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Nonconvex spectral optimization algorithms for reduced-order $\mathcal{H}_\infty$ LPV-LFT controllers

Ye Shi[1], Hoang D. Tuan[1,*,†] and Pierre Apkarian[2]

[1] *Faculty of Engineering and Information Technology, University of Technology, Sydney, NSW, Australia*
[2] *CERT-ONERA, Control System Department, 2, Avenua Edouard Belin, 31055 Toulouse, France*

SUMMARY

A novel sequential semi-definite programming method is developed for optimization subject to rank constraints on matrix-valued nonlinear functions of matrix decision variables, which arise in reduced-order linear parameter varying-linear fractional transformational control synthesis. The global convergence of the method is easily proven without any step size control. An intensive simulation shows the clear advantage of the proposed method over the state-of-the-art nonlinear matrix inequality solvers. Copyright © 2017 John Wiley & Sons, Ltd.

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KEY WORDS: reduced-order LPV-LFT $\mathcal{H}_\infty$ control; matrix-rank constraint; nonlinear programming

1. INTRODUCTION

Rank-constrained optimization is referred to optimization problems involving rank constraints on matrix-valued functions of the decision variables. Initialized by the pioneering work [1], which reformulates the reduced-order $\mathcal{H}_\infty$ control synthesis for linear time invariant (LTI) systems as linear matrix inequality (LMI) optimization subject to a rank constraint on a matrix-valued affine function of the Lyapunov matrix variables, many other important and difficult problems in robust control are also reformulated in similar matrix-rank-constrained optimizations [2]. The simplest approach is to relax or just to drop that rank constraints with hope that the optimal solution of the relaxed (convex) optimization would satisfy these matrix-rank constraints. For instance, matrix trace minimization and nuclear norm minimization were proposed to obtain low matrix rank of positive semi-definite matrix and rectangular matrices, respectively [3, 4]. These techniques are unable to address the matrix-rank constraints. Indeed, just a trace of a matrix or its nuclear norm does not give any adequate indication on the matrix rank. Another attempt is to use a Newton-like method to find a projection of a positive semi-definite matrix to the manifold of fixed-rank matrices [5, 6], which is equally computationally difficult optimization because of complex geometry of this manifold [7], especially for lower fixed-rank matrices of larger size. Realizing the challenge by these matrix-rank constraints on the Lyapunov matrix variables, most later developments in robust control preferred to avoid them in favor of alternative bilinear matrix inequality (BMI) [2, 8–14]. The state-of-the-art BMI solvers [12, 14] initialize from a reduced-order stabilizing controller and then move within a convex feasibility subset containing this initialized point. There are a few difficulties arisen with this kind of feasibility algorithms. Firstly, finding a good reduced-order stabilizing controller is not an easy task because its computation is still an NP-hard problem [15]. Secondly, the feasibility set of

*Correspondence to: Hoang D. Tuan, Faculty of Engineering and Information Technology, University of Technology, Sydney, NSW 2007, Australia.
†E-mail: tuan.hoang@uts.edu.au

stabilizing controllers is highly nonconvex, which is disconnected in general. This means moving within a convex neighborhood of such reduced-order stabilizing controller may be trapped by local minima. Thirdly, usually, the convergence of this kind of algorithms is slow and is dependent very much on the local geometry around such initial point [16], which may be unpredictable. Within the past few years, it has been realized that all BMI solvers [12, 14], which address the control synthesis for LTI systems in state space using Lyapunov functions, could hardly compete with the nonsmooth optimization solver developed earlier in [17], which addresses the problems directly in the frequency domain to bypass the Lyapunov variables of high dimension. Nowadays, the Matlab systune command [18], which is based on [17], is the most powerful tool for control synthesis of LTI systems and is widely used in industry. This means that rank-constrained optimization and BMI should seek applications outside uncertain LTI systems such as linear parameter varying (LPV) systems [19], where Lyapunov function is irreplaceable.

Meanwhile, for solution of indefinite quadratic optimization in signal processing applications, Phan *et al*.[20, 21] and Shi *et al*.[22] developed an approach for optimization on the rank-one constrained positive semi-definite outer product of decision vector variable. Intensive simulations even for large scale indefinite quadratic optimization [22] show that the rank-one matrices can be quickly located, which are turned out to be global optimal solutions of the considered indefinite quadratic problems in most cases. Reduced-order robust LPV controller synthesis is more difficult than indefinite quadratic programming and has not been appropriately considered in literature. The matrix-rank constraints in the former are much more challenging than the rank-one constraint in the latter. Indeed, they are lower fixed-rank constraints on matrix-valued affine functions of larger size with very complex geometry. For instance, k-order robust control synthesis for an LPV plant of order n leads to rank-$(n+k)$ constraint on the positive semi-definite matrix-valued affine function of size $(2n) \times (2n)$ [23, 24]. A novel approach proposed in the present paper is to equivalently express these rank-$(n+k)$ constraints on the positive semi-definite matrix-valued affine function by rank-k constraint on the matrix-valued nonlinear function of size n, which are then exactly expressed by spectral nonlinear functions. We then show a simple but effective optimization technique leading to a path-following optimization procedure for these problems. To the author's best knowledge, spectral nonlinear function optimization was not quite considered in the literature.

The paper is organized as follows. After the introduction, Section 2 is devoted to algorithmic solutions for reduced-order LPV $\mathcal{H}_\infty$ controllers while Section 3 is devoted to static output feedback LPV controllers. An intensive simulation is provided in Section 4 to support the algorithmic development of the previous sections. Section 5 concludes the paper.

Notation. Notation used in this paper is standard. Particularly, $X \succeq 0$, $X \succ 0$, $X \preceq 0$, and $X \prec 0$ mean that a symmetric matrix X is positive semi-definite, positive definite, negative semi-definite, and negative definite, respectively, while $\langle X, Y \rangle$ is the dot product of the matrices X and Y. For simplicity, we also denote $\mathrm{tr}(X)$ as the trace of X. I is the identity matrix but when needed we also use I_n to emphasize the size $n \times n$ of I. In symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for terms that are induced by symmetry, for example,

$$K \begin{bmatrix} S + S^T & M^T \\ M & Q \end{bmatrix} K^T = K \begin{bmatrix} S + (*) & * \\ M & Q \end{bmatrix} *$$

The matrix variables are typed boldfaced in the paper.

2. DYNAMIC REDUCED-ORDER $\mathcal{H}_\infty$ LPV CONTROL SYNTHESIS

Consider a continuous LPV system in linear fractional transformation (LFT) [23–25]

$$\begin{bmatrix} \dot{x}(t) \\ z_\Delta(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_\Delta & B_1 & B_2 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ C_1 & D_{1\Delta} & D_{11} & D_{12} \\ C_2 & D_{2\Delta} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w_\Delta(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{1}$$
$$w_\Delta(t) = \Delta(\alpha(t)) z_\Delta(t)$$

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where

$$\Delta(\alpha(t)) = \sum_{i=1}^{L} \alpha_i(t)\Delta_i, \alpha_i(t) \geqslant 0, \sum_{i=1}^{L} \alpha_i(t) = 1. \tag{2}$$

Here, $x(t) \in R^n$, $y(t) \in R^{n_y}$, $z(t) \in R^{n_z}$, $w(t) \in R^{n_z}$, $z_\Delta(t) \in R^{n_\Delta}$, $w_\Delta(t) \in R^{n_\Delta}$. Note that we assume without loss of generality that $z(t)$ and $w(t)$ ($z_\Delta(t)$ and $w_\Delta(t)$, respectively) have the same dimension. The pair (w_Δ, z_Δ) is regarded as the gain-scheduling channel. All matrices in (1) and (2) are given with appropriate size. Parameters $\alpha_i(t)$ are measured online and exploited by the controller.

The standard $\mathcal{H}_\infty$ LPV control design is to find k-order controller in LFT:

$$\begin{aligned} \begin{bmatrix} \dot{x}_K(t) \\ u(t) \\ z_K(t) \end{bmatrix} &= \begin{bmatrix} A_K & B_{K1} & B_{K\Delta} \\ C_{K1} & D_{K11} & D_{K1\Delta} \\ C_{K\Delta} & D_{K\Delta 1} & D_{K\Delta\Delta} \end{bmatrix} \begin{bmatrix} x_K(t) \\ y(t) \\ w_K(t) \end{bmatrix} \\ w_K(t) &= \Delta_K(\alpha(t)) z_K(t) \end{aligned} \tag{3}$$

with

$$\Delta_K(\alpha(t)) = \sum_{i=1}^{L} \alpha_i \Delta_{Ki} \tag{4}$$

such that the closed-loop system is internally stable and satisfies

$$\int_0^T ||z(t)||^2 dt \leqslant \gamma^2 \int_0^T ||w(t)||^2 dt \quad \forall w(.) \in L_2, T < +\infty, \tag{5}$$

initialized from $x(0) = 0$. Here, $x_K(t) \in R^k$, $z_K(t) \in R^{n_\Delta}$ and $w_K(t) \in R^{n_\Delta}$. k is called the control order, and the pair (w_K, z_K) is regarded as the control's gain-scheduling channel.

Note that (1) and (3) are the following LPV LFTs:

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = &\left(\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} + \begin{bmatrix} B_\Delta \\ D_{1\Delta} \\ D_{2\Delta} \end{bmatrix} (I - \Delta(\alpha(t)) D_{\Delta\Delta})^{-1} \Delta(\alpha(t)) \right. \\ &\left. \times \begin{bmatrix} C_\Delta & D_{\Delta 1} & D_{\Delta 2} \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \begin{bmatrix} \dot{x}_K(t) \\ u(t) \end{bmatrix} = &\left(\begin{bmatrix} A_K & B_{K1} \\ C_{K1} & D_{K11} \end{bmatrix} + \begin{bmatrix} B_{K\Delta} \\ D_{K1\Delta} \end{bmatrix} (I - \Delta_K(\alpha(t)) D_{K\Delta\Delta})^{-1} \Delta_K(\alpha(t)) \right. \\ &\left. \times \begin{bmatrix} C_{K\Delta} & D_{K\Delta 1} \end{bmatrix} \right) \begin{bmatrix} x_K(t) \\ y(t) \end{bmatrix}, \end{aligned} \tag{7}$$

respectively. Figure 1 provides a block diagram for a such system.

Let us state the following result adapted from [24]: the feasibility of the following matrix inequality in $\mathbf{X} \in R^{n\times n}$, $\mathbf{Y} \in R^{n\times n}$, $\mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \mathbf{\Delta}_{Ki}$ and

$$\hat{\mathbf{K}} := \begin{bmatrix} \mathbf{A_K} & \mathbf{B_{K1}} & \mathbf{B_{K_\Delta}} \\ \mathbf{C_{K1}} & \mathbf{D_{K11}} & \mathbf{D_{K1\Delta}} \\ \mathbf{C_{K\Delta}} & \mathbf{D_{K\Delta 1}} & \mathbf{D_{K\Delta\Delta}} \end{bmatrix} \tag{8}$$

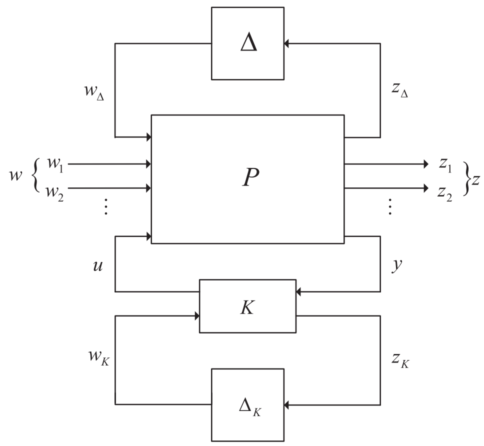

Figure 1. Closed-loop linear parameter varying-linear fractional transformational system.

is sufficient for the existence of such controller

$$\begin{bmatrix} \mathbf{LMI_1} & * \\ \mathbf{LMI_2} & \mathbf{LMI_3} \end{bmatrix} \prec 0, \tag{9}$$

$$\begin{bmatrix} \mathbf{R} & I & \Delta_i^T\mathbf{Q} & \Delta_i^T \\ I & \mathbf{H} & \mathbf{\Delta}_{\mathbf{Ki}}^T & \mathbf{H}\Delta_i^T \\ \mathbf{Q}\Delta_i & \mathbf{\Delta_{Ki}} & -\mathbf{Q} & -I \\ \Delta_i & \Delta_i\mathbf{H} & -I & -\mathbf{E} \end{bmatrix} \succ 0, \quad i = 1, 2, \cdots, L.$$

$$\begin{pmatrix} \mathbf{X} & I_n \\ I_n & \mathbf{Y} \end{pmatrix} \succeq 0, \tag{10}$$

$$\mathrm{rank}(\mathbf{X} - \mathbf{Y}^{-1}) \leqslant k, \tag{11}$$

where

$$\mathbf{LMI}_1 := \begin{bmatrix} \mathbf{X}A + \mathbf{B_{K1}}C_2 + (*) & * & * & * \\ \mathbf{A}_\mathbf{K}^T + A + B_2\mathbf{D_{K11}}C_2 & (A\mathbf{Y} + B_2\mathbf{C_{K1}}) + (*) & * & * \\ B_\Delta^T\mathbf{X} + D_{2\Delta}^T\mathbf{B}_{\mathbf{K1}}^T & B_\Delta^T D_{2\Delta}^T\mathbf{D}_{\mathbf{K11}}^T B_2^T & Q & * \\ \mathbf{B}_{\mathbf{K\Delta}}^T & \mathbf{E}B_\Delta^T + \mathbf{D}_{\mathbf{K1\Delta}}^T B_2^T & -I & \mathbf{E} \end{bmatrix}$$

$$\mathbf{LMI}_2 := \begin{bmatrix} B_1^T\mathbf{X} + D_{21}^T\mathbf{B}_{\mathbf{K1}}^T & B_1^T + D_{21}^T\mathbf{D}_{\mathbf{K11}}^T B_2^T & 0 & 0 \\ \mathbf{R}C_\Delta + \mathbf{D_{K\Delta 1}}C_2 & \mathbf{C_{K\Delta}} & \mathbf{R}D_{\Delta\Delta} + \mathbf{D_{K\Delta 1}}D_{2\Delta} & \mathbf{D_{K\Delta\Delta}} \\ C_\Delta + D_{\Delta 2}\mathbf{D_{K11}}C_2 & C_\Delta\mathbf{Y} + D_{\Delta 2}\mathbf{C_{K1}} & D_{\Delta\Delta} + D_{\Delta 2}\mathbf{D_{K11}}D_{2\Delta} & D_{\Delta\Delta}\mathbf{E} + D_{\Delta 2}\mathbf{D_{K1\Delta}} \\ C_1 + D_{12}\mathbf{D_{K11}}C_2 & C_1\mathbf{Y} + D_{12}\mathbf{C_{K1}} & D_{1\Delta} + D_{12}\mathbf{D_{K11}}D_{2\Delta} & D_{1\Delta}\mathbf{E} + D_{12}\mathbf{D_{K1\Delta}} \end{bmatrix}$$

$$\mathbf{LMI}_3 := \begin{bmatrix} -\gamma I & * & * & * \\ \mathbf{R}D_{\Delta 1} + \mathbf{D_{K\Delta 1}}D_{21} & -\mathbf{R} & * & * \\ D_{\Delta 1} + D_{\Delta 2}\mathbf{D_{K11}}D_{21} & -I & -\mathbf{H} & * \\ D_{11} + D_{12}\mathbf{D_{K11}}D_{21} & 0 & 0 & -\gamma I \end{bmatrix}. \tag{12}$$

Note that (10) and (11) imply that

$$\mathrm{rank}(I_n - \mathbf{XY}) \leqslant k.$$

Without loss of generality, assume $\text{rank}(I_n - \mathbf{XY}) = k$. Then, factorize

$$I_n - \mathbf{XY} = \mathbf{MN}^T$$

with full-rank $\mathbf{M} \in R^{n\times k}$ and $\mathbf{N} \in R^{n\times k}$. Their left-inverse matrices are

$$\mathbf{M}^+ = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T, \mathbf{N}^+ = (\mathbf{N}^T\mathbf{N})^{-1}\mathbf{N}^T.$$

Also, factorize

$$I - \mathbf{RH} = \mathbf{R}_{12}\mathbf{H}_{12}^T \quad \text{and} \quad I - \mathbf{QE} = \mathbf{Q}_{12}\mathbf{E}_{12}^T$$

with invertible matrices $\mathbf{R}_{12}, \mathbf{H}_{12}, \mathbf{Q}_{12}$, and $\mathbf{E}_{12}$. Accordingly, the controller (3) can be recovered as follows [23]:

$$D_{K11} = \mathbf{D_{K11}} \tag{13}$$

$$B_{K1} = \mathbf{M}^+(\mathbf{B_{K1}} - \mathbf{X}B_2D_{K11}) \tag{14}$$

$$C_{K1} = (\mathbf{C_{K1}} - D_{K11}C_2\mathbf{Y})(\mathbf{N}^+)^T, \tag{15}$$

$$A_K = \mathbf{M}^+[\mathbf{A_K} - (\mathbf{X}A\mathbf{Y} + \mathbf{M}B_{K1}C_2\mathbf{Y} + \mathbf{X}B_2C_{K1}\mathbf{N}^T + \mathbf{X}B_2D_{K11}C_2\mathbf{Y})](\mathbf{N}^+)^T \tag{16}$$

$$D_{k1\Delta} = (\mathbf{D_{K1\Delta}} - D_{K11}D_{2\Delta}\mathbf{E})(\mathbf{E}_{12}^{-1})^T \tag{17}$$

$$D_{K\Delta 1} = \mathbf{R}_{12}^{-1}(\mathbf{D_{K\Delta 1}} - \mathbf{R}D_{\Delta 2}D_{K11}) \tag{18}$$

$$\begin{aligned} B_{K\Delta} = \mathbf{M}^+[\mathbf{B_{K\Delta}} - (&\mathbf{X}B_\Delta\mathbf{E} + \mathbf{M}B_{k1}D_{2\Delta}\mathbf{E} + \mathbf{X}B_2D_{K11}D_{2\Delta}\mathbf{E} \\ &+ \mathbf{X}B_2D_{K1\Delta}\mathbf{E}_{12}^T)](\mathbf{E}_{12}^{-1})^T \end{aligned} \tag{19}$$

$$\begin{aligned} C_{K\Delta} = \mathbf{R}_{12}^{-1}[\mathbf{C_{K\Delta}} - (&\mathbf{R}C_\Delta\mathbf{Y} + \mathbf{R}D_{\Delta 2}D_{K11}C_2\mathbf{Y} + \mathbf{R}_{12}D_{K\Delta 1}C_2\mathbf{Y} \\ &+ \mathbf{R}D_{\Delta 2}C_{K1}\mathbf{N}^T)](\mathbf{N}^{-1})^T \end{aligned} \tag{20}$$

$$\begin{aligned} D_{K\Delta\Delta} = \mathbf{R}_{12}^{-1}[\mathbf{D_{K\Delta\Delta}} - (&\mathbf{R}D_{\Delta\Delta}\mathbf{E} + \mathbf{R}D_{\Delta 2}D_{K11}D_{2\Delta}\mathbf{E} + \mathbf{R}_{12}D_{K\Delta 1}D_{2\Delta}\mathbf{E} \\ &+ \mathbf{R}D_{\Delta 2}D_{K1\Delta}\mathbf{E}_{12}^T)]\left(\mathbf{E}_{12}^{-1}\right)^T. \end{aligned} \tag{21}$$

It should be noted that (10) and (11) are equivalent to (10) and

$$\text{rank}\left(\begin{pmatrix} \mathbf{X} & I_n \\ I_n & \mathbf{Y} \end{pmatrix}\right) \leqslant n + k \tag{22}$$

which is a lower fixed-rank constraint on a matrix-valued affine function of $(\mathbf{X}, \mathbf{Y})$. Although our developed algorithms later still work for this constraint (22), we will see that in fact the rank constraint (11) on a nonlinear function of $(\mathbf{X}, \mathbf{Y})$ can be more efficiently handled. The difficulty degree of formulations in [12, 14, 26] is proportional to the dimension of the control variable (A_K, B_K, C_K, D_K) in (3), that is, it is proportional to the control order k. In contrast, by exploring the rank constraint (11), the difficulty degree in our formulation is proportional to $\min\{k, n-k\}$, that is, the computational difficulty with k-order and $(n-k)$-order controllers is the same.

We formulate the k-order LPV-LFT $\mathcal{H}_\infty$ control as

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} \gamma \quad \text{s.t.} \quad (9), (10), (11), \tag{23}$$

where all the nonconvexity of the problem is concentrated in the rank constraint (11), which is automatically satisfied for the (full) n-order control. For $k < n$, as expected (11) is a highly non-convex and discontinuous constraint. Consequently, the feasibility set (9) and (11) is disconnected

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in general, for which locating a feasible point is already not an easy task. As an aside note, the aforementioned formulation (23) is based on parameter-independent Lyapunov function and static multipliers $\mathbf{R}$, $\mathbf{H}$, $\mathbf{Q}$, and $\mathbf{E}$, which may potentially be more conservative than that with either parameter-dependent Lyapunov functions (e.g., [27]) or dynamic multipliers (e.g., [28]). However, efficient formulations for $k-$order LPV-LFT $\mathcal{H}_\infty$ control by using parameter-dependent Lyapunov functions and dynamic multipliers are still very much open for study. The interested reader is also referred to [29] and the references therein for convex relation-based results for fixed-order LPV controllers for LPV systems.

The function $\text{rank}(\mathbf{X}-\mathbf{Y}^{-1})$ in (11) seems to be very complicated. However, we will see shortly that it can be efficiently handled from the following observation. Suppose $f_{[k]}(\mathbf{X}-\mathbf{Y}^{-1})$ is the sum of the k largest eigenvalues of $\mathbf{X}-\mathbf{Y}^{-1}$, which is positive definite ($\mathbf{X}-\mathbf{Y}^{-1}\succeq 0$) thanks to LMI (10). Then, the matrix-rank constraint (11) holds true if and only if

$$\text{tr}(\mathbf{X}-\mathbf{Y}^{-1}) = f_{[k]}(\mathbf{X}-\mathbf{Y}^{-1})$$

because it implies that $\mathbf{X}-\mathbf{Y}^{-1}$ has at least $(n-k)$ zero eigenvalues. Under the LMI (10), the quantity $\text{tr}(\mathbf{X}-\mathbf{Y}^{-1})-f_{[k]}(\mathbf{X}-\mathbf{Y}^{-1})$ is always nonnegative and can therefore be used to measure the degree of satisfaction of the matrix-rank constraint (11). Instead of handling nonconvex constraint (11), we incorporate it into the objective, resulting in the following alternative formulation to (23):

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} F_\mu(\mathbf{X},\mathbf{Y},\gamma) := \gamma + \mu(\text{tr}(\mathbf{X}-\mathbf{Y}^{-1}) - f_{[k]}(\mathbf{X}-\mathbf{Y}^{-1})) \quad \text{s.t.} \quad (9)-(10), \tag{24}$$

where $\mu > 0$ is a penalty parameter. Without squaring on the factor of μ, the aforementioned penalization is exact, meaning that the constraint (11) can be satisfied by a minimizer of (24) with a finite value of μ (e.g., [30, Chapter 16]). This is generally considered as a sufficiently nice property to make such exact penalization attractive. On the other hand, any feasible $(\mathbf{X},\mathbf{Y},\gamma)$ to (23) is also feasible to (24), implying that the optimal value of (24) for any $\mu > 0$ is upper bounded by the optimal value of (23).

Suppose $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$ is a feasible point to the convex feasibility set (9) and (10). Using the following variational principle [31, p. 191]

$$f_{[k]}(\mathbf{X}-\mathbf{Y}^{-1}) = \max_{\text{orthonormal } x_1,\ldots,x_k} \sum_{i=1}^{k} x_i^H(\mathbf{X}-\mathbf{Y}^{-1})x_i$$

it follows that

$$f_{[k]}(\mathbf{X}-\mathbf{Y}^{-1}) \geqslant \sum_{i=1}^{k}(x_i^{(\kappa)})^H(\mathbf{X}-\mathbf{Y}^{-1})x_i^{(\kappa)}, \tag{25}$$

where $x_i^{(\kappa)}$, $i=1,\ldots,k$ are the orthonormal eigenvectors corresponding to k largest eigenvalues of $X^{(\kappa)}-(Y^{(\kappa)})^{-1}$. On the other hand, as $\text{tr}(\mathbf{Y}^{-1})$ is convex in $\mathbf{Y}\succ 0$, it is true that

$$\begin{aligned}\text{tr}(\mathbf{Y}^{-1}) &\geqslant \text{tr}((Y^{(\kappa)})^{-1}) - \text{tr}((Y^{(\kappa)})^{-1}(\mathbf{Y}-Y^{(\kappa)})(Y^{(\kappa)})^{-1})\\ &= 2\text{tr}((Y^{(\kappa)})^{-1}) - \text{tr}((Y^{(\kappa)})^{-1}\mathbf{Y}(Y^{(\kappa)})^{-1}).\end{aligned} \tag{26}$$

The following convex optimization is majorant minimization for (24)

$$\begin{aligned}\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} F_\mu^{(\kappa)}(\mathbf{X},\mathbf{Y},\gamma) &:= \gamma + \mu(\text{tr}(\mathbf{X}-2(Y^{(\kappa)})^{-1}) + \text{tr}((Y^{(\kappa)})^{-1}\mathbf{Y}(Y^{(\kappa)})^{-1})\\ &\quad - \sum_{i=1}^{k}\left(x_i^{(\kappa)}\right)^H(\mathbf{X}-\mathbf{Y}^{-1})x_i^{(\kappa)}) \quad \text{s.t.} \quad (9)-(10).\end{aligned} \tag{27}$$

because by (25) and (26), function $F_\mu^{(\kappa)}$ obeys the two following crucial properties:

$$F_\mu^{(\kappa)}(\mathbf{X},\mathbf{Y},\gamma) \geqslant F_\mu(\mathbf{X},\mathbf{Y},\gamma)\forall(\mathbf{X},\mathbf{Y},\gamma) \quad \text{on} \quad (9)-(10)$$

and

$$F_\mu^{(\kappa)}(X^{(\kappa)},Y^{(\kappa)},\gamma^{(\kappa)}) = F_\mu(X^{(\kappa)},Y^{(\kappa)},\gamma^{(\kappa)}).$$

Therefore, for the optimal solution $(X^{(\kappa+1)},Y^{(\kappa+1)},\gamma^{(\kappa+1)})$ of the convex program (27), it is true that

$$\begin{aligned} F_\mu(X^{(\kappa+1)},Y^{(\kappa+1)},\gamma^{(\kappa+1)}) &\leqslant F_\mu^{(\kappa)}(X^{(\kappa+1)},Y^{(\kappa+1)},\gamma^{(\kappa+1)}) \\ &\leqslant F_\mu^{(\kappa)}(X^{(\kappa)},Y^{(\kappa)},\gamma^{(\kappa)}) \\ &= F_\mu(X^{(\kappa)},Y^{(\kappa)},\gamma^{(\kappa)}), \end{aligned}$$

implying that $(X^{(\kappa+1)},Y^{(\kappa+1)},\gamma^{(\kappa+1)})$ is better than $(X^{(\kappa)},Y^{(\kappa)},\gamma^{(\kappa)})$ toward optimizing the objective in (24). By using [32], we can prove the following result of global convergence.

Proposition 1
Initialized by any feasible point $(X^{(0)},Y^{(0)},\gamma^{(0)})$ for LMIs (9) and (10), $\{(X^{(\kappa)},Y^{(\kappa)},\gamma^{(\kappa)})\}$ is a sequence of improved feasible points of the nonconvex program (24), which converges to a point satisfying first-order necessary optimality conditions.

Proof
The sequence $\{(X^{(\kappa)},Y^{(\kappa)},\gamma^{(\kappa)})\}$ terminates (whenever $F_\mu(X^{(\kappa+1)},Y^{(\kappa+1)},\gamma^{(\kappa+1)}) = F_\mu^{(\kappa)}(X^{(\kappa+1)},Y^{(\kappa+1)},\gamma^{(\kappa+1)})$) or convergence to $\{(\bar{X},\bar{Y},\bar{\gamma}\}$, which is the optimal solution of the convex program:

$$\begin{aligned} \min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\boldsymbol{\Delta}_{Ki}} \bar{F}_\mu(\mathbf{X},\mathbf{Y},\gamma) &:= \gamma + \mu(\text{tr}(\mathbf{X} - 2\bar{Y}^{-1}) + \text{tr}(\bar{Y}^{-1}\mathbf{Y}\bar{Y}^{-1}) \\ &\quad - \sum_{i=1}^{k}(\bar{x}_i^H(\mathbf{X}-\mathbf{Y}^{-1})\bar{x}_i) \quad \text{s.t.} \quad (9)-(10). \end{aligned} \tag{28}$$

where $\bar{x}_i$, $i = 1,\ldots,k$ are the orthonormal eigenvectors corresponding to k largest eigenvalues of $\bar{X} - \bar{Y}^{-1}$. Therefore, $(\bar{X},\bar{Y},\bar{\gamma})$ satisfies Kuh–Tucker condition for the convex program (28), which is also the first-order necessary optimality condition for the nonconvex program (24) [32]. □

Algorithm 1 is pseudocode for implementing the aforementioned procedure.

Alternatively, we can also use the following formulation instead of (24):

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\boldsymbol{\Delta}_{Ki},\hat{\mathbf{K}}} \gamma + \mu[\text{tr}(\mathbf{Y}-\mathbf{X}^{-1}) - f_{[k]}(\mathbf{Y}-\mathbf{X}^{-1})] \quad \text{s.t.} \quad (9)-(10), \tag{31}$$

for which the Algorithm 1 can be easily adjusted for solution. Usually, the initial point $(X^{(0)},Y^{(0)},\gamma^{(0)})$ is taken as the optimal solution of the full-order controller program

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\boldsymbol{\Delta}_{Ki},\hat{\mathbf{K}}} \gamma \quad \text{s.t.} \quad (9),(10). \tag{32}$$

and the preference of using (24) or (31) goes to whichever smaller among $\text{tr}(X^{(0)} - Y^{(0)}) - f_{[k]}(X^{(0)} - Y^{(0)})$ and $\text{tr}(Y^{(0)} - X^{(0)}) - f_{[k]}(Y^{(0)} - X^{(0)})$.

On the other hand, we can seek a k-order control to satisfy the $\mathcal{H}_\infty$-gain condition (5) for given γ by solving the following nonconvex program

$$\min_{\mathbf{X},\mathbf{Y},\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\boldsymbol{\Delta}_{Ki},\hat{\mathbf{K}}} F(\mathbf{X},\mathbf{Y}) := \text{tr}(\mathbf{X}-\mathbf{Y}^{-1}) - f_{[k]}(\mathbf{X}-\mathbf{Y}^{-1}) \quad \text{s.t.} \quad (9)-(10), \tag{33}$$

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Algorithm 1 Nonconvex Spectral Optimization Algorithm for optimized k-order $\mathcal{H}_\infty$ controllers

Initialize $\kappa := 0$ and solve SDP (32) to find its optimal solution $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$. For k normalized eigenvectors corresponding to k largest eigenvalues of $X^{(\kappa)} - (Y^{(\kappa)})^{-1}$ stop the algorithm if

$$\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - \sum_{i=1}^{k} \left(x_i^{(\kappa)}\right)^H (X^{(\kappa)} - (Y^{(\kappa)})^{-1})x_i^{(\kappa)}) \leqslant \epsilon_2$$

and accept $(X^{(0)}, Y^{(0)}, \gamma^{(0)})$ as the optimal solution of the nonconvex program (23). Otherwise set $\mu = 0.5$.

repeat

  if

$$\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - \sum_{i=1}^{k} \left(x_i^{(\kappa)}\right)^H (X^{(\kappa)} - (Y^{(\kappa)})^{-1})x_i^{(\kappa)}) \geqslant \epsilon_2 \tag{29}$$

  for k normalized eigenvectors corresponding to k largest eigenvalues of $X^{(\kappa)} - (Y^{(\kappa)})^{-1}$ **then** reset $\mu \to 2\mu$ and solve SDP (27) to find the optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)})$.

  else Solve SDP (27) with additional convex constraint

$$\mathrm{tr}(\mathbf{X} - 2(Y^{(\kappa)})^{-1}) + \mathrm{tr}((Y^{(\kappa)})^{-1}\mathbf{Y}(Y^{(\kappa)})^{-1}) - \sum_{i=1}^{k} \left(x_i^{(\kappa)}\right)^H (\mathbf{X} - \mathbf{Y}^{-1})x_i^{(\kappa)}) \leqslant \epsilon_2 \tag{30}$$

  end if

  Set $\kappa := \kappa + 1$.

until $\gamma^{(\kappa)} - \gamma^{(\kappa-1)} \leqslant \epsilon_1$

Accept $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$ as a found solution of (23) if $\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - f_{[k]}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leqslant \epsilon_2$.

where the penalty parameter μ is not needed. The pseudocode for solving (33) is provided by Algorithm 2, which is terminated when the zero value (with some tolerance) of the objective in (33), is found so the rank condition (11) is fulfilled leading to the construction of k-order control.

Again, an alternative formulation to (33)

$$\min_{\mathbf{X},\mathbf{Y},\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} F(\mathbf{X},\mathbf{Y}) := \mathrm{tr}(\mathbf{Y} - \mathbf{X}^{-1}) - f_{[k]}(\mathbf{Y} - \mathbf{X}^{-1}) \quad \text{s.t.} \quad (9)-(10), \tag{34}$$

is preferred if $\mathrm{tr}(Y^{(0)} - X^{(0)}) - f_{[k]}(Y^{(0)} - X^{(0)})$ is smaller than $\mathrm{tr}(X^{(0)} - Y^{(0)}) - f_{[k]}(X^{(0)} - Y^{(0)})$, where $(X^{(0)}, Y^{(0)})$ is the optimal solution of the following full-order controller program

$$\min_{\mathbf{X},\mathbf{Y},\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} \mathrm{tr}(\mathbf{X} + \mathbf{Y}) \quad \text{s.t.} \quad (9), (10). \tag{35}$$

Remarks on less reduced-order controllers. The computational difficulty degree in the formulation in [12, 14] (for LTI systems) is proportional to the control order k. Particularly, less reduced-order controllers may pose more computational challenges than highly reduced-order ones. In contrast, we now show that using the rank constraints (11) helps us solve them at the same computational efficiency.

Indeed, less order reduction means that $n-k$ is small. Suppose $\lambda_{[n-k]}(\mathbf{X} - \mathbf{Y}^{-1})$ is the sum of the $n-k$ smallest eigenvalues of $\mathbf{X} - \mathbf{Y}^{-1}$. Then, (11) holds true if and only if $\lambda_{[n-k]}(\mathbf{X} - \mathbf{Y}^{-1}) = 0$. Therefore, we propose the following alternative formulation for (23):

$$\min_{\mathbf{X}}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \mathbf{\Delta}_{Ki}, \hat{\mathbf{K}}\gamma + \mu\lambda_{[n-k]}(\mathbf{X} - \mathbf{Y}^{-1}) \quad \text{s.t.} \quad (9)-(10). \tag{37}$$

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Algorithm 2 Nonconvex Spectral Optimization Algorithm for feasible k-order $\mathcal{H}_\infty$ controllers

1: Initialize $\kappa := 0$ and solve SDP (35) to find its optimal solution $(X^{(\kappa)}, Y^{(\kappa)})$. Stop the algorithm if

$$\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - \sum_{i=1}^{k} \left(x_i^{(\kappa)}\right)^H (X - Y^{-1})x_i^{(\kappa)}) \leqslant \epsilon_2$$

and accept $(X^{(0)}, Y^{(0)})$ as the solution of the nonconvex program (33).

2: **repeat**

3: Solve SDP

$$\min_{\mathbf{X},\mathbf{Y}} \mathrm{tr}(\mathbf{X} - 2(Y^{(\kappa)})^{-1}) + \mathrm{tr}((Y^{(\kappa)})^{-1}\mathbf{Y}(Y^{(\kappa)})^{-1}) - \sum_{i=1}^{k} \left(x_i^{(\kappa)}\right)^H (\mathbf{X} - \mathbf{Y}^{-1})x_i^{(\kappa)}) \quad \text{s.t.}(9), (10) \tag{36}$$

to find its optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)})$

4: Set $\kappa := \kappa + 1$.

5: **until** $\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - \sum_{i=1}^{k}(x_i^{(\kappa)})^H (X^{(\kappa)} - (Y^{(\kappa)}) - 1)x_i^{(\kappa)}) \leqslant \epsilon_2$ or $F(X^{(\kappa-1)}, Y^{(\kappa-1)}) - F(X^{(\kappa)}, Y^{(\kappa)}) \leqslant \epsilon_1$

6: Accept $(X^{(\kappa)}, Y^{(\kappa)})$ as a found feasible solution of (23) under fixed γ if $\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - f_{[k]}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leqslant \epsilon_2$.

Using the following variational principle

$$\lambda_{[n-k]}(X - Y^{-1}) = \min_{\text{orthornomal } x_1,\ldots,x_{n-k}} \sum_{i=1}^{n-k} x_i^H (\mathbf{X} - \mathbf{Y}^{-1})x_i$$

the following optimization is majorant optimization for (37)

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} \gamma + \mu \sum_{i=1}^{n-k} \left(x_i^{(\kappa)}\right)^H (\mathbf{X} - \mathbf{Y}^{-1})x_i^{(\kappa)} \quad \text{s.t.} \quad (9) - (10), \tag{38}$$

where $x_i^{(\kappa)}$, $i = 1, \ldots, n-k$ are the orthonormal eigenvectors corresponding to $(n-k)$ smallest eigenvectors of $X^{(\kappa)} - (Y^{(\kappa)})^{-1}$. Because each $(x_i^{(\kappa)})^H Y^{-1} x_i^{(\kappa)}$ is convex in $Y > 0$, the following convex optimization is majorant minimization for (38) and (37):

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} \gamma + \mu \sum_{i=1}^{n-k} \left(x_i^{(\kappa)}\right)^H (\mathbf{X} - 2(Y^{(\kappa)})^{-1} + (Y^{(\kappa)})^{-1}\mathbf{Y}(Y^{(\kappa)})^{-1})x_i^{(\kappa)} \quad \text{s.t.} \quad (9) - (10), \tag{39}$$

which provides an alternative to κ-th iteration (27). This iteration is more efficient than (27) for larger $n-k$, that is, for lower order k of the controllers.

3. STATIC OUTPUT FEEDBACK LPV-LFT $\mathcal{H}_\infty$ CONTROLLER

The static output feedback LPV-LFT controller corresponds to $k = 0$, that is, the control in (7) is in the form

$$u(t) = (D_{K11} + D_{K1\Delta}(I - \Delta_K(\alpha(t))D_{K\Delta\Delta})^{-1}\Delta_K(\alpha(t))D_{K\Delta 1})y(t) \tag{40}$$

leading to the following optimization formulation for its synthesis:

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} \gamma \quad \text{s.t.} \quad (9) - (10), \tag{41}$$

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$$\mathbf{X} = \mathbf{Y}^{-1}, \tag{42}$$

for

$$\hat{\mathbf{K}} := \begin{bmatrix} \mathbf{D_{K11}} & \mathbf{D_{K1\Delta}} \\ \mathbf{D_{K\Delta 1}} & \mathbf{D_{K\Delta\Delta}} \end{bmatrix}$$

and setting $\mathbf{A_K} = 0$, $\mathbf{B_{K1}} = 0$, $\mathbf{B_{K_\Delta}} = 0$, $\mathbf{C_{K1}} = 0$, $\mathbf{C_{K\Delta}} =$ in (12). The controller (40) is recovered by (13), (17), (18), and (21). To the author's best knowledge, such simple structured controller (40) has not been considered in literature so far.

The first attractive reformulation of nonlinear constraint (42) is given back in [33]

$$(10), \text{Trace}(\mathbf{XY}) \leqslant n \tag{43}$$

where the nonconvexity is concentrated at the last indefinite quadratic constraint in $(\mathbf{Y}, \mathbf{X})$. Alternating optimization between $\mathbf{X}$ and $\mathbf{Y}$ is applied in handling (43).

Later, Nguyen *et al.* [14] also addressed the static output feedback controller problem for LTI systems by developing the so-called convex-concave inequality approach for a solution of the corresponding BMI reformulation. All these results must start from a feasible point of a nonconvex feasible set, which is not easily located.

Note that $\mathbf{X} \succeq \mathbf{Y}^{-1}$ by LMI (10), which yields $\mathbf{X} - \mathbf{Y}^{-1} \succeq 0$. Hence, the nonlinear equality (42) holds if and only if

$$\text{tr}(\mathbf{X}) - \text{tr}(\mathbf{Y}^{-1}) = 0.$$

In other words, the nonnegative quantity $\text{tr}(\mathbf{X}) - \text{tr}(\mathbf{Y}^{-1})$ can be used to measure the degree of satisfaction of the nonlinear equality (42). Instead of the formulation (24), we consider a simpler nonconvex program

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} \gamma + \mu(\text{tr}(\mathbf{X}) - \text{tr}(\mathbf{Y}^{-1})) \quad \text{s.t.}(9) - (10). \tag{44}$$

Using (26), at $(X^{(\kappa)}, Y^{(\kappa)})$ feasible to LMIs (44)–(10), the following convex program is a majorant minimization for the nonconvex program (44):

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} \gamma + \mu(\text{tr}(\mathbf{X}) - 2\text{tr}(Y^{(\kappa)}) + \text{tr}((Y^{(\kappa)})^{-1}\mathbf{Y}(Y^{(\kappa)})^{-1})) \quad \text{s.t.} \quad (9) - (10), \tag{45}$$

The pseudocode using (45) in κth iteration is given by Algorithm 3.

Alternatively, $\mathbf{Y} \succeq \mathbf{X}^{-1}$ by LMI (10), so whenever

$$\text{tr}(Y^{(0)}) - \text{tr}((X^{(0)})^{-1}) < \text{tr}(X^{(0)}) - \text{tr}((Y^{(0)})^{-1}) \tag{46}$$

for the initial point $(X^{(0)}, Y^{(0)})$, we use

$$\min_{\mathbf{X},\mathbf{Y},\gamma,\mathbf{R},\mathbf{H},\mathbf{Q},\mathbf{E},\mathbf{\Delta}_{Ki},\hat{\mathbf{K}}} \gamma + \mu(\text{tr}(\mathbf{Y}) - \text{tr}(\mathbf{X}^{-1})) \quad \text{s.t.} \quad (9) - (10) \tag{47}$$

instead of the formulation (44), for which Algorithm 3 can be easily adjusted for solution.

Similarly to (33), we address the design of a static output feedback $\mathcal{H}_\infty$ controller to satisfy the $\mathcal{H}_\infty$-gain condition (5) for given γ by solving the following nonconvex program:

$$\min_{\mathbf{X},\mathbf{Y}} \text{tr}(\mathbf{X}) - \text{tr}(\mathbf{Y}^{-1}) \quad \text{s.t.} \quad (9) - (10). \tag{50}$$

Its κth iteration is

$$\min_{\mathbf{X},\mathbf{Y}} \text{tr}(\mathbf{X}) - 2\text{tr}(Y^{(\kappa)}) + \text{tr}((Y^{(\kappa)})^{-1}\mathbf{Y}(Y^{(\kappa)})^{-1}) \quad \text{s.t.} \quad (9) - (10), \tag{51}$$

and Algorithm 4 is the pseudocode for the implementation.

Alternatively, whenever (46), we use

$$\min_{\mathbf{X},\mathbf{Y}} \text{tr}(\mathbf{Y}) - \text{tr}(\mathbf{X}^{-1}) \quad \text{s.t.} \quad (9) - (10) \tag{52}$$

instead of (50), for which Algorithm 4 can be easily adjusted for computation.

Algorithm 3 Nonconvex Spectral Optimization Algorithm for static LPV-LFT $\mathcal{H}_\infty$ controllers

1: Initialize $\kappa := 0$ and solve SDP (41) to find its optimal solution $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$. Stop the algorithm if

$$\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leqslant \epsilon_2 \tag{48}$$

and accept $(X^{(0)}, Y^{(0)}, \gamma^{(0)})$ as the optimal solution of the nonconvex program (23). Otherwise set $\mu = 0.5$.

2: **repeat**

3: **if** $\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \geqslant \epsilon_2$ **then** reset $\mu \to 2\mu$ and solve SDP (45) to find the optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)})$.

4: **else** Solve SDP (45) with additional convex constraint

$$\mathrm{tr}(\mathbf{X} - 2(Y^{(\kappa)})^{-1}) + \mathrm{tr}((Y^{(\kappa)})^{-1}\mathbf{Y}(Y^{(\kappa)})^{-1}) \leqslant \epsilon_2 \tag{49}$$

to find the optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)})$

5: **end if**

6: Set $\kappa := \kappa + 1$.

7: **until** $\gamma^{(\kappa)} - \gamma^{(\kappa-1)} \leqslant \epsilon_1$

8: Accept $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$ as a found suboptimal solution of (41)- (42) if (48) is fulfilled.

Algorithm 4 Nonconvex Spectral Optimization Algorithm for feasible static output feedback LPV-LFT $\mathcal{H}_\infty$ controllers

1: Initialize $\kappa := 0$ and solve SDP (41) (for fixed $\gamma = \bar{\gamma}$ to find its optimal solution $(X^{(\kappa)}, Y^{(\kappa)})$. Stop the algorithm if

$$\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leqslant \epsilon_2 \tag{53}$$

and accept $(X^{(0)}, Y^{(0)})$ as the optimal solution of the nonconvex program (23).

2: **repeat**

3: Solve SDP (51) to find the optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)})$.

4: Set $\kappa := \kappa + 1$.

5: **until** $\mathrm{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leqslant \epsilon_2$ or $\mathrm{tr}(X^{(\kappa-1)} - Y^{(\kappa-1)}) - \mathrm{tr}(X^{(\kappa)} - Y^{(\kappa)}) \leqslant \epsilon_1$

6: Accept $(X^{(\kappa)}, Y^{(\kappa)})$ as a found feasible solution of (41)- (42) under fixed γ if (53) is fulfilled.

4. SIMULATION RESULTS

The hardware and software facilities for our computational implementation are as follows:

- Processor: Intel(R) Core(TM) i5-3470 CPU @3.20 GHz;
- Software: Matlab version R2015b;
- Matlab toolbox: Yalmip [34] with SeDumi 1.3 [35] solver for SDP;
- Data: The data in Section 4.1 are from [24], while the state-space data in Sections 4.2, 4.3, and 4.4 are from [36];
- Criterion: The stop and rank check criterion ϵ_1 and ϵ_2 are set as 10^{-4}.

4.1. Rotational-translational actuator control

Consider the nonlinear benchmark model [37] of rotational-translational actuator. The regulated output is the tracking performance of the translational and angular positions and control

$$z = (0.1x_1, 0.1x_3, u)^T$$

The system can be represented by LPV-LFT (1) and (2) [24, Appendix] with the numerical values of the matrices in (1 and (2) recalled in Appendix A.

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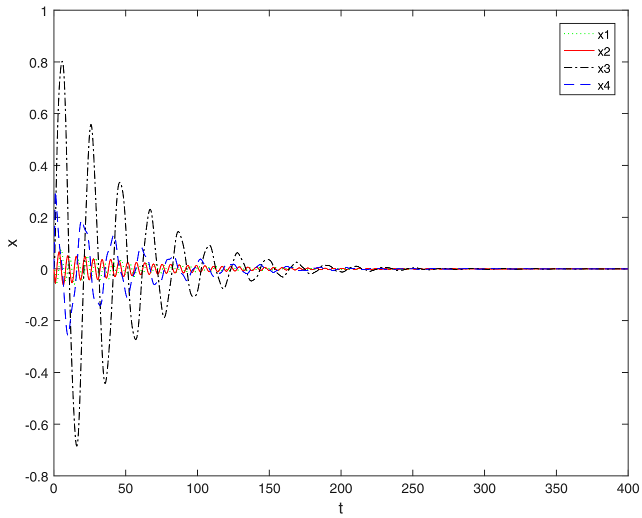

Figure 2. Tracking performance of the first-order linear parameter varying-linear fractional transformational controller in the absence of disturbance. [Colour figure can be viewed at wileyonlinelibrary.com]

By solving SDP (32) for step 1 of Algorithm 1, we found $\gamma^{(0)} = 8.1909$ with $X^0 - (Y^0)^{-1}$ of rank-four, which lead to full-order control (3) and (4) [24]. Implementing Algorithm 1 with $\mu = 1$ for the first-order controller, $\gamma = 9.3785$ was found and the following numerical data for control (3) and (4) are obtained:

$$A_K = -4.2617,\ B_{K1} = \begin{bmatrix} 0.4767 & -2.0207 \end{bmatrix}, C_{K1} = -0.1327, \quad D_{K11} = \begin{bmatrix} -0.0670 & 0.0284 \end{bmatrix},$$

$$B_{K\Delta} = \begin{bmatrix} 0 & -0.0030 & -0.0132 \end{bmatrix}, D_{K1\Delta} = \begin{bmatrix} 0 & 0 & 0.0013 \end{bmatrix}, D_{K\Delta\Delta} = \begin{bmatrix} -0.0002 & -0.3509 & -1.4991 \\ 0 & -0.0003 & -0.0015 \\ 0 & 0.0002 & 0.0008 \end{bmatrix},$$

$$C_{K\Delta} = \begin{bmatrix} -30.6379 \\ 4.2867 \\ -1.6715 \end{bmatrix}, D_{K\Delta 1} = \begin{bmatrix} -11.5584 & -2.5580 \\ 0.1977 & 0.0127 \\ 0.0034 & -0.0107 \end{bmatrix},$$

$$\Delta_{K1} = \begin{bmatrix} -0.3118 & 4.8064 & -25.2717 \\ 0.3425 & 4.8514 & 64.0377 \\ -0.0932 & -27.9300 & -84.1243 \end{bmatrix}, \Delta_{K2} = \begin{bmatrix} -0.3125 & 3.5657 & 51.1251 \\ 0.3434 & 6.5093 & -39.3112 \\ -0.0934 & -28.3084 & -59.9568 \end{bmatrix},$$

$$\Delta_{K3} = \begin{bmatrix} 0.3118 & -4.8064 & 25.2717 \\ -0.3425 & -4.8514 & -64.0377 \\ 0.0932 & 27.9300 & 84.1243 \end{bmatrix}, \Delta_{K4} = \begin{bmatrix} 0.3125 & -3.5657 & -51.1251 \\ -0.3434 & -6.5093 & 39.3112 \\ 0.0934 & 28.3084 & 59.9568 \end{bmatrix},$$

Under the condition $x(0) = (0.5, 0, 0, 0)^T$, the simulation given by Figures 2–4 clearly shows that our first-order LPV-LFT stabilizes the system well.

4.2. *Reduced-order LPV-LFT controllers*

We modify the LTI examples in [12, Section 10] by adding the gain-scheduling channel (w_Δ, z_Δ) to have LPV-LFT system (1). The randomly generated matrix sets for the gain-scheduling channel are provided in Appendix B.

The computational results by implementing Algorithm 1 are provided by Table I. The full order of LPV-LFT control, which is equal to the system state dimension is given in the second row with the initial γ obtained by solving (32) (for full-order LPV-LFT control) given in the third row. The fourth column indicates the value of initial μ used. The fifth column is the found value of γ for the controller of order indicated in the sixth column.

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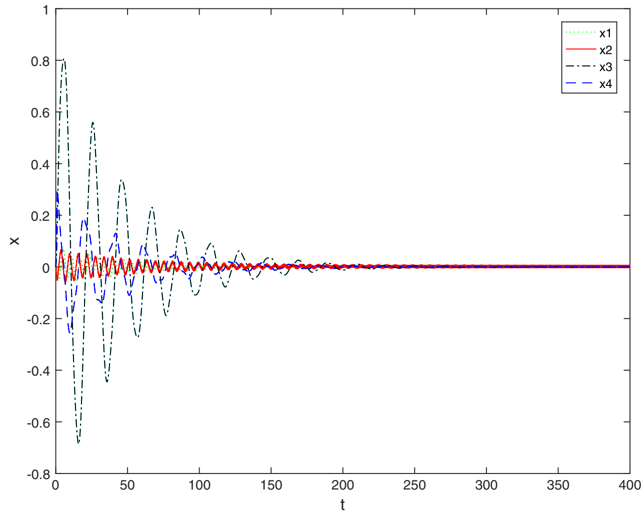

Figure 3. Tracking performance of the first-order linear parameter varying-linear fractional transformational controller with the disturbance $w = 0.1\sin(5\pi t)$. [Colour figure can be viewed at wileyonlinelibrary.com]

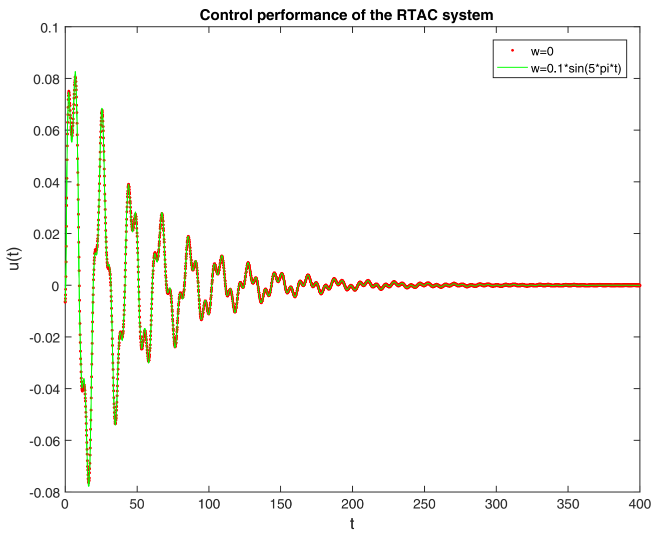

Figure 4. The behaviour of the first-order linear parameter varying-linear fractional transformational controller in the absence of disturbance (dot) and with disturbance $w = 0.1\sin(5\pi t)$(solid). [Colour figure can be viewed at wileyonlinelibrary.com]

Table I. Computational results for reduced-order linear parameter varying-linear fractional transformational controllers by Algorithm 1.

	Full order	Lower bound by (32)	μ	γ	Order k
VTOL helicopter	4	0.0871	0.05	0.2958	1
Chemical reactor	4	0.8653	0.05	0.8653	1
Transport airplane	10	1.7042	10	2.5302	1

4.3. Static output feedback LPV-LFT controller

We modify the LTI examples in [14, Tab. III] by adding the matrices relating to the gain-scheduling channel (w_Δ, z_Δ) as provided in Appendix C. The computational results by implementing Algorithm 3 are provided in Table II, which is formatted similarly to Table I. Tables I and II reveal that very low-order (first-order or static) controllers, which lead to very efficient online control realization, work well in these examples.

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Table II. Computational results for static linear parameter varying-linear fractional transformational controllers by Algorithm 3.

	Full order	Lower bound by (32)	μ	γ
AC1	5	2.76E-08	0.1	6.68E-07
AC2	5	0.117675	1	2.113598
AC3	5	3.095253	1	4.891439
AC6	7	3.683339	5	4.948768
AC8	9	8.727383	20	9.57084
AC9	10	1.000091	5	1.002586
AC15	4	16.30176	5	18.91505
AC17	4	6.686188	5	6.688397
HE4	8	28.81522	5	37.26205

Table III. Numerical results by Algorithms 1 and 3 compared with [12].

Case	Lower bound	Obj	μ	γ	Order	# iter	γ in [12]	Order in [12]
VTOL helicopter	0.0737	(24)	1	0.118713	2	1	0.133	2
VTOL helicopter	0.0737	(44)	0.7	0.1539	0	20	0.1542	0
Chemical reactor	0.8617	(24)	1	0.8617	2	1	1.142	2
Chemical reactor	0.8617	(44)	1	0.8937	0	28	1.183	0
Transport aircraft	0.0417	(31)	10	0.349	1	42	2.86	1
Transport aircraft	0.0417	(31)	1	0.2167	2	6	failed	2
Piezoelectric actuator	3.11E-05	(31)	5	0.0048	2	3	0.03	2
Piezoelectric actuator	3.11E-05	(47)	100	0.0213	0	3	0.0578	0
Coupled springs model	0.01996	(24)	1	0.01997	2	4	0.0235	4

Table IV. Distillation tower case with γ fixed by Algorithm 4 compared with [12].

Case	Fixed γ	Obj	Order	# iter	Trace$(X - Y^{-1})$	γ in [12]	Order in [12]
Distillation tower	0.8000	(44)	0	64	2.24E-05	1.0722	0

4.4. LTI systems

In LTI systems, there are no gain-scheduling channel (w_Δ, z_Δ) in system (1) and no gain-scheduling channel (w_K, z_K) in controller (3). Accordingly,

$$\hat{\mathbf{K}} := \begin{bmatrix} \mathbf{A_K} & \mathbf{B_{K1}} \\ \mathbf{C_{K1}} & \mathbf{D_{K11}} \end{bmatrix}$$

in (8) and LMI (9) becomes

$$\begin{bmatrix} \mathbf{X}A + \mathbf{B_{K1}}C_2 + (*) & * & * & * \\ \mathbf{A_K}^T + A + B_2\mathbf{D_{K11}}C_2 & (A\mathbf{Y} + B_2\mathbf{C_{K1}}) + (*) & * & * \\ B_1^T\mathbf{X} + D_{21}^T\mathbf{B_{K1}}^T & B_1^T + D_{21}^T\mathbf{D_{K11}}^T B_2^T & -\gamma I & * \\ C_1 + D_{12}\mathbf{D_{K11}}C_2 & C_1\mathbf{Y} + D_{12}\mathbf{C_{K1}} & D_{11} + D_{12}\mathbf{D_{K11}}D_{21} & -\gamma I \end{bmatrix} \prec 0 \qquad (54)$$

with control recovered by (13)–(16) [38].

As mentioned in the Section 1, the Matlab command systune [18] is the most efficient tool for LTI systems. Our purpose in this subsection is not to show any advantage of the Lyapunov matrix-rank-constrained approach over the frequency approach by using the Matlab command systune [18]. We consider numerical examples from [12, section 10] and [14, table III] for LTI systems to only show the efficiency of our approach in handling the rank-reduced constraints.

4.4.1. Dynamic and static output feedback controllers in [12]. All cases in [12, section 10] were tested. The computational results are summarized in Table III, where the first column is the case name, the second column is the initial γ obtained by solving (32), that is, it is the optimal $\mathcal{H}_\infty$ by

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Table V. Numerical results of static output feedback controllers by Algorithm 3 compared with [14].

Case	Lower bound	Obj	μ	γ	# iter	γ in [14]	# iter	γ by systune [18]	# iter
AC1	2.68E-06	(47)	0.1	3.42E-05	7	0.0177	93	6.97E-05	85
AC2	0.1115	(47)	0.1	0.1115	1	0.1140	99	0.1115	53
AC3	2.9675	(47)	0.15	3.4696	300	3.4859	210	3.4056	111
AC4	0.5573	(44)	0.3	1.0064	300	69.9900	2	0.9355	45
AC6	3.4275	(47)	0.1	4.1208	132	4.1954	167	4.114	59
AC7	0.0396	(47)	0.1	0.0657	150	0.0548	300	0.0651	28
AC8	1.6165	(47)	2	2.0508	16	3.052	247	2.005	37
AC9	1.0000	(47)	1	1.003	1	0.9237 (wrong)	300	1.0006	102
AC11	2.8079	(44)	10	2.9261	300	3.0104	68	2.818	95
AC12	0.0225	(44)	1	0.4706	14	2.3025	300	0.0537	300
AC15	14.8628	(47)	0.2	15.1730	116	15.1995	105	15.1689	54
AC16	14.8556	(44)	0.09	15.0012	24	14.9881	186	14.858	54
AC17	6.6124	(47)	1	6.6124	1	6.6373	129	6.6124	28
HE1	0.0737	(44)	0.7	0.1539	20	0.1807	300	0.1538	43
HE2	2.4181	(47)	10	4.4162	272	6.7846	177	3.8958	58
HE4	22.8382	(47)	3	22.8431	203	22.8713	252	22.8382	69
REA1	0.8617	(47)	0.2	0.8911	189	0.8815	96	0.8656	54
REA2	1.1341	(44)	1	1.1895	1	1.4188	300	1.149	45
REA3	74.2513	(47)	1	74.2513	4	74.5478	2	74.2513	24
DIS1	4.1593	(44)	5	4.5625	276	4.1943	93	4.1606	43
DIS3	1.0423	(47)	0.1	1.0933	150	1.1382	285	1.0624	117
DIS4	0.7315	(44)	0.1	0.7556	64	0.7498	126	0.7353	76
AGS	8.1732	(47)	1	8.1732	5	8.1732	24	8.1732	20
WEC2	3.5981	(47)	100	5.9166	128	6.6082	300	4.2483	95
WEC3	3.7685	(47)	100	6.2305	107	6.8402	300	4.4496	101
BDT1	0.2653	(47)	0.1	0.331	195	0.8562	29	0.2662	30
MFP	4.1865	(47)	300	31.5978	300	31.6079	171	31.5899	42
IH	1.26E-06	(47)	1	1.40E-05	1	1.1858	114	0.002	300
CSE1	0.02	(44)	1	0.02	1	0.022	3	0.02	34
PSM	0.9202	(44)	0.1	0.9206	15	0.9227	87	0.9202	18
EB1	3.1041	(44)	20	3.142	1	2.2076 (wrong)	300	3.1225	21
EB2	1.7676	(44)	1	2.0205	24	0.8148 (wrong)	84	2.0201	22
EB3	1.7976	(44)	1	2.058	26	0.8153 (wrong)	84	2.0575	22
NN1	13.1299	(47)	1	17.2732	4	18.4813	300	13.8474	45
NN2	1.7645	(44)	1	2.2217	27	2.2216	9	2.2216	20
NN8	2.3576	(47)	0.47	3.074	312	2.9345	180	2.8854	47
NN9	13.6461	(47)	40	30.0387	300	32.1222	300	28.6673	77
NN11	0.0181	(47)	50	0.1981	648	0.1566	9	0.0914	92
NN15	0.0977	(44)	1	0.0993	2	0.1194	6	0.0981	38
NN17	2.6386	(44)	15	11.2182	165	11.2381	117	11.2182	26

the full-order controller, the third column is the objective function, which is either (24) or (31) for k-order controller and either (44) or (47) for static output feedback controller. The fourth column is the value of initial μ. The fifth column is the found value of γ. The sixth column provides the controller order. The seventh column is the iterations by our method; the last two column are the found γ and corresponding order in [12], respectively.

Compared with [12], it can be seen that our optimal γ are better than [12] in all cases provided. For the transport aircraft example [12], failed to obtain two-order controller, although it was found by our Algorithm after six iterations. The Piezoelectric actuator example poses the most difficulty for [12], but it is easily solved by our algorithm with three iterations for both order 2 controllers and statistic output feedback controller.

The last example in [12] for static output feedback for a plant with state dimension 82. The computational results by implementing Algorithm 2 are summarized in Table IV, whose format is similar to Table III, but the second column is the fixed value of γ, which is better than the value provided by [12] in the seventh column. The sixth column is the value of trace$(X - Y^{-1})$ at the last iteration.

Table VI. Numerical results of static output feedback controllers by Algorithm 4 compared with [14].

Case	Lower bound	Fixed γ	Obj	# iter	γ in [14]	# iter	γ by systune [18]	# iter
HE3	0.7990	0.9200	(44)	257	0.9243	105	0.8052	70
DIS1	4.1593	4.1700	(47)	10	4.1943	93	4.1606	43
TG1	3.4652	12.8000	(44)	264	12.9336	45	12.8462	36
NN4	1.2862	1.3600	(44)	283	1.3802	156	1.3598	79
NN11	0.0181	0.1000	(44)	178	0.1566	9	0.0914	92
NN16	0.9556	0.9600	(44)	53	0.9656	48	0.9558	65

4.4.2. Static output feedback controllers in [14]. There are 45 cases in [14, table III]. The computational results are summarized in Tables V and VI by Algorithms 3 and 4, respectively, whose format is in similar style to Table III. Anyway, the best γ obtained from systune [18], which is referred to what is achievable, is also provided in the ninth column. In AC9, EB1, EB2, and EB3, the results by [14] are obviously incorrect as its provided values of γ in the seventh column is even smaller than their lower bound in the second column. It should be mentioned that the value of μ is increased to regulate the convergence speed, but a larger μ results in larger γ as well. According to [14], the iteration threshold to stop its solver is 300. The solver [14] is trapped by local minima in AC4 and AC12 as its found values are much bigger than that found by our Algorithm 3. The former is also heading to a wrong minima in the case AC12 as the value found after 300 iterations is still very far from that found by the latter. In AC1, AC2, AC8, AC11, AC16, AC17, HE1, HE2, REA2, DIS3, DIS4, WEC2, WEC3, IH, PSM, and NN1, the later clearly outperforms the former in both computational performance and convergence. Note that the result for HE3, DIS1, TG1, NN4, NN11, and NN16 in Table VI was obtained by using Algorithm 4. Our simulation results are better than or consistent with [14]. It should be realized that the results in these tables also depend on the setting of stopping parameters in the algorithms, that is, the presented results are not necessarily the local minima when the algorithms stop because of slow convergence.

5. CONCLUSIONS

We have proposed new algorithms for solving matrix-rank-constrained optimization arising in reduced-order $\mathcal{H}_\infty$ LPV-LFT controller design. Unlike the previous developments, we formulate the problem as minimization of nonconvex objective function over a convex feasibility set. The global convergence of the proposed algorithms to a local minima follows immediately from their path-following nature, while there is no difficulty for initial solutions, which are found from a semi-definite program for full-order controller synthesis. The numerical results reported for the benchmark collections have shown their merit. Their application to solutions of reduced-order generalized $\mathcal{H}_2$ LPV-LFT controllers is obvious. Their extensions to multi-objective and structured controller design are currently under development.

APPENDIX A: LPV-LFT DATA OF ROTATIONAL-TRANSLATIONAL ACTUATOR SYSTEM

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.0365 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.1946 & 0 & 0 & 0 \end{bmatrix}, B_\Delta = \begin{bmatrix} 0 & 0 & 0 \\ -0.5 & 0.5 & 1.0365 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & -0.1946 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1.0365 \\ 0 \\ -0.1946 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -0.1946 \\ 0 \\ 1.03654 \end{bmatrix}$$

$$C_\Delta = \begin{bmatrix} 1.5157 & 0 & 0 & 0 \\ 0.7088 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D_{\Delta\Delta} = \begin{bmatrix} 1.2311 & 0 & -1.5157 \\ 0 & -0.8419 & -0.7088 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D_{\Delta 1} = \begin{bmatrix} -1.5157 \\ -0.7088 \\ 0 \end{bmatrix}, D_{\Delta 2} = \begin{bmatrix} 1.5157 \\ -0.7088 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.31622 & 0 & 0 & 0 \\ 0 & 0 & 0.3162 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} 0.01224 & 0 & 0 \\ 0 & 0.01224 & 0 \\ 0 & 0 & 0.04794 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.01224 & 0 & 0 \\ 0 & 0.01224 & 0 \\ 0 & 0 & -0.04794 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} -0.01224 & 0 & 0 \\ 0 & -0.01224 & 0 \\ 0 & 0 & -0.04794 \end{bmatrix}, \Delta_4 = \begin{bmatrix} -0.01224 & 0 & 0 \\ 0 & -0.01224 & 0 \\ 0 & 0 & 0.04794 \end{bmatrix}.$$

$$\alpha_1(t) = \frac{1}{4a_2a_5}(a_2 - \delta_1(t))(a_5 - \delta_2(t)), \alpha_2(t) = \frac{1}{4a_2a_5}(a_2 - \delta_1(t))(a_5 + \delta_2(t))$$

$$\alpha_3(t) = \frac{1}{4a_2a_5}(a_2 + \delta_1(t))(a_5 + \delta_2(t)), \alpha_4(t) = \frac{1}{4a_2a_5}(a_2 + \delta_1(t))(a_5 - \delta_2(t))$$

where

$$a_1 = \epsilon\frac{\cos 0.5 + 1}{2} \quad a_2 = \epsilon\frac{1 - \cos 0.5}{2}, a_3 = 1 - a_1, a_4 = 1 + a_1, a_5 = \epsilon 0.5 \sin 0.5$$

$$\delta_1 = \epsilon \cos x_3 - a_1, -a_2 \leqslant \delta_1 \leqslant a_2, \delta_2 = \epsilon x_4 \sin x_3, -a_5 \leqslant \delta_2 \leqslant a_5,\ \epsilon = 0.2.$$

APPENDIX B

B.1 Modified VTOL helicopter system

$$B_\Delta = \begin{bmatrix} 0.5243 & 0.4413 \\ 0.3440 & 0.1393 \\ 0.1109 & 0.5365 \\ 0.2478 & 0.1764 \end{bmatrix}, C_\Delta = \begin{bmatrix} 0.3031 & 0.5501 & 0.5605 & 0.0255 \\ 0.3292 & 0.2303 & 0.3367 & 0.1256 \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0.0740 & 0.0618 \\ 0.0217 & 0.0152 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0305 & 0.0611 \\ 0.0097 & 0.0724 \end{bmatrix}, \Delta_3 = \begin{bmatrix} 0.0477 & 0.0562 \\ 0.0371 & 0.0565 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0506 & 0.0704 \\ 0.0474 & 0.0156 \end{bmatrix}.$$

B.2 Modified chemical reactor system

$$B_\Delta = \begin{bmatrix} 0.3992 & 0.2357 & 0.1968 & 0.2560 \\ 0.3125 & 0.0069 & 0.3433 & 0.0130 \\ 0.2363 & 0.0491 & 0.0851 & 0.2498 \\ 0.3772 & 0.3506 & 0.2244 & 0.1473 \end{bmatrix}, C_\Delta = \begin{bmatrix} 0.1869 & 0.3511 & 0.2284 & 0.2440 \\ 0.3123 & 0.2240 & 0.0962 & 0.2576 \\ 0.0888 & 0.3294 & 0.2597 & 0.2844 \\ 0.1941 & 0.2879 & 0.0325 & 0.3472 \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0.0032 & 0.0133 & 0.0410 & 0.0322 \\ 0.0316 & 0.0095 & 0.0182 & 0.0346 \\ 0.0124 & 0.0122 & 0.0348 & 0.0287 \\ 0.0079 & 0.0028 & 0.0449 & 0.0080 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0232 & 0.0099 & 0.0097 & 0.0067 \\ 0.0403 & 0.0267 & 0.0448 & 0.0079 \\ 0.0413 & 0.0303 & 0.0039 & 0.0294 \\ 0.0128 & 0.0197 & 0.0050 & 0.0271 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} 0.0019 & 0.0024 & 0.0319 & 0.0148 \\ 0.0346 & 0.0320 & 0.0292 & 0.0050 \\ 0.0271 & 0.0347 & 0.0191 & 0.0011 \\ 0.0274 & 0.0366 & 0.0066 & 0.0349 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0144 & 0.0309 & 0.0408 & 0.0085 \\ 0.0141 & 0.0012 & 0.0166 & 0.0316 \\ 0.0159 & 0.0402 & 0.0213 & 0.0158 \\ 0.0223 & 0.0267 & 0.0026 & 0.0429 \end{bmatrix}.$$

B.3 Modified transport airplane system

$$B_\Delta^T = \begin{bmatrix} 0.1454 & 0.0308 & 0.2485 & 0.2151 & 0.0160 & 0.0712 & 0.1049 & 0.1605 & 0.2354 & 0.1063 \\ 0.1822 & 0.2487 & 0.1115 & 0.1627 & 0.2309 & 0.2232 & 0.0366 & 0.0514 & 0.1590 & 0.1985 \\ 0.0632 & 0.0946 & 0.0874 & 0.1169 & 0.0438 & 0.1910 & 0.0251 & 0.2280 & 0.0210 & 0.2002 \\ 0.2174 & 0.1155 & 0.1550 & 0.0908 & 0.1944 & 0.1510 & 0.1566 & 0.2160 & 0.1170 & 0.2261 \end{bmatrix},$$

$$C_\Delta = \begin{bmatrix} 0.1991 & 0.0018 & 0.2215 & 0.2496 & 0.1639 & 0.1131 & 0.0108 & 0.0643 & 0.2234 & 0.1669 \\ 0.0184 & 0.2225 & 0.1165 & 0.2177 & 0.1031 & 0.0067 & 0.1392 & 0.1817 & 0.1950 & 0.2448 \\ 0.2484 & 0.2162 & 0.0930 & 0.0000 & 0.1192 & 0.1064 & 0.1530 & 0.1785 & 0.1238 & 0.0293 \\ 0.1942 & 0.1746 & 0.0208 & 0.1702 & 0.0972 & 0.0265 & 0.1849 & 0.1261 & 0.1906 & 0.2175 \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0.0191 & 0.0212 & 0.0033 & 0.0055 \\ 0.0241 & 0.0303 & 0.0386 & 0.0324 \\ 0.0217 & 0.0173 & 0.0281 & 0.0281 \\ 0.0279 & 0.0353 & 0.0216 & 0.0161 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0136 & 0.0429 & 0.0119 & 0.0158 \\ 0.0156 & 0.0105 & 0.0376 & 0.0239 \\ 0.0289 & 0.0252 & 0.0393 & 0.0138 \\ 0.0342 & 0.0211 & 0.0192 & 0.0078 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} 0.0181 & 0.0437 & 0.0088 & 0.0238 \\ 0.0130 & 0.0085 & 0.0322 & 0.0011 \\ 0.0111 & 0.0416 & 0.0298 & 0.0010 \\ 0.0294 & 0.0260 & 0.0152 & 0.0365 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0037 & 0.0241 & 0.0188 & 0.0333 \\ 0.0234 & 0.0321 & 0.0223 & 0.0058 \\ 0.0263 & 0.0101 & 0.0321 & 0.0320 \\ 0.0269 & 0.0338 & 0.0076 & 0.0325 \end{bmatrix}$$

APPENDIX C

C.1 Modified AC1 system

$$B_\Delta = \begin{bmatrix} 0.2972 & 0.1612 \\ 0.4326 & 0.0854 \\ 0.1907 & 0.2157 \\ 0.4383 & 0.2318 \\ 0.5171 & 0.3084 \end{bmatrix}, C_\Delta = \begin{bmatrix} 0.2360 & 0.5001 & 0.1001 & 0.5307 & 0.0602 \\ 0.1685 & 0.2330 & 0.4901 & 0.2377 & 0.1398 \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0.1028 & 0.0531 \\ 0.0173 & 0.1622 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0046 & 0.1142 \\ 0.1453 & 0.0764 \end{bmatrix}, \Delta_3 = \begin{bmatrix} 0.0936 & 0.0742 \\ 0.0384 & 0.1558 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.1177 & 0.0499 \\ 0.1122 & 0.1052 \end{bmatrix}.$$

C.2 Modified AC2 system

$$B_\Delta = \begin{bmatrix} 0.1860 & 0.1471 & 0.1931 & 0.1790 & 0.3075 \\ 0.3197 & 0.2706 & 0.0325 & 0.0686 & 0.2026 \\ 0.2126 & 0.2984 & 0.0404 & 0.1789 & 0.3360 \\ 0.0559 & 0.2856 & 0.0493 & 0.0534 & 0.2518 \\ 0.0722 & 0.1151 & 0.2453 & 0.0199 & 0.2107 \end{bmatrix}, C_\Delta = \begin{bmatrix} 0.0063 & 0.2641 & 0.1275 & 0.2541 & 0.2679 \\ 0.1064 & 0.1382 & 0.2599 & 0.3052 & 0.2471 \\ 0.1364 & 0.2854 & 0.2427 & 0.1053 & 0.0538 \\ 0.0869 & 0.1257 & 0.1213 & 0.2158 & 0.2771 \\ 0.0633 & 0.2472 & 0.0694 & 0.1410 & 0.3182 \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0.0464 & 0.0349 & 0.0130 & 0.0420 & 0.0356 \\ 0.0500 & 0.0563 & 0.0284 & 0.0334 & 0.0376 \\ 0.0563 & 0.0300 & 0.0513 & 0.0140 & 0.0415 \\ 0.0000 & 0.0273 & 0.0327 & 0.0379 & 0.0507 \\ 0.0493 & 0.0456 & 0.0481 & 0.0048 & 0.0559 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0630 & 0.0099 & 0.0452 & 0.0041 & 0.0120 \\ 0.0476 & 0.0706 & 0.0516 & 0.0401 & 0.0155 \\ 0.0760 & 0.0396 & 0.0026 & 0.0158 & 0.0035 \\ 0.0475 & 0.0692 & 0.0503 & 0.0101 & 0.0520 \\ 0.0014 & 0.0171 & 0.0297 & 0.0168 & 0.0231 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} 0.0396 & 0.0091 & 0.0153 & 0.0697 & 0.0457 \\ 0.0511 & 0.0361 & 0.0415 & 0.0060 & 0.0422 \\ 0.0367 & 0.0627 & 0.0471 & 0.0078 & 0.0038 \\ 0.0394 & 0.0643 & 0.0307 & 0.0104 & 0.0685 \\ 0.0327 & 0.0199 & 0.0151 & 0.0122 & 0.0536 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0496 & 0.0578 & 0.0090 & 0.0224 & 0.0376 \\ 0.0043 & 0.0528 & 0.0021 & 0.0314 & 0.0574 \\ 0.0579 & 0.0345 & 0.0632 & 0.0436 & 0.0234 \\ 0.0628 & 0.0119 & 0.0203 & 0.0017 & 0.0300 \\ 0.0662 & 0.0268 & 0.0199 & 0.0566 & 0.0036 \end{bmatrix}.$$

C.3 Modified AC3 system

$$B_\Delta = \begin{bmatrix} 0.0421 & 0.1351 & 0.2372 & 0.3017 & 0.4218 \\ 0.0606 & 0.0929 & 0.0787 & 0.2381 & 0.0728 \\ 0.0718 & 0.1072 & 0.0905 & 0.1338 & 0.1101 \\ 0.0838 & 0.3812 & 0.0330 & 0.0710 & 0.1694 \\ 0.1355 & 0.3002 & 0.3901 & 0.2657 & 0.0316 \end{bmatrix}, C_\Delta = \begin{bmatrix} 0.2106 & 0.2908 & 0.1364 & 0.0399 & 0.2294 \\ 0.3321 & 0.1855 & 0.3337 & 0.0034 & 0.0470 \\ 0.1710 & 0.1951 & 0.3008 & 0.1827 & 0.2729 \\ 0.1179 & 0.2634 & 0.0541 & 0.2831 & 0.0546 \\ 0.0161 & 0.0257 & 0.0562 & 0.3122 & 0.0580 \end{bmatrix},$$

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$$\Delta_1=\begin{bmatrix}0.0495&0.0112&0.0254&0.0433&0.0260\\0.0291&0.0276&0.0497&0.0243&0.0404\\0.0712&0.0117&0.0213&0.0217&0.0538\\0.0291&0.0549&0.0384&0.0328&0.0307\\0.0450&0.0631&0.0603&0.0306&0.0311\end{bmatrix},\ \Delta_2=\begin{bmatrix}0.0086&0.0659&0.0523&0.0319&0.0113\\0.0017&0.0644&0.0510&0.0146&0.0459\\0.0200&0.0315&0.0512&0.0068&0.0616\\0.0219&0.0166&0.0073&0.0567&0.0356\\0.0450&0.0526&0.0469&0.0121&0.0484\end{bmatrix}$$

$$\Delta_3=\begin{bmatrix}0.0107&0.0562&0.0380&0.0014&0.0640\\0.0663&0.0520&0.0277&0.0642&0.0552\\0.0376&0.0084&0.0289&0.0454&0.0401\\0.0472&0.0365&0.0126&0.0648&0.0306\\0.0025&0.0226&0.0178&0.0114&0.0179\end{bmatrix},\ \Delta_4=\begin{bmatrix}0.0471&0.0448&0.0199&0.0398&0.0395\\0.0143&0.0402&0.0510&0.0595&0.0222\\0.0040&0.0262&0.0494&0.0278&0.0624\\0.0480&0.0245&0.0534&0.0038&0.0140\\0.0420&0.0511&0.0317&0.0543&0.0409\end{bmatrix}.$$

C.4 Modified AC6 system

$$B_\Delta^T=\begin{bmatrix}0.0327&0.0447&0.3191&0.3238&0.1948&0.0202&0.0795\\0.1196&0.2781&0.0052&0.0146&0.0572&0.2198&0.2478\\0.2194&0.1527&0.1853&0.1004&0.2522&0.0640&0.2326\\0.0622&0.1248&0.2119&0.2642&0.0275&0.3148&0.2627\end{bmatrix}$$

$$C_\Delta=\begin{bmatrix}0.3139&0.3081&0.1451&0.0664&0.3260&0.1220&0.2818\\0.2094&0.2247&0.0274&0.0867&0.3412&0.3250&0.1408\\0.1986&0.1267&0.0866&0.1507&0.1773&0.1334&0.0873\\0.0523&0.1854&0.0445&0.0179&0.1767&0.0402&0.1459\end{bmatrix},$$

$$\Delta_1=\begin{bmatrix}0.0394&0.0411&0.0521&0.0284\\0.0352&0.0413&0.0306&0.0759\\0.0361&0.0661&0.0656&0.0708\\0.0248&0.0643&0.0431&0.0445\end{bmatrix},\ \Delta_2=\begin{bmatrix}0.0677&0.0512&0.0246&0.0338\\0.0638&0.0251&0.0186&0.1004\\0.0226&0.0918&0.0248&0.0468\\0.0328&0.0212&0.0474&0.0201\end{bmatrix},$$

$$\Delta_3=\begin{bmatrix}0.0883&0.0252&0.0588&0.0290\\0.0956&0.0399&0.0694&0.0311\\0.0428&0.0581&0.0216&0.0414\\0.0108&0.0256&0.0115&0.0496\end{bmatrix},\ \Delta_4=\begin{bmatrix}0.0075&0.0820&0.0209&0.0460\\0.0232&0.0645&0.0405&0.0204\\0.0707&0.0431&0.0850&0.0431\\0.0026&0.0511&0.0483&0.0551\end{bmatrix}.$$

C.5 Modified AC8 system

$$B_\Delta^T=\begin{bmatrix}0.0964&0.3048&0.2852&0.3392&0.3409&0.1208&0.1274&0.2153&0.1339\\0.3424&0.3336&0.2293&0.1083&0.2802&0.0962&0.1879&0.1583&0.2217\end{bmatrix}$$

$$C_\Delta=\begin{bmatrix}0.4599&0.0189&0.0804&0.3147&0.1684&0.2111&0.3167&0.0977&0.1679\\0.2118&0.3876&0.0747&0.1322&0.2087&0.2006&0.0867&0.0492&0.3997\end{bmatrix}$$

$$\Delta_1=\begin{bmatrix}0.1237&0.1223\\0.0942&0.0293\end{bmatrix},\ \Delta_2=\begin{bmatrix}0.0958&0.1372\\0.0093&0.1091\end{bmatrix},\ \Delta_3=\begin{bmatrix}0.1035&0.1122\\0.1193&0.0495\end{bmatrix},\ \Delta_4=\begin{bmatrix}0.0001&0.0671\\0.1746&0.0708\end{bmatrix}$$

B.6 Modified AC9 system

$$B_\Delta^T=\begin{bmatrix}0.2911&0.3110&0.1462&0.2964&0.3551&0.0966&0.0516&0.0896&0.1393&0.1143\\0.3691&0.0204&0.2359&0.0648&0.3337&0.0667&0.1999&0.3977&0.1414&0.0187\end{bmatrix}$$

$$C_\Delta=\begin{bmatrix}0.1202&0.2758&0.2005&0.3071&0.3533&0.1315&0.1279&0.2939&0.0463&0.0093\\0.0354&0.2762&0.1248&0.2039&0.3294&0.2016&0.2298&0.2752&0.3034&0.1529\end{bmatrix}$$

$$\Delta_1=\begin{bmatrix}0.0704&0.1100\\0.1312&0.0757\end{bmatrix},\ \Delta_2=\begin{bmatrix}0.1194&0.1227\\0.0871&0.0558\end{bmatrix},$$
$$\Delta_3=\begin{bmatrix}0.1365&0.0886\\0.0008&0.1163\end{bmatrix},\ \Delta_4=\begin{bmatrix}0.0315&0.1032\\0.1260&0.1117\end{bmatrix}$$

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C.7 Modified AC15 system

$$B_{\Delta}^{T} = \begin{bmatrix} 0.0637 & 0.3367 & 0.0498 & 0.3052 \\ 0.2307 & 0.3636 & 0.3338 & 0.4218 \\ 0.4159 & 0.1560 & 0.3262 & 0.0923 \end{bmatrix}, C_{\Delta} = \begin{bmatrix} 0.3155 & 0.4589 & 0.4242 & 0.1216 \\ 0.1837 & 0.0175 & 0.3698 & 0.1558 \\ 0.1707 & 0.4112 & 0.0459 & 0.3157 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} 0.0031 & 0.0480 & 0.0618 \\ 0.0745 & 0.0905 & 0.0860 \\ 0.0500 & 0.0610 & 0.0806 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0672 & 0.1033 & 0.0196 \\ 0.0213 & 0.0033 & 0.1141 \\ 0.0280 & 0.0571 & 0.0831 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} 0.0726 & 0.0990 & 0.0757 \\ 0.0684 & 0.0062 & 0.0140 \\ 0.0087 & 0.0104 & 0.1187 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0804 & 0.0649 & 0.0639 \\ 0.0711 & 0.0510 & 0.0788 \\ 0.0147 & 0.0957 & 0.0447 \end{bmatrix}$$

C.8 Modified AC17 system

$$B_{\Delta} = \begin{bmatrix} 0.1462 & 0.2304 & 0.3153 & 0.0251 \\ 0.3333 & 0.1848 & 0.2733 & 0.1295 \\ 0.1619 & 0.3464 & 0.2865 & 0.0017 \\ 0.0925 & 0.3603 & 0.3468 & 0.3192 \end{bmatrix}, C_{\Delta} = \begin{bmatrix} 0.0576 & 0.1217 & 0.3314 & 0.4104 \\ 0.0607 & 0.3968 & 0.1556 & 0.3572 \\ 0.3994 & 0.0267 & 0.1841 & 0.0318 \\ 0.2767 & 0.2101 & 0.2418 & 0.1279 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} 0.0563 & 0.0321 & 0.0455 & 0.0393 \\ 0.0406 & 0.0076 & 0.0137 & 0.0133 \\ 0.0252 & 0.0094 & 0.0492 & 0.0632 \\ 0.0594 & 0.0076 & 0.0998 & 0.0971 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0319 & 0.0677 & 0.0523 & 0.0801 \\ 0.0038 & 0.0418 & 0.0575 & 0.0428 \\ 0.0130 & 0.0611 & 0.0803 & 0.0129 \\ 0.0070 & 0.0640 & 0.0606 & 0.0062 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} 0.0572 & 0.0426 & 0.0399 & 0.0193 \\ 0.0247 & 0.0573 & 0.0750 & 0.0549 \\ 0.0538 & 0.0405 & 0.0312 & 0.0673 \\ 0.0523 & 0.0391 & 0.0094 & 0.0777 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0192 & 0.0456 & 0.0270 & 0.0693 \\ 0.0679 & 0.0443 & 0.0080 & 0.0160 \\ 0.0522 & 0.0582 & 0.0738 & 0.0098 \\ 0.0360 & 0.0665 & 0.0348 & 0.0809 \end{bmatrix}$$

C.9 Modified HE4 system

$$B_{\Delta}^{T} = \begin{bmatrix} 0.1677 & 0.0014 & 0.0621 & 0.1371 & 0.1744 & 0.1203 & 0.2021 & 0.1830 \\ 0.2395 & 0.0546 & 0.1572 & 0.1196 & 0.3053 & 0.2925 & 0.2790 & 0.1386 \\ 0.2348 & 0.2842 & 0.1166 & 0.0363 & 0.1641 & 0.1066 & 0.2654 & 0.0582 \\ 0.0652 & 0.1657 & 0.0263 & 0.0752 & 0.1983 & 0.0347 & 0.1898 & 0.2780 \end{bmatrix}$$

$$C_{\Delta} = \begin{bmatrix} 0.1227 & 0.2765 & 0.1709 & 0.1227 & 0.0710 & 0.0542 & 0.0562 & 0.1699 \\ 0.2542 & 0.1658 & 0.2155 & 0.1823 & 0.1005 & 0.0276 & 0.1711 & 0.2431 \\ 0.1595 & 0.1418 & 0.1693 & 0.0498 & 0.0855 & 0.1956 & 0.2001 & 0.1906 \\ 0.2060 & 0.1499 & 0.2037 & 0.2471 & 0.2728 & 0.0271 & 0.2510 & 0.2213 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} 0.0433 & 0.0570 & 0.0391 & 0.0547 \\ 0.0277 & 0.0428 & 0.0431 & 0.0079 \\ 0.0673 & 0.0387 & 0.0760 & 0.0099 \\ 0.0842 & 0.0164 & 0.0632 & 0.0508 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0472 & 0.0643 & 0.0811 & 0.0197 \\ 0.0557 & 0.0492 & 0.0298 & 0.0826 \\ 0.0039 & 0.0164 & 0.0519 & 0.0068 \\ 0.0592 & 0.0771 & 0.0243 & 0.0200 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} 0.0838 & 0.0086 & 0.0144 & 0.0314 \\ 0.0495 & 0.0756 & 0.0556 & 0.0346 \\ 0.0502 & 0.0767 & 0.0155 & 0.0077 \\ 0.0817 & 0.0184 & 0.0100 & 0.0588 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0444 & 0.0540 & 0.0086 & 0.0591 \\ 0.0801 & 0.0604 & 0.0718 & 0.0232 \\ 0.0173 & 0.0307 & 0.0362 & 0.0482 \\ 0.0746 & 0.0386 & 0.0543 & 0.0301 \end{bmatrix}$$

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