6^{th} **World Congress on Structural and Multidisciplinary Optimization**
Rio de Janeiro, 30 May - 03 June 2005, Brazil

Nonsmooth H_∞ Synthesis

Pierre Apkarian[‡†] **Dominikus Noll**[†] **Paulo C. Pellanda**[¶]

[‡]ONERA, Control Dept.
[†] Paul Sabatier University, Maths. Dept.
2, av. Ed. Belin, 31055 - Toulouse , FRANCE
apkarian@cert.fr, noll@mip.ups-tlse.fr, http://www.cert.fr/dcsd/cdin/apkarian/

[¶]IME, Electrical Engineering Dept.
Praca Gen. Tibúrcio, 80, Rio de Janeiro, BRAZIL
pellanda@ime.eb.br

1. Abstract
We develop nonsmooth optimization techniques to solve H_∞ synthesis problems under additional structural constraints on the controller. Our approach avoids the use of Lyapunov variables and therefore leads to moderate size optimization programs even for very large systems. The proposed framework is very versatile and can accommodate a number of challenging design problems including static, fixed-order, fixed-structure, decentralized control, design of PID controllers and simultaneous design and stabilization problems. Our algorithmic strategy uses generalized gradients and bundling techniques suited for the H_∞-norm and other nonsmooth performance criteria. Convergence to a critical point from an arbitrary starting point is proved (full version) and numerical tests are included to validate our methods.
2. Keywords: H_∞-synthesis, nonsmooth optimization, Clarke subdifferential, BMI.

3. Introduction
In this paper we consider H_∞-synthesis problems with additional structural constraints on the controller. This includes static and reduced-order H_∞-output feedback control, structured, sparse or decentralized synthesis, simultaneous stabilization problems, multiple performance channels, and much else. We propose to solve these problems with a nonsmooth optimization method exploiting the structure of the H_∞-norm.

In nominal H_∞-synthesis, feedback controllers are computed via semidefinite programming (SDP) [13, 1] or algebraic Riccati equations [10]. When structural constraints on the controller are added, the H_∞-synthesis problem is no longer convex. Some of the problems above have even been recognized as NP-hard [19] or as rationally undecidable [5]. These mathematical concepts indicate at least the inherent difficulty of H_∞-synthesis under constraints on the controller.

Even with structural constraints, the bounded real lemma may still be brought into play. The difference with customary H_∞ synthesis is that it no longer produces LMIs, but bilinear matrix inequalities, BMIs, which are genuinely non-convex. Optimization code for BMI problems is currently developed by several groups, see e.g. [16, 3, 24, 18, 11], but it appears that the BMI approach runs into numerical difficulties even for problems of moderate size. This is mainly due to the presence of Lyapunov variables, whose number grows quadratically with the number of states.

Out present approach does *not* use the bounded real lemma and thereby avoids Lyapunov variables. This leads to moderate size optimization programs even for very large systems. In exchange, the cost functions are nonsmooth and require special optimization techniques. We evaluate the H_∞-norm via the Hamiltonian bisection algorithm [7, 6, 12] and exploit it further to compute subgradients, which are then used to compute descent steps.

This present paper is a contraction of a full version where additional algorithmic details, a convergence proof and further examples can be found. The reader is also referred to [20] and [21] for a comprehensive discussion on convergence and further technical details. In the sequel, we shall use notions from nonsmooth analysis covered by [9].

4. H_∞ synthesis
The general setting of the H_∞ synthesis problem is as follows. We consider a linear time-invariant plant

described in standard form by the state-space equations:

$$P(s): \qquad \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{m_2}$ the vector of control inputs, $w \in \mathbb{R}^{m_1}$ a vector of exogenous inputs, $y \in \mathbb{R}^{p_2}$ the vector of measurements and $z \in \mathbb{R}^{p_1}$ the controlled or performance vector. Without loss of generality, it is assumed throughout that $D_{22} = 0$.

Let $u = K(s)y$ be a dynamic output feedback control law for the open loop plant (1), and let $T_{w\to z}(K)$ denote the closed-loop transfer function of the performance channel mapping w into z. Our aim is to compute $K(s)$ such that the following design requirements are met:

- *Internal stability:* For $w = 0$ the state vector of the closed-loop system (1) and (2) tends to zero as time goes to infinity.
- *Performance:* The H_∞ norm $\|T_{w\to z}(K)\|_\infty$ is minimized.

We assume that the controller K has the following frequency domain representation:

$$K(s) = C_K(sI - A_K)^{-1}B_K + D_K, \qquad A_K \in \mathbb{R}^{k\times k}, \tag{2}$$

where k is the order of the controller, and where the case $k = 0$ of a static controller $K(s) = D_K$ is included. Often practical considerations dictate additional challenging structural constraints. For instance it may be desired to design low-order controllers ($0 \leq k \ll n$) or controllers with prescribed-pattern, sparse controllers, decentralized controllers, observed-based controllers, PID control structures, synthesis on a finite set of transfer functions, and much else. Formally, the synthesis problem may then be represented as

$$\begin{array}{ll} \text{minimize} & \|T_{w\to z}(K)\|_\infty \\ \text{subject to} & K \text{ stabilizes (1)} \\ & K \in \mathcal{K} \end{array} \tag{3}$$

where $K \in \mathcal{K}$ represents a structural constraint on the controller (2) like one of the above.

Without the restriction $K \in \mathcal{K}$, and under standard stabilizability and detectability conditions, it is customary to synthesize $K(s)$ using Riccati equations or LMI techniques [14]. This scenario changes dramatically as soon as constraints $K \in \mathcal{K}$ are added. Then the problem may no longer be transformed into an LMI or any other convex program, and alternative algorithmic strategies are required.

Also, it is important to pay attention to the fact that even genuine stabilization problems can be cast as H_∞ synthesis problems. Indeed, under standard assumptions, a system is stable if and only if a well chosen closed-loop transfer function has finite H_∞ norm (see full paper). Therefore, the proposed techniques also cover stabilization problems as a special case.

5. H_∞-norm subdifferentials

In this section, we start characterizing the subdifferential of the H_∞-norm, and derive expressions for the Clarke subdifferential of several nonconvex composite functions $f(x) = \|\mathcal{G}(x)\|_\infty$, where $\mathcal{G}$ is a smooth operator defined on some $\mathbb{R}^n$ with values in the space of stable matrix transfer functions $\mathbf{H}_\infty$.

Consider the H_∞-norm of a nonzero transfer matrix function $G(s)$:

$$\|G\|_\infty = \sup_{\omega\in\mathbb{R}} \overline{\sigma}\left(G(j\omega)\right),$$

where G is stable and $\overline{\sigma}(X)$ is the maximum singular value of X. Suppose $\|G\|_\infty = \overline{\sigma}\left(G(j\omega)\right)$ is attained at some frequency ω, where the case $\omega = \infty$ is allowed. Let $G(j\omega) = U\Sigma V^H$ be a singular value decomposition. Pick u the first column of U, v the first column of V, that is, $u = G(j\omega)v/\|G\|_\infty$. Then the linear functional $\phi = \phi_{u,v,\omega}$ defined as

$$\phi(H) = \|G\|_\infty^{-1} \mathrm{Re}\,\mathrm{Tr}\, G(j\omega)^H uu^H H(j\omega)$$

is continuous on the space $\mathbf{H}_\infty$ of stable transfer functions and is a subgradient of $\|\cdot\|_\infty$ at G [8]. More generally, assume that the columns of Q_u form an orthonormal basis of the eigenspace of $G(j\omega)G(j\omega)^H$ associated with the largest eigenvalue $\lambda_1\left(G(j\omega)G(j\omega)^H\right) = \overline{\sigma}(G(j\omega))^2$. Then for all complex Hermitian matrices $Y_v \succeq 0$, $Y_u \succeq 0$ with $\operatorname{Tr}(Y_v) = 1$ and $\operatorname{Tr}(Y_u) = 1$,

$$\phi(H) = \|G\|_\infty^{-1} \operatorname{Re} \operatorname{Tr} G(j\omega)^H Q_u Y_u Q_u^H H(j\omega) \tag{4}$$

is a subgradient of $\|\cdot\|_\infty$ at G. Finally, with $G(s)$ rational and assuming that there exist finitely many frequencies $\omega_1, \dots, \omega_p$ where the supremum $\|G\|_\infty = \overline{\sigma}(G(j\omega_\nu))$ is attained, all subgradients of $\|\cdot\|_\infty$ at G are precisely of the form

$$\phi(H) = \|G\|_\infty^{-1} \operatorname{Re} \sum_{\nu=1}^{p} \operatorname{Tr} G(j\omega_\nu)^H Q_\nu Y_\nu Q_\nu^H H(j\omega_\nu),$$

where the columns of Q_ν form an orthonormal basis of the eigenspace of $G(j\omega_\nu)G(j\omega_\nu)^H$ associated with the leading eigenvalue $\|G\|_\infty^2$, and where $Y_\nu \succeq 0$, $\sum_{\nu=1}^{p} \operatorname{Tr}(Y_\nu) = 1$. See [9, Prop. 2.3.12 and Thm. 2.8.2] and [2] for this.

Suppose now we have a smooth operator $\mathcal{G}$, mapping $\mathbb{R}^n$ onto the space $\mathbf{H}_\infty$ of stable transfer functions G. Then the composite function $f(x) = \|\mathcal{G}(x)\|_\infty$ is Clarke subdifferentiable at x with

$$\partial f(x) \;=\; \mathcal{G}'(x)^\star[\partial\|\cdot\|_\infty(\mathcal{G}(x))], \tag{5}$$

where $\partial\|\cdot\|_\infty$ is the subdifferential of the H_∞-norm obtained above, and where $\mathcal{G}'(x)^\star$ is the adjoint of $\mathcal{G}'(x)$, mapping the dual of $\mathbf{H}_\infty$ into $\mathbb{R}^n$. In the sequel, we will compute this adjoint $\mathcal{G}'(x)^\star$ for special classes of closed-loop transfer functions. Suitable chain rules covering this case are for instance given in [9, section 2.3].

6. Clarke subdifferentials in closed-loop

Given a stabilizing controller $K(s)$ and a plant with the usual partition

$$P(s) := \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix},$$

the closed-loop transfer function is obtained as

$$T_{w\to z}(K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21},$$

where the state-space data of P_{11}, P_{12}, P_{21} and P_{22} are given in (1) and the dependence on s is omitted for brevity. Our aim is to compute the subdifferential $\partial f(K)$ of $f := \|\cdot\|_\infty \circ T_{w\to z}$ at K. We first notice that the derivative $T'_{w\to z}(K)$ of $T_{w\to z}$ at K is

$$T'_{w\to z}(K)\delta K := P_{12}(I - KP_{22})^{-1}\delta K(I - P_{22}K)^{-1}P_{21},$$

where δK is an element of the same matrix space as K.

Now let $\phi = \phi_Y$ be a subgradient of $\|\cdot\|_\infty$ at $T_{w\to z}(K)$ of the form (4), specified by $Y \succeq 0$, $\operatorname{Tr}(Y) = 1$ and with $\|T_{w\to z}(K)\|_\infty$ attained at frequency ω. According to the chain rule, the subgradients Φ_Y of f at K are of the form $\Phi_Y := T'_{w\to z}(K)^\star\phi_Y \in \mathbb{M}_{m_2,p_2}$, where the adjoint $T'_{w\to z}(K)^\star$ acts on ϕ_Y through $\langle T'_{w\to z}(K)^\star\phi_Y, \delta K\rangle = \langle T'_{w\to z}(K)\delta K, \phi_Y\rangle =$

$$\begin{aligned} &\|T_{w\to z}(K)\|_\infty^{-1} \operatorname{Re}\operatorname{Tr}\big((I - P_{22}(j\omega)K(j\omega))^{-1}P_{21}(j\omega) \\ &T_{w\to z}(K, j\omega)^H QYQ^H P_{12}(j\omega) \\ &(I - K(j\omega)P_{22}(j\omega))^{-1}\delta K(j\omega)\big)\,. \end{aligned} \tag{6}$$

In consequence, for a static K, the Clarke subdifferential of $f(K) := \|T_{w\to z}(K)\|_\infty$ at K consists of all subgradients Φ_Y of the form

$$\begin{aligned} &\|T_{w\to z}(K)\|_\infty^{-1} \operatorname{Re}\big((I - P_{22}(j\omega)K)^{-1}P_{21}(j\omega) \\ &T_{w\to z}(K, j\omega)^H\, QYQ^H\, P_{12}(j\omega)(I - KP_{22}(j\omega))^{-1}\,\big)^T\,, \end{aligned} \tag{7}$$

where $Y \succeq 0$ and $\text{Tr}\,(Y) = 1$. Recall that Φ_Y is now an element of the same matrix space as K and acts on test vectors δK through $\langle \Phi_Y, \delta K \rangle = \text{Tr}(\Phi_Y^T \delta K)$.

This formula is easily adapted if the H_∞-norm is attained at a finite number of frequencies $\omega_1, \ldots, \omega_q$. In this more general situation, subgradients Φ_Y of f at K are of the form

$$\begin{array}{l} \|T_{w\to z}(K)\|_\infty^{-1} \sum_{\nu=1}^{q} \text{Re}\, \big((I - P_{22}(j\omega_\nu)K)^{-1} P_{21}(j\omega_\nu) \\ T_{w\to z}(K, j\omega_\nu)^H Q Y_\nu Q^H \, P_{12}(j\omega_\nu) \\ (I - K P_{22}(j\omega_\nu))^{-1} \,\big)^T \;, \end{array} \tag{8}$$

where $Y \in \mathcal{P}$ with

$$\mathcal{P} := \left\{ (Y_1, \ldots, Y_q), Y_\nu \succeq 0, \sum_{\nu=1}^{q} \text{Tr}(Y_\nu) = 1 \right\}.$$

At this stage, it is important to stress that expressions (6), (7) and (8) are general and can accommodate any problem such as static, dynamic, PID, matrix fraction controllers and also multiple performance channels.

7. Steepest descent method

Nonsmooth techniques have been used before in algorithms for controller synthesis. For instance, E. Polak and co-workers have proposed a variety of techniques suited for eigenvalue or singular-value optimization and for extensions to the semi-infinite case, covering in particular the H_∞-norm (see [22], [23] and the citations given there). Another reference is [8], where the authors exploit the Youla parameterization via convex nondifferentiable analysis to derive the cutting plane and ellipsoid algorithms.

Let us consider the problem of minimizing $f(x) = \|\mathcal{G}(x)\|_\infty$, where x regroups the controller data, referred to as K in the previous section, and where $\mathcal{G}$ maps $\mathbb{R}^n$ smoothly into a space $\mathbf{H}_\infty$ of stable transfer functions. We write $\mathcal{G}(x, s)$ or $\mathcal{G}(x, j\omega)$ when the complex argument of $\mathcal{G}(x) \in \mathbf{H}_\infty$ needs to be specified.

A necessary condition for optimality is $0 \in \partial f(x) = \mathcal{G}'(x)^\star \partial \| \cdot \|_\infty (\mathcal{G}(x))$. It is therefore reasonable to consider the program

$$d = -\frac{g}{\|g\|}, \qquad g = \text{argmin}\{\|\phi_Y\| : Y \in \mathcal{P}\} \tag{9}$$

which either shows $0 \in \partial f(x)$, or produces the direction d of steepest descent at x if $0 \notin \partial f(x)$, and where the ϕ_Y are as in (8). If we vectorize $y = \text{vec}(Y)$, $Y = (Y_1, \ldots, Y_q)$, then we may represent ϕ_Y by a matrix vector product, $\phi_Y = \Phi y$, with a suitable matrix Φ. Program (9) is then equivalent to the following SDP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} t & y^T \Phi^T \\ \Phi y & tI \end{bmatrix} \succeq 0 \\ & Y_i \succeq 0, i = 1, \ldots, q \\ & e^T y = 1 \end{array} \tag{10}$$

where $e^T y = 1$ encodes the constraint $\sum_i \text{Tr}(Y_i) = 1$. The direction d of steepest descent at x is then obtained as $d = -\Phi\, y / \|\Phi\, y\|$, where (t, y) is solution of (10) with $y \neq 0$. This suggests the following algorithm:

1. If $0 \in \partial f(x)$ stop. Otherwise:
2. Solve (10) and compute the direction d of steepest descent at x.
3. Perform a line search and find a descent step $x^+ = x + t\, d$.
4. Replace x by x^+ and go back to step 1.

The drawback of this approach is that it may fail to converge due to the nonsmoothness of f. We believe that a descent method should at least give the weak convergence certificate that accumulation points of the sequence of iterates are critical. This is not guaranteed by the above scheme. The reason is that the steepest descent direction at x does not depend continuously on x. In the full version of this paper, we discuss two variants of the basic descent algorithm and we establish convergence to a local minimum. This is omitted due to space limitation.

8. Numerical experiments

In this section we test our nonsmooth algorithms on a variety of synthesis problems from the $COMPl_eib$ collection by F. Leibfritz [17]. Computations were performed on a (low-level) SUN-Blade Sparc with 256 RAM and a 650 MHz sparcv9 processor. LMI-related computations for search directions used the LMI Control Toolbox [15] or our home made SDP code [3].

Our algorithm is a first-order method. Not surprisingly, it may be slow in the neighborhood of a local solution. We have implemented various stopping criteria to ensure that an adequate approximation of a solution has been found and to avoid unwarranted computational efforts as is often the case with a first-order algorithm. The first of these termination criteria is an absolute stopping test, which provides a criticality assessment

$$\inf\{\|g\| : g \in \partial f(x)\} < \varepsilon_1, \tag{11}$$

which is readily performed using (10).

This is reasonable, as $0 \in \partial f(x)$ indicates a critical point. It is also mandatory to use relative stopping criteria to reduce the dependence on the problem scaling. The test

$$\|T_{w\to z}(K)\|_\infty - \|T_{w\to z}(K^+)\|_\infty < \varepsilon_2(1 + \|T_{w\to z}(K)\|_\infty)\,, \tag{12}$$

compares the progress achieved relatively to the current H_∞ performance, while

$$\|K^+ - K\| < \varepsilon_3(1 + \|K\|) \tag{13}$$

compares the step-length to the controller gains. The tolerances

$$\varepsilon_1 = 1e{-}5,\ \varepsilon_2 = 1e{-}3,\ \varepsilon_3 = 1e{-}3$$

have been used in our numerical testing. For stopping we required that either the first two tests or the third one are satisfied.

The synthesis procedure is based on the scheme (3) and must be initialized with a stabilizing controller. This initial phase I is described in the full paper and in [2].

We compare the results of our nonsmooth algorithm variant II in columns 'nonsmooth H_∞' to older results obtained with the specialized augmented Lagrangian (AL) algorithm described in [4], displayed in columns 'H_∞ AL' (see Table). In column 'H_∞ full' we also display the gain obtained with a full-order feedback controller, synthesized by LMI-methods or via the algebraic Riccati equation solver. This is a lower bound for the gain in column 'nonsmooth H_∞'. The results obtained with our present technique are close to those obtained in [4], except for problems with large state dimension as 'AC10' (55 states), 'BDT2' (82 states) and 'HF1' (130 states) where the augmented Lagrangian method fails, while the present nonsmooth method is still functional. In the same vein, we have observed that even customary Riccati or LMI solvers encounter serious difficulties or even break down when solving the full-order (hence convex) problem for 'AC10', 'BDT2' and 'HF1'.

9. Conclusion

We have proposed several new algorithms to minimize the H_∞-norm subject to structural constraints on the controller dynamics. The proposed method uses nonsmooth techniques suited for H_∞ synthesis and for semi-infinite eigenvalue or singular value optimization programs. Variant I and variant II of our algorithm are supported by global convergence theory, a crucial parameter for the reliability of an algorithm in practice. Variant II has been shown to perform satisfactorily on a number of difficult examples. In particular, three examples with large state dimension ($n = 55$ $n = 82$ and $n = 130$)

Table 1: H_∞ synthesis with nonsmooth algorithm algorithmic variant II - $\varepsilon_\omega = 0.05$.

problem	(n, m, p)	order	iter	cpu (sec.)	nonsmooth H_∞	H_∞ AL	H_∞ full
AC8	$(9, 1, 5)$	0	20	45	2.005	2.02	1.62
HE1	$(4, 2, 1)$	0	4	7	0.154	0.157	0.073
REA2	$(4, 2, 2)$	0	31	51	1.192	1.155	1.141
AC10	$(55, 2, 2)$	0	15	294	13.11	intractable	3.23
AC10	$(55, 2, 2)$	1	46	408	10.21	intractable	3.23
BDT2	$(82, 4, 4)$	0	44	1501	0.8364	intractable	0.2340
HF1	$(130, 1, 2)$	0	11	1112	0.447	intractable	0.447

have been solved. More importantly, our present techniques and tools pave the way for investigating an even larger scope of synthesis problems, characterized through frequency domain inequalities of the form $\lambda_1(H(x,\omega)) \leq 0$, $\omega \geq 0$, where $H(x,\omega)$ is Hermitian-valued and x stands for controller parameters and possibly multiplier variables, as is the case when IQC formulations are used. This is a strong incentive for further developments.

10. References

[**1**] P. Apkarian and P. Gahinet, *A Convex Characterization of Gain-Scheduled H_∞ Controllers*, IEEE Trans. Aut. Control, 40 (1995), pp. 853–864. See also pp. 1681.

[**2**] P. Apkarian, D. Noll, and D. Alazard, *Controller Design via Nonsmooth Multi-Directional Search*, in IFAC Conf. on System Structure and Control, Oaxaca, Mexico, Dec. 2004.

[**3**] P. Apkarian, D. Noll, J. B. Thevenet, and H. D. Tuan, *A Spectral Quadratic-SDP Method with Applications to Fixed-Order $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Synthesis*, in Asian Control Conference, Melbourne, AU, 2004.

[**4**] P. Apkarian, D. Noll, and H. D. Tuan, *Fixed-order $\mathcal{H}_\infty$ control design via an augmented Lagrangian method* , Int. J. Robust and Nonlinear Control, 13 (2003), pp. 1137–1148.

[**5**] V. Blondel and M. Gevers, *Simultaneous stabilizability question of three linear systems is rationally undecidable*, Mathematics of Control, Signals, and Systems, 6 (1994), pp. 135–145.

[**6**] S. Boyd and V. Balakrishnan, *A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its $\mathbf{L}_\infty$-norm*, Syst. Control Letters, 15 (1990), pp. 1–7.

[**7**] S. Boyd, V. Balakrishnan, and P. Kabamba, *A bisection method for computing the $\mathbf{H}_\infty$ norm of a transfer matrix and related problems*, Mathematics of Control, Signals, and Systems, 2 (1989), pp. 207–219.

[**8**] S. Boyd and C. Barratt, *Linear Controller Design: Limits of Performance*, Prentice-Hall, 1991.

[**9**] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Math. Soc. Series, John Wiley & Sons, New York, 1983.

[**10**] J. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, *State-space solutions to standard $\mathbf{H}_2$ and $\mathbf{H}_\infty$ control problems*, in Proc. American Control Conf., 1988, pp. 1691–1696.

[**11**] F. Leibfritz and E. M. E. Mostafa, *Trust region methods for solving the optimal output feedback design problem*, International Journal of Control, 76 (2000), pp. 501 – 519.

[**12**] P. Gahinet and P. Apkarian, *Numerical Computation of the L_∞ Norm Revisited*, in Proc. IEEE Conf. on Decision and Control, 1992.

[13] ——, *A LMI-based parametrization of all* $\mathbf{H}_\infty$ *controllers with applications*, in Proc. IEEE Conf. on Decision and Control, San Antonio, Texas, Dec. 1993, pp. 656–661.

[14] ——, *A Linear Matrix Inequality Approach to* H_∞ *Control*, Int. J. Robust and Nonlinear Control, 4 (1994), pp. 421–448.

[15] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox* , The MathWorks Inc., 1995.

[16] D. Henrion, M.Kocvara, and M. Stingl, *Solving simultaneous stabilization BMI problems with PENNON*, in IFIP Conference on System Modeling and Optimization, vol. 7, Sophia Antipolis, France, July 2003.

[17] F. Leibfritz, *COMPL$_e$IB, COnstraint Matrix-optimization Problem LIbrary - a collection of test examples for nonlinear semidefinite programs, control system design and related problems*, tech. report, Universität Trier, 2003.

[18] F. Leibfritz and E. M. E. Mostafa, *An interior point constrained trust region method for a special class of nonlinear semi-definite programming problems*, SIAM J. on Optimization, 12 (2002), pp. 1048–1074.

[19] A. Nemirovskii, *Several NP-Hard problems arising in robust stability analysis*, Mathematics of Control, Signals, and Systems, 6 (1994), pp. 99–105.

[20] D. Noll and P. Apkarian, *Spectral Bundle Methods for Non-Convex Maximum Eigenvalue Functions. Part 1: First-Order Methods*, to appear in Maths. Programming Series B, (2005).

[21] ——, *Spectral Bundle Methods for Non-Convex Maximum Eigenvalue Functions. Part 2: Second-Order Methods*, to appear in Maths. Programming Series B, (2005).

[22] E. Polak, *On the mathematical foundations of nondifferentiable optimization in engineering design*, SIAM Rev., 29 (1987), pp. 21–89.

[23] ——, *Optimization : Algorithms and Consistent Approximations*, Applied Mathematical Sciences, 1997.

[24] J. Thevenet, D. Noll, and P. Apkarian, *Nonlinear spectral SDP method for BMI-constrained problems: Applications to control design*, in ICINCO proceedings, Portugal, 2004.