$\mathcal{H}_\infty$ Optimal Controller Design With Closed-Loop Positive Real Constraints

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Positive real constraints on the closed-loop of linear systems guarantee stable interaction with arbitrary passive environments. Two such methods of $\mathcal{H}_\infty$ optimal controller synthesis subject to a positive real constraint are presented and demonstrated on numerical examples. The first approach is based on an established multi-objective optimal control framework using linear matrix inequalities and is shown to be overly restrictive and ultimately infeasible. The second method employs a sector transformation to substitute the positive real constraint with an equivalent $\mathcal{H}_\infty$ constraint. In two examples, this method is shown to be more reliable and displays little change in the achieved $\mathcal{H}_\infty$ norm compared to the unconstrained design, making it a promising tool for passivity-based controller design. [DOI: 10.1115/1.4036073]

1 Introduction

Closed-loop passivity is often employed to guarantee interaction stability with uncertain passive environments. In the context of tactile human-robot interaction, for example, passivity of the mechanical impedance is prescribed to enable safe interaction with a human user. Traditionally, this is achieved by exploiting controller structure, such as proportional-derivative-type control of Euler–Lagrange systems [1], or finding the passivity preserving parameter regions of fixed structure controllers and plants [2]. For linear time invariant (LTI) systems, the correspondence of passivity to positive real transfer functions if well known and can be expressed through linear matrix inequalities [3]. This has recently been exploited in the design of $\mathcal{H}_2$ optimal controllers with positive real transfer functions, which are thus robust to arbitrary uncertainty in a passive plant [4]. In this contribution, we consider robust optimal control under the constraint that the closed-loop transfer function remains positive real, in order to guarantee stability in interaction with passive environments. An approach using optimal robust control was presented in Ref. [5]. There, a $\mathcal{H}_2$ controller design that satisfies a positive real constraint is described, in which loop performance can be shaped by solving a guaranteed cost problem. In Ref. [6], the $\mathcal{H}_\infty$ optimal control with closed-loop positive realness is considered for the special case of symmetric state space systems. Another approach is presented in Ref. [7], where the $\mathcal{H}_\infty$ norm of the distance to passive transfer function is minimized. This, however, cannot guarantee closed-loop passivity a priori.

This paper aims to augment the established methods of $\mathcal{H}_\infty$ optimal control with a passivity constraint. This enables controller design free of restrictive controller structure with the additional benefit of established robustness properties of the $\mathcal{H}_\infty$ design.

Two approaches are pursued in this paper. The first is based on a multi-objective controller synthesis framework presented by Scherer et al. [8]. Here, the possibility is given to augment an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ objective with a QSR-Dissipativity constraint, including positive realness constraints. The resulting procedure, however, restricts the solution-space during controller synthesis due to a common matrix Lyapunov variable. In this contribution, the feasibility of this approach is evaluated.

The second approach relies on a nonlinear sector transformation called the Cayley transformation. In Safonov et al. [9], the authors have shown how to employ similar transformations for controller synthesis to ensure that the controlled plant's transfer function be positive real [9]. The approach, however, does not consider a performance objective, making controller design with regard to performance goals difficult. Bao et al. presented similar ideas to robustify $\mathcal{H}_\infty$ controllers for arbitrary passive uncertainties [10,11]. They rely on an optimization over the parameterized central $\mathcal{H}_\infty$ controller, which necessitates an iterative and often unstructured outer optimization.

This contribution presents direct approaches to the constraint problem by solving constrained optimization problems and is structured as follows: Section 2 introduces the considered problem formulation and states necessary definitions and Lemmas. In Sec. 3, the investigated approaches are individually presented and finally evaluated on numeric examples in Sec. 4. Section 5 summarizes the results in a short conclusion.

2 Preliminaries

Consider a stable linear time invariant (LTI) system

$$\mathbf{G} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \tag{1}$$

with states $\mathbf{x} \in \mathbb{R}^n$, inputs $\mathbf{u} \in \mathbb{R}^m$, outputs $\mathbf{y} \in \mathbb{R}^p$, and transfer function matrix

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{2}$$

Positive realness for LTI systems, which entails passivity, can be established through the following Lemma.

LEMMA 1 *(Positive Real Lemma). The LTI system $\mathbf{G}$ has a (extended strictly) positive real transfer function $\mathbf{G}(s)$ if and only if there exists a positive definite solution in $\mathbf{X}_{pr}$ to the following linear matrix inequality (LMI):*

$$\begin{bmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{X}_{pr} + \mathbf{X}_{pr}\mathbf{A} & \mathbf{X}_{pr}\mathbf{B} - \mathbf{C}^{\mathrm{T}} \\ \mathbf{B}^{\mathrm{T}}\mathbf{X}_{pr} - \mathbf{C} & -\mathbf{D} - \mathbf{D}^{\mathrm{T}} \end{bmatrix} \preceq 0 \tag{3}$$

where it is extended strictly positive definite if the strict inequality holds.

A passive system in negative feedback with any passive environment will maintain passivity. An LTI system is furthermore asymptotically stable if at least one system is strictly passive [12]. For single-input single-output (SISO) systems, positive realness corresponds to the fact that the real part of the systems frequency response $\mathrm{Re}(T(j\omega))$ is nonnegative for every $\omega \in \mathbb{R}$.

The $\mathcal{H}_\infty$ norm of the system $\mathbf{G}$ can be evaluated using the Bounded Real Lemma.

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Lemma 2 *(Bounded Real Lemma). The LTI system* $\mathbf{G}$ *has a transfer function* $\mathbf{G}(s)$ *with limited gain* $\|\mathbf{G}(s)\|_\infty < \gamma$ *if and only if there exist a positive definite solution in* $\mathbf{X}_\infty$ *to the LMI*

$$\begin{bmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{X}_\infty + \mathbf{X}_\infty\mathbf{A} & \mathbf{X}_\infty\mathbf{B} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{B}^{\mathrm{T}}\mathbf{X}_\infty & -\gamma\mathbf{I} & \mathbf{D}^{\mathrm{T}} \\ \mathbf{C} & \mathbf{D} & -\gamma\mathbf{I} \end{bmatrix} \prec 0 \tag{4}$$

This linear matrix inequality description has frequently been exploited for $\mathcal{H}_\infty$ optimal controller synthesis [13,14].

It is possible to express the positive realness constraint as an equivalent constraint on a $\mathcal{H}_\infty$ norm using the Cayley transformation [10,15].

Lemma 3 *(Cayley Transformation). Consider the LTI system* $\mathbf{G}$ *with transfer function* $\mathbf{G}(s)$*. Let the Cayley transformation be given by*

$$\mathbf{G}'(s) = (\mathbf{I} - \xi\mathbf{G}(s))(\mathbf{I} + \xi\mathbf{G}(s))^{-1} \tag{5}$$

where ξ *is any positive real number. System* $\mathbf{G}$ *is (extended strictly) positive real if and only if*

$$\|\mathbf{G}'(s)\|_\infty \le 1$$

where it is extended strictly positive real if the strict inequality holds.

We will consider a multi-objective generalized plant configuration as presented in Fig. 1, which consists of one channel for the $\mathcal{H}_\infty$ objective and one for a positive real constraint.

The state-space representation of the generalized plant can be expressed as

$$\mathbf{P}(s) \stackrel{s}{=} \left[\begin{array}{c|ccc} \mathbf{A} & \mathbf{B}_\infty & \mathbf{B}_{pr} & \mathbf{B}_2 \\ \hline \mathbf{C}_\infty & \mathbf{D}_\infty & \mathbf{D}_{\infty,pr} & \mathbf{D}_{\infty,2} \\ \mathbf{C}_{pr} & \mathbf{D}_{pr,\infty} & \mathbf{D}_{pr} & \mathbf{D}_{pr,2} \\ \mathbf{C}_2 & \mathbf{D}_{2,\infty} & \mathbf{D}_{2,pr} & 0 \end{array}\right] \tag{6}$$

where we assume $\mathbf{D}_2 = 0$ without loss of generality. The systems dimensions are given by

$$\begin{aligned} &\mathbf{A} \in \mathbb{R}^{n\times n}, \quad \mathbf{B}_\infty \in \mathbb{R}^{n\times m_\infty}, \quad \mathbf{B}_{pr} \in \mathbb{R}^{n\times m_{pr}}, \mathbf{B}_2 \in \mathbb{R}^{n\times m_2}, \\ &\qquad \mathbf{C}_\infty \in \mathbb{R}^{p_\infty\times n}, \quad \mathbf{C}_{pr} \in \mathbb{R}^{p_{pr}\times n}, \quad \mathbf{C}_2 \in \mathbb{R}^{p_2\times n} \end{aligned} \tag{7}$$

With a state-space description of the controller

$$\mathbf{K}(s) \stackrel{s}{=} \left[\begin{array}{c|c} \mathbf{A}_c & \mathbf{B}_c \\ \hline \mathbf{C}_c & \mathbf{D}_c \end{array}\right] \tag{8}$$

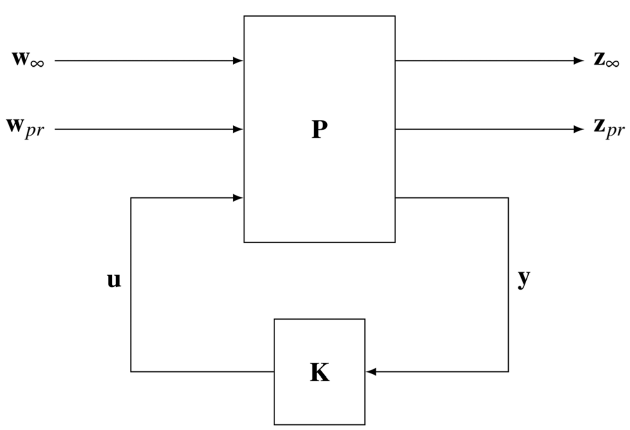

Fig. 1 Multi-objective generalized plant configuration

the closed-loop system is given by

$$\mathbf{P}_{cl}(s) \stackrel{s}{=} \left[\begin{array}{c|cc} \mathbf{A}_{cl} & \mathbf{B}_{cl,\infty} & \mathbf{B}_{cl,pr} \\ \hline \mathbf{C}_{cl,\infty} & \mathbf{D}_{cl,\infty} & (\cdot) \\ \mathbf{C}_{cl,pr} & (\cdot) & \mathbf{D}_{cl,pr} \end{array}\right] \tag{9}$$

where $(\cdot)$ are cross coupling terms that are irrelevant in the multichannel design. A realization of the closed-loop matrices can be obtained using the lower linear fractional transformation of $\mathbf{P}(s)$ and $\mathbf{K}(s)$

$$\mathbf{P}_{cl}(s) = \mathcal{F}_l(\mathbf{P}(s), \mathbf{K}(s))$$

which has a state-space realization

$$\begin{aligned} &\mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A} + \mathbf{B}_2\mathbf{D}_c\mathbf{C}_2 & \mathbf{B}_2\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C}_2 & \mathbf{A}_c \end{bmatrix}, \quad \mathbf{B}_{cl,i} = \begin{bmatrix} \mathbf{B}_i + \mathbf{B}_2\mathbf{D}_c\mathbf{D}_{2,i} \\ \mathbf{B}_c\mathbf{D}_{2,i} \end{bmatrix}, \\ &\mathbf{C}_{cl,i} = \begin{bmatrix} \mathbf{C}_i + \mathbf{D}_{i,2}\mathbf{D}_c\mathbf{C}_2 & \mathbf{D}_{i,2}\mathbf{C}_c \end{bmatrix}, \quad \mathbf{D}_{cl,i} = \begin{bmatrix} \mathbf{D}_i + \mathbf{D}_{i,2}\mathbf{D}_c\mathbf{D}_{2,i} \end{bmatrix} \end{aligned} \tag{10}$$

where $i \in \{pr, \infty\}$ denotes either the positive-real (PR) or the $\mathcal{H}_\infty$ channel.

3 Controller Design With Closed-Loop Positive Real Constraints

3.1 Linear Matrix Inequality Approach. Applying Lemmas 1 and 2 to the closed-loop system of Eq. (9) yields matrix inequalities that are nonlinear in the controller parameters $\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c$ and Lyapunov variables $\mathbf{X}_{pr}, \mathbf{X}_\infty$. A nonlinear variable transformation to linearize the resulting inequalities was presented, which facilitates effective optimization. In order to apply the transformation to the multi-objective case, however, one needs to restrict the Lyapunov variables to be equal

$$\mathbf{X}_\infty = \mathbf{X}_{pr} = \mathbf{X} \tag{11}$$

This was demonstrated to be of little conservatism for combined $\mathcal{H}_2 - \mathcal{H}_\infty$ applications [8]. The change of variables transforms the controller parameters and the Lyapunov variable $\mathbf{X}$ to obtain linearized synthesis equations

$$\begin{bmatrix} \mathbf{A}_c & \mathbf{B}_c \\ \mathbf{C}_c & \mathbf{D}_c \end{bmatrix} \to \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hat{\mathbf{C}} & \hat{\mathbf{D}} \end{bmatrix} \text{ and } \mathbf{X} \to \mathbf{Y}, \tilde{\mathbf{Y}} \tag{12}$$

For details on the linearizing variable transformation and controller reconstruction refer to the cited material [8,16].

With this change of variables, the positive real constraint (Lemma 1) is transformed to the following linear synthesis LMIs

$$\begin{bmatrix} \mathbf{A}\mathbf{Y} + \mathbf{B}_2\hat{\mathbf{C}} + (\mathbf{A}\mathbf{Y} + \mathbf{B}_2\hat{C})^{\mathrm{T}} & \mathbf{A} + \mathbf{B}_2\hat{\mathbf{D}}\mathbf{C}_2 + \hat{\mathbf{A}}^{\mathrm{T}} & (1,3) \\ \star & \tilde{\mathbf{Y}}\mathbf{A} + \hat{\mathbf{B}}\mathbf{C}_2 + (\tilde{\mathbf{Y}}\mathbf{A} + \hat{\mathbf{B}}\mathbf{C}_2)^{\mathrm{T}} & (2,3) \\ \star & \star & (3,3) \end{bmatrix} \preceq 0 \tag{13}$$

with

$$\begin{bmatrix} (1,3) \\ (2,3) \\ (3,3) \end{bmatrix} = \begin{bmatrix} \mathbf{B}\hat{\mathbf{D}}\mathbf{D}_{2,pr} - \mathbf{C}_{pr}\mathbf{Y} + \mathbf{D}_{pr,2}\hat{\mathbf{C}})^{\mathrm{T}} \\ \tilde{\mathbf{Y}}\mathbf{B}_{pr} + \hat{\mathbf{B}}\mathbf{D}_{2,pr} - (\mathbf{C}_{pr} + \mathbf{D}_{pr,2}\hat{\mathbf{D}}\mathbf{C}_2)^{\mathrm{T}} \\ -(\mathbf{D}_{pr} + \mathbf{D}_{pr,2}\hat{\mathbf{D}}\mathbf{D}_{2,pr}) - (\mathbf{D}_{pr} + \mathbf{D}_{pr,2}\hat{\mathbf{D}}\mathbf{D}_{2,pr})^{\mathrm{T}} \end{bmatrix}$$

and $\star$ representing terms easily deduced by symmetry.

Equally, application of the transformation to the $\mathcal{H}_\infty$ objective (Lemma 2) yields

$$
\begin{bmatrix}
\mathbf{AY}+\mathbf{B}_2\hat{\mathbf{C}}+(\mathbf{AY}+\mathbf{B}_2\hat{\mathbf{C}})^{\mathrm{T}} & \star & \star & \star \\
(\mathbf{A}+\mathbf{B}_2\hat{\mathbf{D}}\mathbf{C}_2)^{\mathrm{T}}+\hat{\mathbf{A}} & \tilde{\mathbf{Y}}\mathbf{A}+\hat{\mathbf{B}}\mathbf{C}_2+(\tilde{\mathbf{Y}}\mathbf{A}+\hat{\mathbf{B}}\mathbf{C}_2)^{\mathrm{T}} & \star & \star \\
(\mathbf{B}_\infty+\mathbf{B}_2\hat{\mathbf{D}}\mathbf{D}_{2,\infty})^{\mathrm{T}} & (\tilde{\mathbf{Y}}\mathbf{B}_\infty+\hat{\mathbf{B}}\mathbf{D}_{2,\infty})^{\mathrm{T}} & -\gamma\mathbf{I} & \star \\
\mathbf{C}_\infty\mathbf{Y}+\mathbf{D}_{i,2}\hat{\mathbf{C}}^{\mathrm{T}} & \mathbf{C}_\infty+\mathbf{D}_{i,2}\hat{\mathbf{D}}\mathbf{C}_2 & \mathbf{D}_\infty+\mathbf{D}_{\infty,2}\hat{\mathbf{D}}\mathbf{D}_{2,\infty} & -\gamma\mathbf{I}
\end{bmatrix} \prec 0 \qquad (14)
$$

and the positive definiteness condition on $\mathbf{X}$ results in

$$
\begin{bmatrix} \mathbf{Y} & \mathbf{I} \\ \mathbf{I} & \tilde{\mathbf{Y}} \end{bmatrix} \succ 0 \qquad (15)
$$

A closed-loop system that satisfies Eqs. (13)–(15) is guaranteed to have positive real transfer function from $\mathbf{w}_{pr}$ to $\mathbf{z}_{pr}$ and has $\mathcal{H}_\infty$ norm smaller γ from $\mathbf{w}_\infty$ to $\mathbf{z}_\infty$. Because the equations are linear in all unknowns $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$, $\tilde{\mathbf{D}}$, $\mathbf{Y}$, $\tilde{\mathbf{Y}}$ and γ, it is suitable for optimization with regard to γ, which can be achieved using established methods [17]. The resulting semidefinite program (SDP) is

$$
\min_{\hat{\mathbf{A}},\hat{\mathbf{B}},\hat{\mathbf{C}},\hat{\mathbf{D}},\mathbf{Y},\tilde{\mathbf{Y}}} \quad \gamma\,, \qquad \text{subject to} \begin{cases} \text{Eq. (13)}, \\ \text{Eq. (14)}, \\ \text{Eq. (15)} \end{cases} \qquad (16)
$$

3.2 Non-Smooth Optimization With the Cayley Approach.

As an alternative to fairly standard LMI techniques, we have decided to evaluate tailored non-smooth programming techniques as discussed in Refs. [18,19]. Such techniques indeed offer full flexibility to designers.

(1) Controller complexity is kept under control as they can be used to compute reduced-order or even highly structured controllers such as proportional–integral–derivatives (PIDs) controller, observer-based controllers or any controller architecture made of PIDs, transfer functions and/or state-space models. More formally, a controller $\mathbf{K}$ of the form (8) is called structured if the state-space matrices $\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c$ depend smoothly on a design parameter vector κ varying in some parameter space $\mathbb{R}^{n_\kappa}$, or in a constrained subset of $\mathbb{R}^{n_\kappa}$. In other words, a controller structure $K(\cdot)$, or $K(\kappa)$, consists of four smooth mappings $\mathbf{A}_c(\cdot) : \mathbb{R}^{n_\kappa} \to \mathbb{R}^{k\times k}$, $\mathbf{B}_c(\cdot) : \mathbb{R}^{n_\kappa} \to \mathbb{R}^{k\times p_2}$, $\mathbf{C}_c(\cdot) : \mathbb{R}^{n_\kappa} \to \mathbb{R}^{m_2\times k}$, and $\mathbf{D}_c(\cdot) : \mathbb{R}^{n_\kappa} \to \mathbb{R}^{m_2\times p_2}$.

(2) Non-smooth optimization techniques are inherently multi-objective and can encompass general casts of the form

$$
\begin{aligned}
&\text{minimize } \|\mathcal{F}_l(\mathbf{P}_1, \mathbf{K}(\kappa))\|_\infty \\
&\text{subject to } \|\mathcal{F}_l(\mathbf{P}_i, \mathbf{K}(\kappa))\|_\infty \le \gamma_i,\ i = 2, \ldots, N \\
&\qquad\qquad \mathbf{K} \text{ stabilizes } P_i \text{ internally},\ i = 1, \ldots, N \\
&\qquad\qquad \mathbf{K} \text{ has a fixed structure } \mathbf{K}(\kappa)
\end{aligned} \qquad (17)
$$

where γ_i are some threshold limiting the $\mathcal{H}_\infty$ norm of $\mathbf{z}_i$ in response to $\mathbf{w}_i$. Furthermore, the plants in the objective and $\mathcal{H}_\infty$ constraints of Eq. (17) need not be identical, such that multimodel problems are readily accessible.

For completeness, we should stress that program (17) is NP-hard [20]. This means global solutions are not generally reachable in polynomial time. A more modest and practical goal consists in computing local solutions. The approaches in Refs. [18,19] and implemented in the MATLAB routines hinfstruct and systune [21] have been developed for this purpose. Abundant testing since 2010 indicates local solutions makes sense in practice. To partly overcome nonconvextiy issues, multiple starting points can be used to somewhat globalize the approach. It is also important to note that the program (17) involves a mixture or soft and hard constraints. Hard constraints correspond to classical constraints in mathematical programming and prevail over soft constraint. Practically speaking, this means a controller has to satisfy hard constraints to be acceptable. A local solution is by definition a locally optimal controller in the set of hard constraint feasible controllers. Refer to the cited material [18,19] for details on the nonsmooth optimization.

Given the cast (17), our second approach to PR constraints is based on the Cayley transformation (Lemma 3). We aim to employ the transformation to change the mixed $\mathcal{H}_\infty$ -PR problem into a multichannel $\mathcal{H}_\infty$ problem. The Cayley transformation

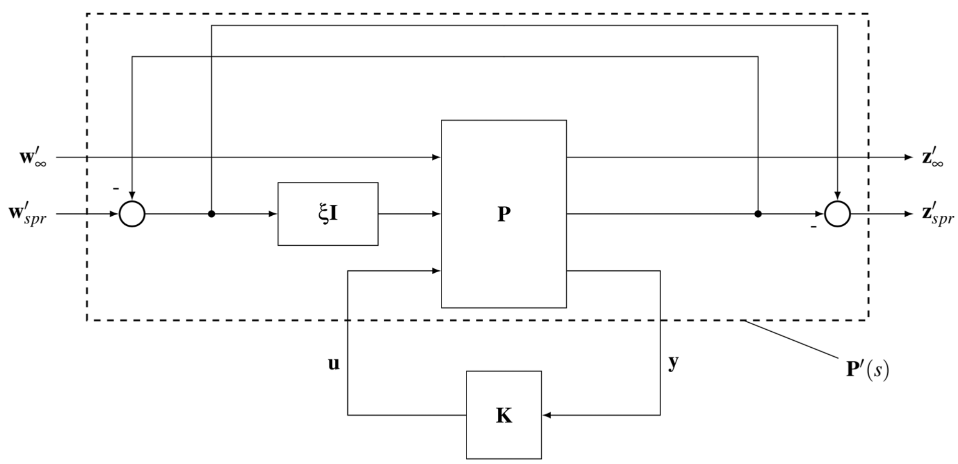

Fig. 2 Block diagram representation of the multi-objective Cayley transformation

$\mathbf{G}'_{pr}(s) = (\mathbf{I} - \xi \mathbf{G}_{pr}(s))(\mathbf{I} + \xi \mathbf{G}_{pr}(s))^{-1}$ can be expressed through simple feedback and feed-forward operations. The application of the procedure to the multi-objective plant (Fig. 1) is displayed in Fig. 2 in form of a block diagram, where the $\mathbf{P}'$ refers to the transformed plant.

Applying linear fractional transformations, we can identify the system equations of the plant after transformation. In transfer function representation, this yields for the $\mathcal{H}_\infty$ part of the plant (for readability, we drop the dependency on s in the following equations when convenient)

$$\begin{aligned}\mathbf{P}'_\infty &= \begin{bmatrix} \mathbf{G}'_\infty & \mathbf{G}'_{\infty,2} \\ \mathbf{G}'_{2,\infty} & \mathbf{G}'_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{G}_\infty + \xi \mathbf{G}_{\infty,pr}(\mathbf{I} + \xi \mathbf{G}_{pr})^{-1}\mathbf{G}_{pr,\infty} & \mathbf{G}_{\infty,2} + \xi \mathbf{G}_{\infty,pr}(\mathbf{I} + \xi \mathbf{G}_{pr})^{-1}\mathbf{G}_{pr,2} \\ \mathbf{G}_{2,\infty} + \xi \mathbf{G}_{2,pr}(\mathbf{I} + \xi \mathbf{G}_{pr})^{-1}\mathbf{G}_{pr,2} & \mathbf{G}_2 + \xi \mathbf{G}_{2,pr}(\mathbf{I} + \xi \mathbf{G}_{pr})^{-1}\mathbf{G}_{pr,2} \end{bmatrix}\end{aligned} \tag{18}$$

and the positive real part

$$\begin{aligned}\mathbf{P}'_{pr} &= \begin{bmatrix} \mathbf{G}'_{pr} & \mathbf{G}'_{pr,2} \\ \mathbf{G}'_{2,pr} & \mathbf{G}'_2 \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{I} + \xi \mathbf{G}_{pr})^{-1}(\mathbf{I} - \xi \mathbf{G}_{pr}) & -2(\mathbf{I} + \xi \mathbf{G}_{pr})^{-1}\mathbf{G}_{pr,2} \\ \xi \mathbf{G}_{2,pr}(\mathbf{I} + \xi \mathbf{G}_{pr})^{-1} & \mathbf{G}_2 + \xi \mathbf{G}_{2,pr}(\mathbf{I} + \xi \mathbf{G}_{pr})^{-1}\mathbf{G}_{pr,2} \end{bmatrix}\end{aligned} \tag{19}$$

while the entire plant can be given by

$$\mathbf{P}' = \begin{bmatrix} \mathbf{G}'_\infty & (\cdot) & \mathbf{G}'_{\infty,2} \\ (\cdot) & \mathbf{G}'_{pr} & \mathbf{G}'_{pr,2} \\ \mathbf{G}'_{2,\infty} & \mathbf{G}'_{2,pr} & \mathbf{G}'_2 \end{bmatrix} \tag{20}$$

where again the terms in $(\cdot)$ can be neglected because cross-coupling terms are not considered in the optimization. A derivation of Eq. (20) is given in the Appendix.

It is obvious from Eq. (18) that the $\mathcal{H}_\infty$ channel is not unaffected by the transformation. The individual terms consist of the sum original terms and terms that are due to the feedback connection of the transformation. The gain ξ can, however, theoretically be chosen to be arbitrarily small to reduce the influence of the feedback connection. In practice, however, this will result in a badly conditioned problem and numerical difficulties, such that a tradeoff has to be found. The resulting $\mathcal{H}_\infty$ optimization procedure is given by

$$\begin{aligned} &\min_{\mathbf{K}} \quad \gamma, \\ &\text{subject to} \quad \begin{cases} \left\| \dfrac{\mathbf{z}'_\infty}{\mathbf{w}'_\infty} \right\|_\infty < \gamma, \\ \left\| \dfrac{\mathbf{z}'_{pr}}{\mathbf{w}'_{pr}} \right\|_\infty \le 1 \end{cases} \end{aligned} \tag{21}$$

which is in the form of problem (17).

4 Numerical Examples

In the following, two examples are presented to evaluate the feasibility and possible conservativeness of the proposed controller synthesis methods. First, we consider a simple disturbance rejection problem. We show, that a classic $\mathcal{H}_\infty$ design renders the closed-loop transfer function nonpassive and demonstrate that this can be overcome with the additional positive realness constraint.

4.1 First Example

4.1.1 Setup. Given a simple first-order plant G with transfer function $T(s) = (1/(1+s))$, we wish to design a controller for disturbance rejection using $\mathcal{H}_\infty$ methods. To this end, we employ a performance weight W_p and an input weight W_u in the configuration shown in Fig. 3.

To limit controller effort, we choose an input weight of

$$W_u(s) = \frac{s+5}{s+50} \tag{22}$$

and as a performance weight, we have

$$W_p(s) = \frac{5}{s/2+1} \tag{23}$$

This way, the $\mathcal{H}_\infty$ objective becomes

$$\left\| \begin{bmatrix} W_p T(1+KT)^{-1} \\ W_u KT(1+KT)^{-1} \end{bmatrix} \right\|_\infty < \gamma \tag{24}$$

where $T(1+KT)^{-1}$ is the closed-loop disturbance transfer function.

Additionally, we wish to ensure positive realness of the closed-loop system and define the transfer function from d to y as the

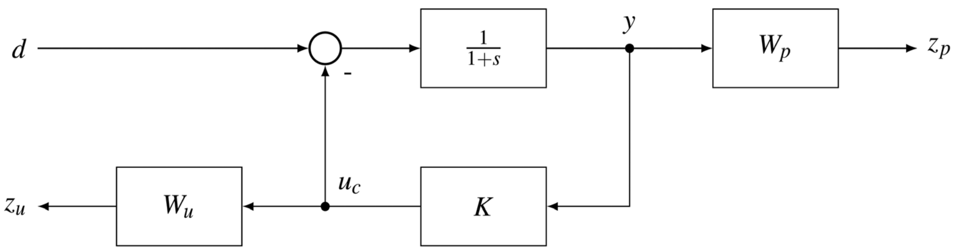

Fig. 3 Block diagram of a simple $\mathcal{H}_\infty$ disturbance rejection problem

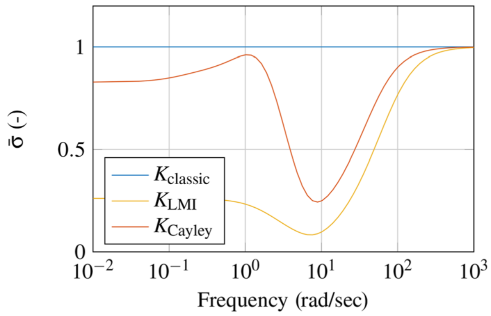

Fig. 4 Singular value plot of the $\mathcal{H}_\infty$ channel with different controllers

positive real objective. The generalized plant for the multi-objective approach is

$$\mathbf{P} = \begin{bmatrix} W_p T(1+KT)^{-1} & (\cdot) & -W_p T(1+KT)^{-1} \\ W_u KT(1+KT)^{-1} & (\cdot) & -W_u KT(1+KT)^{-1} \\ (\cdot) & T(1+KT)^{-1} & -T(1+KT)^{-1} \\ T(1+KT)^{-1} & T(1+KT)^{-1} & -T(1+KT)^{-1} \end{bmatrix} \tag{25}$$

with

$$\begin{bmatrix} w_\infty \\ w_{pr} \\ u \end{bmatrix} = \begin{bmatrix} d \\ d \\ u \end{bmatrix}, \text{ and } \begin{bmatrix} \mathbf{z}_\infty \\ z_{pr} \\ y \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} z_p \\ z_u \end{bmatrix} \\ y \\ y \end{bmatrix} \tag{26}$$

Note that certain signals are duplicated. For example, d is the input to both the $\mathcal{H}_\infty$ channel and the PR channel. Similarly, y is both the feedback signal and objective of the positive real channel.

4.1.2 Linear Matrix Inequality Approach. The plant in Eq. (25) is readily put into state space form for application in Eqs. (13) and (14). Solutions to the optimization problem are investigated using the MATLAB toolboxes Yalmip [22] and SeDuMi [23]. Yalmip acts as an interface to the solver toolbox SeDuMi and allows for the direct use of the LMIs in the presented form.

The optimization problem is successfully solved for the transformed variables and the controller K_{LMI} can be reconstructed. The maximum singular value plot of the closed-loop system is given in Fig. 4. For reference, a classic $\mathcal{H}_\infty$ controller K_{classic} is designed using same LMI procedure without the additional positive real constraint.

The multi-objective controller does not decrease the $\mathcal{H}_\infty$ performance in this example. As can be observed in Fig. 5, the positive real condition is also satisfied whereas the conventional $\mathcal{H}_\infty$ approach does not provide a positive real port transfer function.

A weakness of the procedure, however, is revealed when checking the objective function of the optimization problem. The optimization procedure terminates with a value of

$$\gamma = 87.95$$

which is highly conservative with respect to the actually achieved $\mathcal{H}_\infty$ norm of

$$\left\| \begin{bmatrix} W_p T(1+K_{\mathrm{LMI}}T)^{-1} \\ W_u K_{\mathrm{LMI}} T(1+K_{\mathrm{LMI}}T)^{-1} \end{bmatrix} \right\|_\infty = 1$$

Since the only source of conservatism in the approach is the constraint on the Lyapunov variable $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{X}$, this implies that the constraint severely diminishes the set of feasible solutions which can result in conservative controller designs or unfeasible optimization procedures.

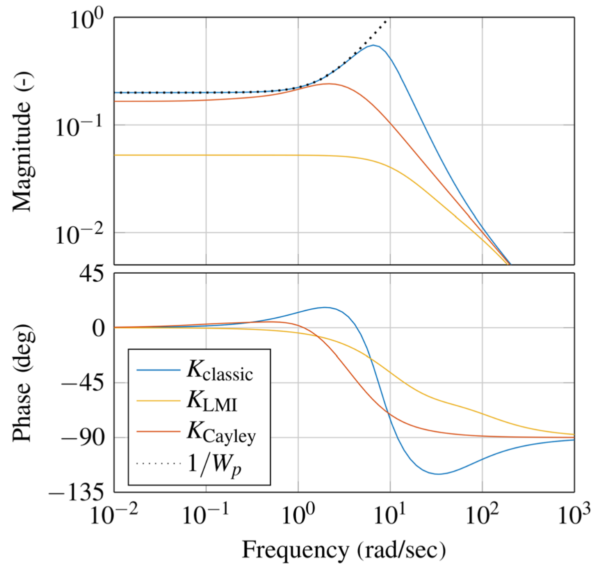

Fig. 5 Bode plot of the closed-loop frequency response $T_{cl}(j\omega) = y(j\omega)/d(j\omega)$

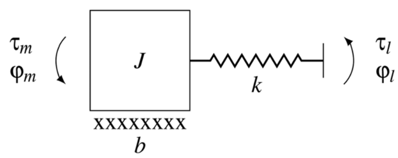

Fig. 6 Simple series elastic actuator (SEA) model consisting of motor inertia *J*, viscous friction coefficient *b*, and ideal spring constant *k*

4.1.3 Cayley Approach. As discussed earlier, the Cayley approach can conveniently be implemented using the nonsmooth $\mathcal{H}_\infty$ optimization procedure systune in MATLAB. The optimization procedure furthermore requires a fixed structure controller. For better comparability, a general full (fourth)-order state-space controller is chosen. Because of the nature of the Cayley transformation, the positive real objective has to be solved with high accuracy; the tolerance is set to `eps = 1e-10` and a scaling factor for the positive real channel of $\xi = 10^{-2}$ is chosen.

As presented in Fig. 4, the multi-objective Cayley solution also introduces no conservatism on the $\mathcal{H}_\infty$ channel in this example. The positive real constraint is met as shown in Fig. 5. The problem is reliably solved by the optimization procedure. It is, however, nonconvex and depends on initial values of the controller and error tolerances.

4.2 Second Example. As a second example, we consider the torque control of a series elastic actuator. They are frequently used in rehabilitation robotics because of their inherent compliance and their ability to physically decouple patient and robot [24]. These cases necessitate stable interaction with human users, which can be achieved by guaranteeing a passive mechanical load impedance

$$T_{imp}(s) = -\frac{\tau_l(s)}{\dot{\varphi}_l(s)} \tag{27}$$

We will consider a simple actuator model as depicted in Fig. 6 consisting of motor inertia J, viscous motor friction $F_f = b\dot{\varphi}_m$,

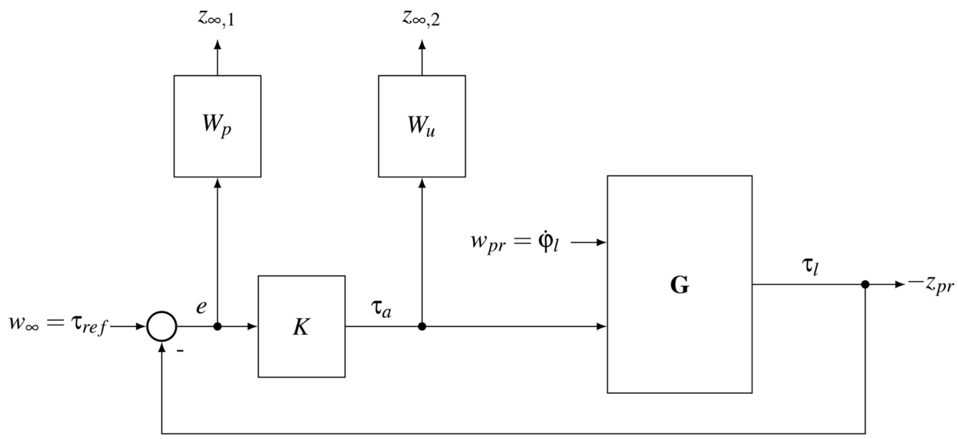

Fig. 7 Augmented plant for controller design of simple SEA

and an ideal spring as the series elastic element with $F_{\text{SEA}} = -k(\varphi_l - \varphi_m)$.

A minimal order state-space realization is given by $\mathbf{G}(s) \stackrel{s}{=} \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$, with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/J & -b/J \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1/J \end{bmatrix}, \quad \mathbf{C} = [k \quad 0], \quad \mathbf{D} = 0 \tag{28}$$

and the states $\mathbf{x} = [\varphi_m - \varphi_l \quad \dot{\varphi}_m]^{\mathrm{T}}$, inputs $\mathbf{u} = [\dot{\varphi}_l \quad \tau_m]^{\mathrm{T}}$, and output $y = \tau_l$. The plant parameters used for the controller design are $J = 3.25\,\mathrm{kg/m^2}$, $b = 17.5\,\mathrm{N\,s/rad}$, $k = 200\,\mathrm{N/rad}$.

We wish to design a torque tracking controller, and therefore, introduce weights on tracking performance and control effort

$$W_p(s) = \frac{s+30}{s+0.001}, \quad W_u(s) = \frac{1}{5}\frac{s+50}{s+200} \tag{29}$$

With these weights and the additional PR constraint on the mechanical impedance, the generalized plant is given as in Fig. 7, with inputs $[w_\infty \quad w_{pr} \quad u]^{\mathrm{T}} = [\tau_{\text{ref}} \quad \dot{\varphi}_l \quad u]^{\mathrm{T}}$ and outputs $[\mathbf{z}_\infty \quad z_{pr} \quad y]^{\mathrm{T}} = [[W_p e \quad W_u \tau_m] \quad -\tau_l \quad \tau_l]^{\mathrm{T}}$.

4.2.1 Linear Matrix Inequality Approach. The system is again brought to the form of Eqs. (13) and (14), and optimized using the toolboxes Yalmip and SeDuMi. Solving these linear matrix inequalities, however, is not successful, with the solver reporting either infeasibility or numerical problems.

To investigate this apparent failure of the procedure, we manually design a controller that fulfills the positive real constraint and stabilizes the system. For this controller, the closed-loop system can be analyzed with regard to the positive real and bounded real Lemma (Lemmas 1 and 2). We discover that we can find solutions for both LMIs of Eqs. (3) and (4) in individual Lyapunov variables $\mathbf{X}_{pr}$ and $\mathbf{X}_\infty$, respectively. If, however, we apply the restriction $\mathbf{X}_{pr} = \mathbf{X}_\infty$ necessary for the controller synthesis, the problem again becomes infeasible.

This suggests that the restriction on the Lyapunov variable critically reduces the solution space of the problem, that is, there seems to exist no Lyapunov variable $\mathbf{X}$ to solve the positive real and $\mathcal{H}_\infty$ problem simultaneously.

4.2.2 Cayley Approach. As before, controller design is performed for a full (fourth)-order state-space controller to enable better comparison with a classic $\mathcal{H}_\infty$ controller design, which is carried out using the LMI approach without PR constraint.

Figure 8 shows the closed-loop Nyquist plots of the impedance transfer function for both the classic unconstrained controller and the one designed using the Cayley approach. It is apparent, that the designed Cayley controller satisfies the PR constraint, while the classic $\mathcal{H}_\infty$ controller fails to do so.

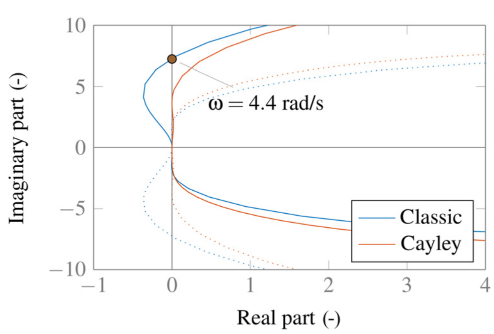

Fig. 8 Nyquist plot of the impedance transfer function for simple SEA example with either classically designed $\mathcal{H}_\infty$ controller or positive real constraint Cayley approach $\mathcal{H}_\infty$ controller

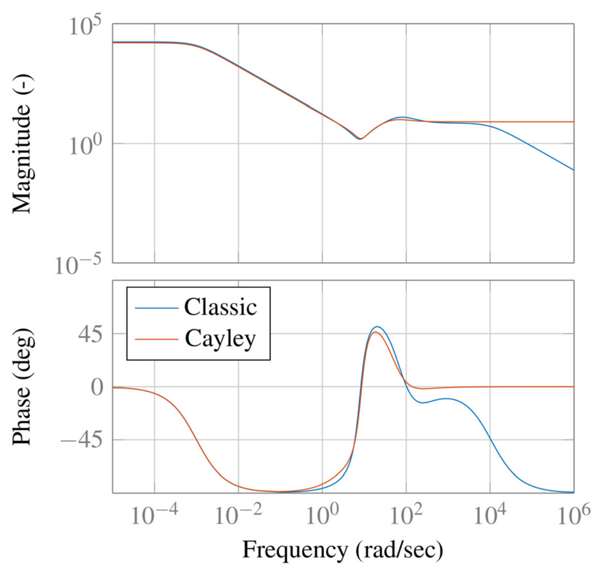

Fig. 9 Bode plot of the controller transfer function for both classic $\mathcal{H}_\infty$ controller and positive real constraint Cayley approach $\mathcal{H}_\infty$ controller

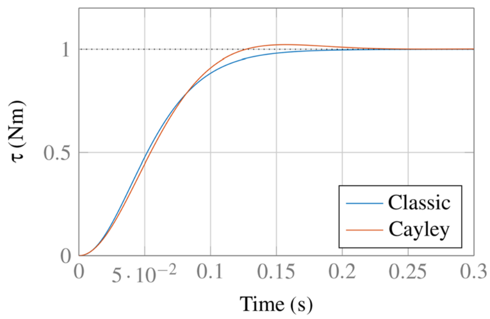

Fig. 10 Reference tracking step response for simple SEA example with either classically designed $\mathcal{H}_\infty$ controller or positive real constraint Cayley approach $\mathcal{H}_\infty$ controller

Table 1 Pole zero location of the controllers designed for the simple SEA example

Cayley		Classic	
Poles	Zeros	Poles	Zeros
$-2.53+7.95i$	$-2.77+7.93i$	$-2.70+7.37i$	$-2.70+7.36i$
$-2.53-7.95i$	$-2.77-7.93i$	$-2.70-7.37i$	$-2.70-7.36i$
-3.15	-3.35	32.5	-200
$-21.6+24.2i$	-99.8	$-43.6-28.2i$	-1.88×10^{7}
$-21.6-24.2i$		$-43.6-28.2i$	
-50.7		-9.72×10^{5}	

The controllers achieve a $\mathcal{H}_\infty$ norm of

$$\gamma_{\text{Cayley}} = 1.901$$
$$\gamma_{\text{Classic}} = 1.745$$

which corresponds to an increase of roughly 9% from the unconstrained to the constrained cased.

Generally, both controllers show similar structure, as evident from their bode plot in Fig. 9. Most noticeably, the Cayley controller does not have low pass characteristics toward very high frequencies, as the classic $\mathcal{H}_\infty$ controller provides. Results in the time domain are slightly stronger oscillatory behavior and overshoot, as depicted in Fig. 10. This results from an imperfect pole zero cancelation, which is forced by the PR constraint and is evident in Table 1.

5 Conclusion

We addressed two approaches for $\mathcal{H}_\infty$ optimal controller design under closed-loop positive real constraints.

The first one, based on a multi-objective design framework [8], yielded infeasible results in example applications, due to a restriction on the Lyapunov variable that limits the solution space of the synthesis equations. However, it should be noted that parameterization approaches were presented for multi-objective control problems (see for example Refs. [25,26]). These approaches are intended to reduce conservatism by employing finite dimensional Q-parameterizations that lead to an increase in the controller states. Moreover, only the multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ case is considered in the literature and solutions are presented in discrete time only. We, therefore, did not include a comparison to our study.

The second approach was based on a nonlinear sector transformation, the Cayley transformation, that transforms the positive real problem in an equivalent $\mathcal{H}_\infty$ problem. The approach extends results in Ref. [9], by augmenting it with a $\mathcal{H}_\infty$ performance objective. It was successfully demonstrated in two examples, in which it led to little conservatism compared to the unconstrained design. Due to the reformulation as an $\mathcal{H}_\infty$ problem, it also enables the use of established robust control technique. As such, it can be a powerful tool toward a robust controller design with regard to plant interaction with passive environments.

Appendix: Multi-Objective Cayley Transformation

The transformation with respect to the individual signals is defined through

$$\mathbf{w}_{spr} = \xi(\mathbf{w}'_{spr} - \mathbf{z}_{spr}) \tag{A1}$$

$$\mathbf{z}'_{spr} = -\mathbf{z}_{spr} + \mathbf{w}'_{spr} \tag{A2}$$

and using Eq. (A1) on the system equations of **P,** we find

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_\infty\mathbf{w}_\infty + \mathbf{B}_{spr}\xi(\mathbf{w}'_{spr} - \mathbf{z}_{spr}) \tag{A3}$$

$$\mathbf{z}_{spr} = \mathbf{C}_{spr}\mathbf{x} + \mathbf{D}_{spr,\infty}\mathbf{w}_\infty + \xi\mathbf{D}_{spr}(\mathbf{w}'_{spr} - \mathbf{z}_{spr}) + \mathbf{D}_{spr,2}\mathbf{u} \tag{A4}$$

$$\mathbf{z}_\infty = \mathbf{C}_\infty\mathbf{x} + \mathbf{D}_\infty\mathbf{w}_\infty + \xi\mathbf{D}_{\infty,spr}(\mathbf{w}'_{spr} - \mathbf{z}_{spr}) + \mathbf{D}_{\infty,2}\mathbf{u} \tag{A5}$$

$$\mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{D}_{2,\infty}\mathbf{w}_\infty + \xi\mathbf{D}_{2,spr}(\mathbf{w}'_{spr} - \mathbf{z}_{spr}) + \mathbf{D}_2\mathbf{u} \tag{A6}$$

Solving Eq. (A4) for $\mathbf{z}_{spr}$ yields

$$\mathbf{z}_{spr} = (\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}(\mathbf{C}_{spr}\mathbf{x} + \mathbf{D}_{spr,\infty}\mathbf{w}_\infty + \xi\mathbf{D}_{spr}\mathbf{w}'_{spr} + \mathbf{D}_{spr,2}) \tag{A7}$$

Substituting this in Eq. (A3) yields after reorganizing the terms

$$\mathbf{A}' = \mathbf{A} - \xi\mathbf{B}_{spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{C}_{spr} \tag{A8}$$

$$\mathbf{B}'_\infty = \mathbf{B}_\infty - \xi\mathbf{B}_{spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{D}_{spr,\infty} \tag{A9}$$

$$\mathbf{B}'_{spr} = \xi\mathbf{B}_{spr} - \xi\mathbf{B}_{spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\xi\mathbf{D}_{spr} = \xi\mathbf{B}_{spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1} \tag{A10}$$

$$\mathbf{B}'_2 = \mathbf{B}_2 - \xi\mathbf{B}_{spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{C}_{spr} \tag{A11}$$

Substituting Eq. (A7) in Eq. (A5)

$$\mathbf{C}'_\infty = \mathbf{C}_\infty - \xi\mathbf{D}_{\infty,spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{C}_{spr} \tag{A12}$$

$$\mathbf{D}'_\infty = \mathbf{D}_\infty - \xi\mathbf{D}_{\infty,spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{D}_{spr,\infty} \tag{A13}$$

$$\begin{aligned}\mathbf{D}'_{\infty,spr} &= \xi\mathbf{D}_{\infty,spr} - \xi\mathbf{D}_{\infty,spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\xi\mathbf{D}_{spr}\\ &= \xi\mathbf{D}_{\infty,spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\end{aligned} \tag{A14}$$

$$\mathbf{D}'_{\infty,2} = \mathbf{D}_{\infty,2} - \xi\mathbf{D}_{\infty,spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{D}_{spr,2} \tag{A15}$$

Substituting Eqs. (A7) and (A1) in Eq. (A2) yields

$$\mathbf{C}'_{spr} = -2(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{C}_{spr} \tag{A16}$$

$$\mathbf{D}'_{spr,\infty} = -2(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{D}_{spr,\infty} \tag{A17}$$

$$\mathbf{D}'_{spr} = -2(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\xi\mathbf{D}_{spr} + \mathbf{I} = (\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}(\mathbf{I} - \xi\mathbf{D}_{spr}) \tag{A18}$$

$$\mathbf{D}'_{spr,2} = -2(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{D}_{spr,2} \tag{A19}$$

Substituting Eq. (A7) and Eq. (A1) in Eq. (A6) yields

$$\mathbf{C}'_2 = \mathbf{C}_2 - \xi\mathbf{D}_{2,spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{C}_{spr} \tag{A20}$$

$$\mathbf{D}'_{2,\infty} = \mathbf{D}_{2,\infty} - \xi\mathbf{D}_{2,spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\mathbf{D}_{spr,\infty} \tag{A21}$$

$$\begin{aligned}\mathbf{D}'_{2,spr} &= \xi\mathbf{D}_{2,spr} - \xi\mathbf{D}_{2,spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\xi\mathbf{D}_{spr}\\ &= \xi\mathbf{D}_{2,spr}(\mathbf{I} + \xi\mathbf{D}_{spr})^{-1}\end{aligned} \tag{A22}$$

$$\mathbf{D}_2' = \mathbf{D}_2 - \xi \mathbf{D}_{2,spr}(\mathbf{I} + \xi \mathbf{D}_{spr})^{-1}\mathbf{D}_{spr,2} \tag{A23}$$

with the state-space representation

$$\mathbf{P}'(s) \stackrel{s}{=} \left[\begin{array}{c|ccc} \mathbf{A}' & \mathbf{B}_\infty' & \mathbf{B}_{pr}' & \mathbf{B}_2' \\ \hline \mathbf{C}_\infty' & \mathbf{D}_\infty' & \mathbf{D}_{\infty,pr}' & \mathbf{D}_{\infty,2}' \\ \mathbf{C}_{pr}' & \mathbf{D}_{pr,\infty}' & \mathbf{D}_{pr}' & \mathbf{D}_{pr,2}' \\ \mathbf{C}_2' & \mathbf{D}_{2,\infty}' & \mathbf{D}_{2,pr}' & \mathbf{D}_2' \end{array}\right] \tag{A24}$$

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