controller $C(s)$ but different multipliers are employed, say, in the low, medium and high frequency ranges. Here again the cast is that of synthesizing a specialized structured controller capturing all free components (controller and multipliers) with 3 frequency domain criteria of type (14) or a maximum of such. The limit case of such a strategy would be to use a different multiplier at every frequency M_ω or more realistically piece-wise constant multipliers M_ω on a family of frequency intervals.

If the aim is to achieve best possible performance given the uncertainty set, we will solve a sequence of problems of the form (14) where $P_r(s)$ is replaced with

$$P_{r,\gamma}(s) := L_r \left(\mathcal{B}_N \star \begin{bmatrix} I_{\sum_i r_i} & 0 & 0 \\ 0 & \frac{1}{\gamma} I_{m_1} & 0 \\ 0 & 0 & I_{p_2} \end{bmatrix} P(s) \right) R_r + Q_r$$

where γ is driven downwards using a dichotomy search. Clearly, the same argument holds if best possible robustness is sought given a prescribed level of performance.

It is possible to get rid of the spectral abscissa constraint in (14) through the use of bistable factorizations of the multiplier. We refer the reader to [18, 23, 8] for an in-depth treatment of system analysis using general anti-causal multipliers. In this setting, $M(s)$ is rewritten as $M(s) = M_1(s)^H M_2(s)$ where both $M_1(s)$ and $M_2(s)$ are stable with stable inverse. It can then be shown that the negative real conditions in (12) become

$$M_2(j\omega)\mathcal{F}_l\left(\mathcal{B}_N \star P(j\omega), C(j\omega)\right) M_1(j\omega)^{-1} + (.)^H \prec 0,\ \forall \omega \in \mathbb{R} \cup \{\infty\},$$

and

$$M_1(j\omega)M_2(j\omega)^{-1} + (M_1(j\omega)M_2(j\omega)^{-1})^H \prec 0,\ \forall \omega \in \mathbb{R} \cup \{\infty\}\,.$$

Defining $W_1(s) := M_1(s) - I$ and $W_2(s) := M_2(s) - I$, the cast in (14) can be equivalently formulated as the nonsmooth program

$$\left| \begin{array}{ll} \underset{K(s)}{\text{minimize}} & \max\left\{ \|\mathcal{F}_l\left(P_w(s), K(s)\right)\|_\infty, \quad \|\mathcal{F}_l\left(P_m, \begin{bmatrix} W_1(s) & 0 \\ 0 & W_2(s) \end{bmatrix}\right)\|_\infty \right\} \\ \text{subject to} & \\ & K(s) := \begin{bmatrix} C(s) & 0 & 0 \\ 0 & W_1(s) & 0 \\ 0 & 0 & W_2(s) \end{bmatrix}, \end{array} \right. , \tag{15}$$

with the plants $P_w(s)$ and static P_m defined as

$$P_w(s) := \widetilde{\mathcal{B}}_N \star \left(L_w(\mathcal{B}_N \star P(s))R_w + Q_w\right),\ P_m := \widetilde{\mathcal{B}}_N \star \left[\begin{array}{c|cc} I_N & I_N & -I_N \\ \hline I_N & 0 & -I_N \\ I_N & 0 & -I_N \end{array}\right],$$

where left and right outer factors and constant in P_w

$$L_w := \left[\begin{array}{cc} I_N & 0 \\ \hline 0 & I_{p_2} \\ 0 & 0 \\ I_N & 0 \end{array}\right],\ R_w = \left[\begin{array}{cc} I_N & 0 \\ \hline 0 & I_{m_2} \\ -I_N & 0 \\ 0 & 0 \end{array}\right]^T,\ Q_w = \left[\begin{array}{c|ccc} 0 & 0 & 0 & I_N \\ \hline 0 & 0 & 0 & 0 \\ I_N & 0 & -I_N & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Note the multiplier factors $W_1(s)$ and $W_2(s)$ enjoy the same spatial structure as $M(s)$. As before, the matrix partitioning above materializes the lower LFT structure with feedback gain $K(s)$.

Note the first index of the max-function in (15) captures stability of the nominal plant so that spectral abscissa constraint as in (14) becomes superfluous. Considered jointly the H_∞ indices also ensure that multiplier factors $M_1(s)$ and $M_2(s)$ are bistable. Finally, if a controller $K(s)$ with suitable structure is found such that the max-function is strictly less than unity then controller $C(s)$ meets robust performance constraints as described in (2). A similar comment applies to programs (8) and (14) with the proviso that nominal stability holds. Programs such as (15) aggregate a family of H_∞ objectives and are related to multidisk problems examined in [21].

4 Nonsmooth techniques

μ-synthesis problems discussed in this paper are solved using tailored nonsmooth optimization techniques. Obviously the full details are outside the scope of this paper, and we refer the reader to [3, 5] for comprehensive expositions on line-search-based methods and to [37] and references therein for those adopting a trust region strategy. Alternative nonsmooth optimization methods for solving control-related problems are also discussed in [27, 36, 15]. In the sequel we recall the salient features of the line-search-based method as it is used in the application section.

Problems (8), (14) and (15) involve solving semi-infinite, nonconvex and nonsmooth programs of the form

$$\underset{\kappa\in\mathbb{R}^k}{\text{minimize}}\, f_\infty(\kappa) := \max_{\omega\in[0,\,\infty]} f(\kappa,\omega)\,, \tag{16}$$

where $\kappa \in \mathbb{R}^k$ is the decision vector which gathers all free variables in the (structured) controller $K(s)$. By nature these programs represents challenging mathematical programming problems. It is central to note however that function f_∞ features a composite structure $[\max_\omega \circ\, \overline{\sigma}] \circ T$ where the bracketed function is convex but nonsmooth while T denotes a closed-loop transfer function $(\kappa,\omega) \to T(\kappa,\omega)$ which albeit nonconvex is differentiable in κ for all ω. Such functions are Clarke regular [17] which means that a complete description of the Clarke subdifferential $\partial f(\kappa)$ is accessible. This is a key property as it will allow us to distinguish between critical points including local minima κ^*, that is, those with $0 \in \partial f(\kappa^*)$ from points κ that must be discarded, i.e., $0 \notin \partial f(\kappa)$.

From now on, we restrict our discussion to unconstrained programs of the form (16) since semi-infinite, nonconvex and nonsmooth programs involving constraints of similar nature can be handled through the introduction of a so-called progress function [7] which preserves both Clarke properties and the max structure in (16).

Solving (16) is carried out computationally by constructing a tangent model around the current iterate κ which is a quadratic first-order local approximation of the original problem (16). We introduce

$$\widehat{f}_\infty(\kappa+h,\kappa) := \max_{(\phi_f,\Phi_f)\in W_f} \phi_f - f_\infty(\kappa) + \Phi_f^T h + \frac{1}{2} h^T Q h\,, \tag{17}$$

where h is the displacement in the κ-space. For a given κ, the set W_f collects functions values $\phi_f := f(\kappa,\omega)$ and subgradients $\Phi_f \in \partial f(\kappa,\omega)$ over an extended set of frequencies $\omega \in \Omega_f$. This can be implemented in different ways [3] but a minimal requirement is that Ω_f should at least contain active frequencies ω_a, that is, those achieving the maximum value in (16), we have that

$$f_\infty(\kappa) = f(\kappa,\omega_a)\,.$$

The term extension set comes from the fact that Ω_f may include additional frequencies to improve our tangent model in (17) and thereby promote convergence. See figure 4.

A key ingredient here is that selecting active frequencies (global peaks) is enough for the algorithm to converge. But adding a few more frequencies can accelerate convergence because tangent program (17) forms a better local approximation of the nonsmooth function and better steps are performed at every iteration. Extensive numerical testing to raise the algorithm to professional standards indicates that including frequencies bracketing global maxima and frequencies corresponding to secondary peaks (local maxima) dramatically enhances convergence. Both active and secondary peaks are easily estimated using a standard method for the H_∞-norm computation [14]. Secondary peaks often are good candidates for becoming active peaks on latter iterations and thereby provide anticipated information. Bracketing frequencies on the other side are more heuristic [4, 3] and are chosen to correspond to a 10% below cut of the H_∞-norm value: $f(\kappa, \omega_b) := 0.9 f_\infty(\kappa)$ where ω_b designates bracketing frequencies. Characterized in this way, bracketing frequencies also can be computed using the Hamiltonian method [14]. Of course other strategies are possible.

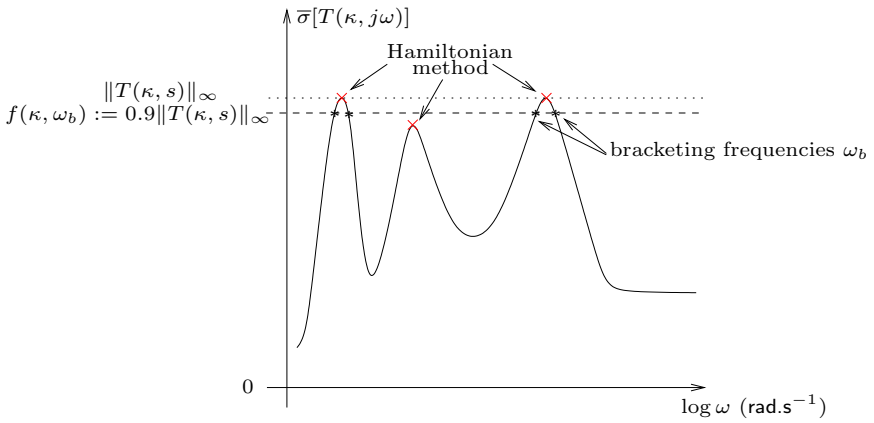

Figure 4: frequency selection to build tangent program

Program (17) is a standard convex quadratic program (CQP), and can be efficiently solved using currently available codes. Note the terminology "quadratic first-order local approximation" marks a subtle difference with the usual "quadratic second-order local approximation". Tangent program (17) indeed does extract first-order information from (Clarke) subgradients whereas the quadratic term is not a (second-order) Hessian in the usual sense. The latter is out of reach because functions are typically nonsmooth. Matrix Q is either a trust region management parameter or some sort of quasi-Newton estimate of the function local curvature. The way matrix Q evolves in the course of the algorithm is discussed in various papers along with the convergence theory. See [6, 11] and references therein. The algorithm used in the application section is a line search algorithm which updates matrix Q using a quasi-Newton BFGS technique where gradients of the BFGS formula correspond to the most active frequency. The key facts about (17) have been established in [3] and we re-state them here without proof:

- The fact that extension sets Ω_f contain active frequencies ensures that the solution h to (17) is a descent direction of f_∞ at κ. If it happens that $h = 0$, then $0 \in \partial f_\infty(\kappa)$, and we are done. Clearly, a stopping test may be based on the solution to the tangent program.

- The direction h can be used in an Armijo or Wolfe line search [10] which terminates after finitely many steps.

We then have all components to build a line search descent algorithm where at each iteration, a descent direction is computed, a line search is performed until a termination test holds. This algorithm has a sound convergence certificate which means that it converges to a critical point (a local minimum in practice) for any even remote starting point. Our point is that this strategy is generally preferable to biconvex $D-K$-iteration schemes or alike since they are prone to stall at points that are not local minima. Reasons of failures of biconvex schemes are analyzed in [30, 29]. Finally, the proposed technique runs in the κ-space of free controller parameters without recourse to auxiliary variables such as Lyapunov variables which result in significant size inflation in BMI- or LMI-based methods.

5 Examples

Computations in the examples hereafter are based on a general purpose nonsmooth programming software. A first cut for structured H_∞ synthesis is made available commercially through the MATLAB R2010b Prerelease *Robust Control Toolbox Version 3.5* developed by The MathWorks, Inc. [33].

5.1 Lightly damped plant with parametric uncertainties

The example below is an excerpt from [48] which has been modified to yield an unstable nominal system. This is a somewhat contrived example as it has been built to stress issues due to (real) parametric uncertainties. The uncertain plant is described as the second-order transfer function

$$G(s,\delta) := \frac{s^2 + 2\zeta\omega_n s + \omega_n^2(1+\delta_1)}{s^2 - 2\zeta\omega_n s + \omega_n^2(1+\delta_2)}\,, \tag{18}$$

where δ_1, δ_2 are uncertain real parameters, i.e. $\delta_1,\, \delta_2 \in \mathbb{R}$ and $\omega_n = 1$, $\zeta = 0.2$. Such descriptions are typical of lightly damped modes with uncertain natural frequency. This can be rearranged into the LFT format of figure 1 upon defining

$$P(s) := \left[\begin{array}{cc|ccc} 0 & 1 & -1/\omega_n^2 & -1/\omega_n^2 & 0 \\ -\omega_n^2 & 2\zeta\omega_n & 0 & 0 & 4\zeta\omega_n \\ \hline 0 & 0 & 0 & 0 & \omega_n^2 \\ 0 & -\omega_n^2 & 0 & 0 & -\omega_n^2 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right],$$

and

$$\Delta = \left[\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array}\right].$$

The aim in this illustration is the synthesis of a robust controller maximizing the parametric margin, that is, the radius of the square in the $(\delta_1,\ \delta_2)$-space where the closed-loop system remains stable. Note that for $(\delta_1,\ \delta_2) = (-1,\ -1)$, system (18) has an unstable non-minimal pole at $s = 0$. We infer the maximum reachable parametric margin for this system is bounded above by one no matter the technique used.

Results are collected in table 1 where various techniques are compared. The classical $D-K$ and $D-G$, K iteration schemes with curve fitting and the nonsmooth approach as discussed in section 3. The nonsmooth method is initialized from scratch using random generation of state-space systems, the 'rss' function from Matlab. Both techniques are run in default mode. Various controller orders were prescribed for the nonsmooth technique and we have also included the case of a proportional derivative controller denoted 'PD' in the table. Multiplier state dimensions were set to 4 for each uncertainty channel. The order of the overall controller $K(s)$ is therefore $\dim x_C + 4 + 4$, where x_C designates the state of $C(s)$.

technique	controller order for $C(s)$	parametric margin
$D-K$ complex μ-synthesis	10	0.59
$D-G, K$ real μ-synthesis	26	0.82
$D-G, K$ reduced	14	0.82
nonsmooth real μ-synthesis	0	0.999
nonsmooth real μ-synthesis	1	0.999
nonsmooth real μ-synthesis	2	0.999
nonsmooth real μ-synthesis	3	0.999
nonsmooth real μ-synthesis	PD	0.999

Table 1: achieved parametric margins with different synthesis techniques

Following is a list of controllers of various orders computed using the nonsmooth technique. As clarified earlier in section 3, the order of $C(s)$ is chosen independently of the multiplier orders which is an advantage in realistic applications. We note the $D-G, K$ controller can be reduced to order 14 without perceptible deterioration of robustness but further reduction results in unstable closed-loop systems.

$$C_0(s) = -174.3,\ C_1(s) = \frac{631.5s + 1302}{s + 2.112},\ C_{\mathrm{PD}} = -3.7830 - \frac{9.5637\,10^{-4}s}{1/74.842s + 1}$$

$$C_2(s) = \frac{-1.632s^2 - 236.9s - 7401}{s^2 + 100.1s + 2.891},\ C_3(s) = \frac{-0.1427s^3 - 140.1s^2 - 1.326\,10^4 s - 2.059\,10^4}{s^3 + 104.1s^2 + 53.04s + 0.8339}$$

As an instance, the corresponding multiplier for the first-order controller $C_1(s)$ is obtained as $M(s) = \mathrm{diag}(M_{11}(s), M_{22}(s))$ where

$$\begin{aligned} M_{11}(s) &= \frac{0.06816s^4 + 1.075s^3 + 29.74s^2 + 675.5s + 1.649\,10^4}{s^4 - 3.138s^3 - 38.23s^2 - 247.6s + 33.24} \\ M_{22}(s) &= \frac{111.9s^4 - 2372s^3 - 4.161\,10^5 s^2 + 7.322\,10^6 s + 2.618\,10^8}{s^4 - 30.69s^3 - 786.7s^2 + 6029s + 849.1} \end{aligned}$$

This is an unstable system whose frequency responses are shown in figure 5. As expected multipliers are especially active in the low frequency range and significant phase variations are observed in the range of variation of lightly damped modes and zeros to achieve positive realness.

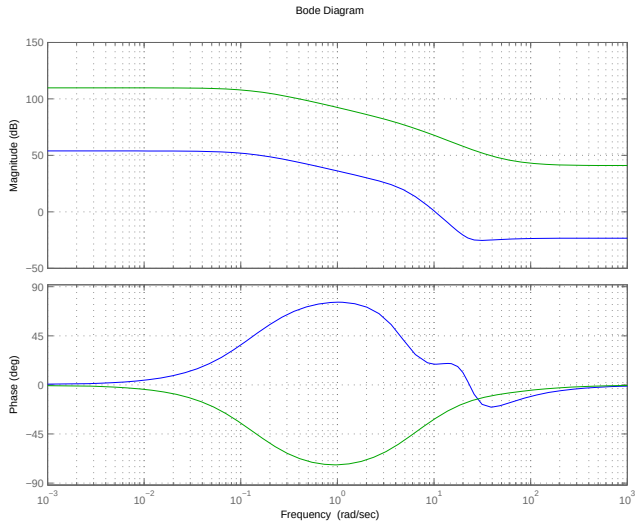

Figure 5: frequency responses of multiplier for first-order controller $C_1(s)$

5.2 Plant with a repeated parameter and one complex block

A major impediment in μ synthesis problems is when repeated scalar blocks are present. Repeated scalar blocks have in correspondance full-block multipliers which entails substantial size inflation in terms of decision variables. Another issue with the conventional approach is that curve fitting is essentially a SISO procedure which means all entries of D and G scalings must be fitted. The overall scaling orders accumulate those of the individual entries so that subsequent synthesis phases become numerically challenging or even may succumb. In the sequel, we consider an example of this type and we show the nonsmooth approach holds promise to overcome the difficulty.

We consider an example borrowed from [46] with plant described as

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \hline y \end{array}\right] = \left[\begin{array}{cccc|ccc} 0.75 & 2 & \cos\rho & \sin\rho & 0 & 0 & 0 \\ 0 & 0.5 & -\sin\rho & \cos\rho & 3 & 0 & 0 \\ 0 & 0 & -10 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & -10 & 0 & 0 & 10 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \hline f \\ u \end{array}\right],$$

where f designates a disturbance signal. Parametric uncertainties are restricted to $\rho \in [-\pi/4,\, \pi/4]$. A reasonably accurate LFT approximation of the model is obtained by using 3rd-order Taylor series expansion of trigonometric functions. This yields an LFT representation with a repeated

real scalar block of size 12. Performance is expressed as minimizing effects of disturbance f on measurements y in the L_2-gain sense. Overall the uncertainty structure can be written as

$$\Delta := \left[\begin{array}{cc} \delta I_{12} & 0 \\ 0 & \Delta_P \end{array} \right]$$

with Δ_P a (complex) performance block and δ a normalization of ρ to the unit interval $[-1, 1]$.

Conventional mixed-μ synthesis techniques based on curve fitting quickly succumb on such problems as they are facing augmented plants of orders 212 and 462 if restricting $D - G$ scalings to orders 1 and 2, respectively.

Our resolution strategy is based on program (14). We have explored various multiplier and controller orders with the nonsmooth technique and results are gathered in table 2.

multiplier order for $M(s)$	controller order for $C(s)$	mixed-μ margin
static	4	0.95
1	4	1.53
2	5	1.67

Table 2: achieved parametric margins with various multiplier/controller combinations

Pure H_∞-synthesis yields a mixed-μ margin of 0.75 and falls short of achieving robust performance. In the first row of table 2, the nonsmooth synthesis was conducted with static multipliers which implicitly assumes that uncertainties are time-varying $\rho = \rho(t)$. The second and third rows involves multipliers and controllers of increasing complexity. In both cases robust performance has been achieved beyond the whole range of parameter variations. Note that mixed-μ related computations were double-checked using the skew-μ toolbox [24].

6 Conclusion

Complex- and mixed-μ syntheses remains very challenging problems if sizeable practical applications is the aim. In this work, we have proposed a different line of attack where complex- and mixed-μ syntheses based on upper bounds are formalized as structured controller design problems which are then solved using a specialized nonsmooth technique with local convergence certificate. In this set-up, both the controller and scalings or multipliers are computed simultaneously as parts of a fictitious block-diagonal controller. The nonsmooth approach allows to bypass a number of issues attached to more conventional methods and seems to offer a valid alternative according to preliminary testing.

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