Robust Pole Placement in LMI Regions

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Abstract

This paper discusses analysis and synthesis techniques for robust pole placement in LMI regions, a class of convex regions of the complex plane that embraces most practically useful stability regions. The focus is on linear systems with static uncertainty on the state matrix. For this class of uncertain systems, the notion of quadratic stability and the related robustness analysis tests are generalized to arbitrary LMI regions. The resulting tests for robust pole clustering are all numerically tractable since they involve solving linear matrix inequalities (LMIs), and cover both unstructured and parameter uncertainty.

These analysis results are then applied to the synthesis of dynamic output-feedback controllers that robustly assign the closed-loop poles in a prescribed LMI region. With some conservatism, this problem is again tractable via LMI optimization. In addition, robust pole placement can be combined with other control objectives such as H_2 or H_∞ performance to capture realistic sets of design specifications. Physically-motivated examples demonstrate the effectiveness of this robust pole clustering technique.

1 Introduction

Stability is a minimum requirement for control systems. However, in most practical situations, a good controller should also deliver sufficiently fast and well-damped time responses. A customary way to guarantee satisfactory transients is to place the closed-loop poles in a suitable region of the complex plane. We refer to this technique as *regional* pole placement, by contrast with pointwise pole placement where the poles are assigned to specific locations in the complex plane. For example, fast decay, good damping, and reasonable controller dynamics can be imposed by confining the poles in the intersection of a shifted half-plane, a sector, and a disk [18, 1, 4, 5]. Regional pole assignment has also been considered in conjunction with other design objectives such as H_∞ or H_2 performance [20, 8, 28, 9, 32].

Because real systems always involve some amount of uncertainty, it is natural to worry about the robustness of pole clustering, i.e., whether the poles remain in the prescribed region when the nominal model is perturbed. Such robustness issues have been thoroughly studied in the context of pointwise pole placement [23, 22, 25]. In comparison, few results are available on robust regional pole clustering. These include a Lyapunov approach to compute explicit robustness bounds for pole clustering in a disk [10], and extensions of the notion of quadratic stability to robust pole placement in a disk or a sector [3, 16, 15].

The present paper extends these results to more general clustering regions and to structured uncertainty. The regions considered here are the LMI regions introduced in [9]. This class of

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regions covers a large variety of useful clustering regions including half-planes, disks, sectors, vertical/horizontal strips, and any intersection thereof. The following analysis and synthesis problems are addressed:

- Robustness of pole clustering within a given LMI region in the face of unstructured or parameter uncertainty in the state matrix
- Synthesis of output-feedback controllers that robustly assign the closed-loop poles in some arbitrary LMI region (assuming static and unstructured uncertainty on the plant matrices).

With some conservatism, these problems are reduced to solving linear matrix inequalities (LMIs). Since LMIs can be solved numerically using efficient optimization algorithms such as those described in [29, 30, 6, 35] or implemented in [14, 2], our approach yields practical analysis and synthesis tools for robust regional pole placement. See [7] for an overview of the applications of LMI techniques in control theory.

The paper is organized as follows. Section 2 recalls the definition of LMI regions and key results on pole clustering in LMI regions. Section 3 contains the main result, a generalization of the Bounded Real Lemma to arbitrary LMI regions. This result gives a sufficient condition in terms of LMIs for robust pole clustering within a given LMI region. Section 4 shows how some standard robustness analysis tests for parameter uncertainty can be generalized to LMI regions, and illustrates the performance of the resulting robust pole clustering tests on a realistic example. Section 5 applies the results in Section 3 to the synthesis of output-feedback controllers that robustly assign the closed-loop poles in a given LMI region. This section also shows how to combine robust pole clustering with other synthesis objectives using the multi-objective design framework developed in [26, 33, 32]. Finally, Section 6 demonstrates the effectiveness of this approach on a physically motivated design example.

2 Background

This section recalls the basics on LMI regions and some useful properties of Kronecker products.

2.1 Notation

$\mathbf{R}$ and $\mathbf{C}$ denote the sets of real and complex numbers, respectively. The notation $\mathbf{C}^-$ stands for the open left-half plane.

For a complex matrix M, M^H denotes the Hermitian transpose of M and $\mathrm{Herm}(M)$ is defined as

$$\mathrm{Herm}(M) := M + M^H .$$

For Hermitian matrices, $M > N$ means that $M - N$ is positive definite and $M \geq N$ means that $M - N$ is positive semi-definite. In symmetric block matrices, we use $\star$ as an ellipsis for terms that are induced by symmetry, e.g.,

$$\begin{bmatrix} S & M \\ \star & Q \end{bmatrix} \equiv \begin{bmatrix} S & M \\ M^T & Q \end{bmatrix} .$$

Finally, we use the shorthand

$$\operatorname*{diag}_{i=1}^{N} X_i := \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_N \end{bmatrix} .$$

2.2 Kronecker products

The Kronecker product is an important tool for the subsequent analysis. Recall that the Kronecker product of two matrices A and B is a block matrix C with generic block entry $C_{ij} = A_{ij}B$, that is,

$$A \otimes B = [A_{ij}B]_{ij} \; .$$

The following properties of the Kronecker product are easily established [17]:

$$\begin{array}{lcl} 1 \otimes A & = & A \\ (A+B) \otimes C & = & A \otimes C + B \otimes C \\ (A \otimes B)(C \otimes D) & = & AC \otimes BD \\ (A \otimes B)^T & = & A^T \otimes B^T \\ (A \otimes B)^{-1} & = & A^{-1} \otimes B^{-1} \end{array}$$

The eigenvalues of $A \otimes B$ are the pairwise products $\lambda_i(A)\lambda_j(B)$ of the eigenvalues of A and B. As a result, the Kronecker product of two positive definite matrices is a positive definite matrix. Finally, the singular values of $A \otimes B$ consist of all pairwise products $\sigma_i(A)\sigma_j(B)$ of singular values of A and B.

2.3 LMI Regions

An LMI region is any subset $\mathcal{D}$ of the complex plane that can be defined as

$$\mathcal{D} = \left\{ z \in \mathbf{C} : \quad L + zM + \bar{z}M^T < 0 \right\} \tag{1}$$

where L and M are real matrices such that $L^T = L$. The matrix-valued function

$$f_{\mathcal{D}}(z) = L + zM + \bar{z}M^T$$

is called the characteristic function of $\mathcal{D}$. Below are a few examples of LMI regions:

- half-plane $\mathrm{Re}(z) < -\alpha$: $f_{\mathcal{D}}(z) = z + \bar{z} + 2\alpha < 0$
- disk centered at $(-q, 0)$ with radius r:

$$f_{\mathcal{D}}(z) = \begin{bmatrix} -r & q+z \\ q+\bar{z} & -r \end{bmatrix} < 0$$

- conic sector with apex at the origin and inner angle 2θ:

$$f_{\mathcal{D}}(z) = \begin{bmatrix} \sin\theta\,(z+\bar{z}) & \cos\theta\,(z-\bar{z}) \\ \cos\theta\,(\bar{z}-z) & \sin\theta\,(z+\bar{z}) \end{bmatrix} < 0 \; .$$

Key facts about LMI regions include [9]:

- Intersections of LMI regions are LMI regions.
- Any convex region that is symmetric with respect to the real axis can be approximated by an LMI region to any desired accuracy.

- A real matrix A is $\mathcal{D}$-stable, i.e., has all its eigenvalues in the LMI region $\mathcal{D}$, if and only if there exists a symmetric matrix X such that

$$M_{\mathcal{D}}(A,X) \;:=\; L\otimes X + M\otimes(XA) + M^T\otimes(A^TX) < 0, \qquad X>0\,. \tag{2}$$

  This result can be seen as a generalization of the Lyapunov theorem since for the usual stability region $f_{\mathcal{D}}(z) = z+\bar{z} < 0$, (2) reduces to

$$1\otimes(XA) + 1\otimes(A^TX) = A^TX + XA < 0\,, \quad X>0\,.$$

Pole clustering in LMI regions can be formulated as an LMI optimization problem, a *convex semi-definite program* that is easily tractable with recently available interior-point techniques. Moreover, it is possible to combine such pole clustering specifications with other design objectives while preserving tractability [9, 32]

3 Robustness of Pole Clustering in LMI Regions

The notions of robust and quadratic stability are useful tools to analyze the stability of uncertain state-space models [7, 24]. These notions are now generalized to pole clustering in arbitrary LMI regions, and a counterpart of the Bounded Real Lemma is derived for LMI regions. While our analysis is restricted to static (real or complex) uncertainty, its implications for more general classes of uncertainty (dynamic or time-varying) are briefly discussed at the end of the section.

3.1 Robust and quadratic $\mathcal{D}$-stability

Consider the uncertain linear system

$$\dot{x} = A(\Delta)x, \qquad A(\Delta) := A + B(I-\Delta D)^{-1}\Delta C, \quad A\in\mathbf{R}^{n\times n} \tag{3}$$

where the state matrix $A(\Delta)$ depends fractionally on the norm-bounded uncertainty matrix

$$\Delta\in\mathbf{E}^{m\times m}, \quad \sigma_{max}\,(\Delta) \le \gamma^{-1} \tag{4}$$

with $\mathbf{E}=\mathbf{R},\mathbf{C}$. The value $\Delta=0$ corresponds to the nominal state matrix A and the parameter γ defines the level of uncertainty. While the uncertain model (3) is physically meaningful only for real uncertainty Δ, we also consider the complex case because of its connection with dynamic uncertainty (see Subsection 3.4 below).

Let

$$\mathcal{D} = \left\{\, z\in\mathbf{C} : \quad f_{\mathcal{D}}(z) = L + zM + \bar{z}M^T < 0 \,\right\} \tag{5}$$

be any LMI region and suppose that the nominal state matrix A is $\mathcal{D}$-stable, i.e., has all its eigenvalues in $\mathcal{D}$. The question of interest here is:

> Given some uncertainty level γ, do the poles of $A(\Delta)$ remain in $\mathcal{D}$ for all Δ satisfying $\sigma_{max}\,(\Delta)\le\gamma^{-1}$?

Definition 3.1 (Robust $\mathcal{D}$-Stability) *The uncertain system (3)–(4) is robustly $\mathcal{D}$-stable if the eigenvalues of $A(\Delta)$ lie in $\mathcal{D}$ for all admissible uncertainties Δ.*

Similar definitions can be found in [7, 24] for the usual stability region and in [4] for uncertain polynomials. Non conservative assessment of robust $\mathcal{D}$-stability is difficult except in very special cases, e.g., complex unstructured Δ for the open left-half plane. While conservative, the following notion of quadratic $\mathcal{D}$-stability proves more practical for analysis and synthesis purposes.

Definition 3.2 (Quadratic $\mathcal{D}$-Stability) *Given any LMI region $\mathcal{D}$ defined by (5), the uncertain system (3)–(4) is said to be quadratically $\mathcal{D}$-stable if there exists a real symmetric matrix $X > 0$ such that*

$$M_{\mathcal{D}}(A(\Delta), X) := L \otimes X + \text{Herm}\left(M \otimes \left\{X(A + B(I - \Delta D)^{-1}\Delta C)\right\}\right) < 0 \qquad (6)$$

for all complex matrices Δ such that $\|\Delta\| \leq \gamma^{-1}$.

Recall from Subsection 2.3 that A is $\mathcal{D}$-stable if and only if there exists $X > 0$ such that $M_{\mathcal{D}}(A, X) < 0$. Hence quadratic $\mathcal{D}$-stability implies robust $\mathcal{D}$-stability, but the converse is generally false since quadratic $\mathcal{D}$-stability requires that a *single* X satisfy $M_{\mathcal{D}}(A(\Delta), X) < 0$ for all admissible Δ's. Note that the assumption "X real" incurs no loss of generality and is motivated by the tractability of the synthesis problem discussed in Section 5.

When $\mathcal{D}$ is the open left half-plane, it is well known that robust stability for complex Δ is equivalent to quadratic stability for real or complex Δ [24], which in turn is completely characterized by the Bounded Real Lemma:

> The uncertain system (3)–(4) is quadratically stable ($\mathcal{D} = \mathbf{C}^-$) if and only if there exists $X > 0$ such that
> $$\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0 \,. \qquad (7)$$

Using a bilinear shift [8], it can be shown that this remains true for vertical half-planes and disks centered on the real axis. Next we show that the Bounded Real Lemma condition for quadratic stability can actually be generalized to arbitrary LMI regions.

3.2 Main result

Given an LMI region $\mathcal{D}$ with characteristic function

$$f_{\mathcal{D}}(z) = L + zM + \overline{z}M^T < 0\,, \quad L, M \in \mathbf{R}^{p\times p}\,, \qquad (8)$$

factorize the matrix M as

$$M = M_1^T M_2 \qquad (9)$$

where M_1, M_2 have full column rank (such a factorization is easily obtained from the SVD of M). If M has rank k, both M_1 and M_2 are $k \times p$ matrices.

We are now ready to state the main result, a sufficient LMI-based condition for quadratic $\mathcal{D}$-stability.

Theorem 3.3 *The system (3) with uncertainty*

$$\Delta \in \mathbf{C}^{m\times m}, \qquad \sigma_{max}(\Delta) \leq \gamma^{-1}$$

is quadratically $\mathcal{D}$-stable if there exist matrices $X \in \mathbf{R}^{n\times n}$ and $P \in \mathbf{R}^{k\times k}$ such that

$$\mathcal{B}_\mathcal{D}(X,P) := \begin{bmatrix} M_\mathcal{D}(A,X) & M_1^T \otimes (XB) & (M_2^T P)\otimes C^T \\ M_1 \otimes (B^T X) & -\gamma P \otimes I & P \otimes D^T \\ (PM_2)\otimes C & P\otimes D & -\gamma P \otimes I \end{bmatrix} < 0 \tag{10}$$

$$P > 0, \quad X > 0 \tag{11}$$

with the notation

$$M_\mathcal{D}(A,X) := L \otimes X + M \otimes (XA) + M^T \otimes (A^T X) .$$

Proof: See Appendix A.

The inequalities (10)–(11) are LMIs with unknown matrices X and P. Hence testing this sufficient condition numerically can be tackled efficiently with LMI solvers. The matrix X plays the role of Lyapunov matrix while P can be viewed as a scaling matrix that accounts for the block-diagonal structure of $I_k \otimes \Delta$ in the relation $p = (I_k \otimes \Delta)q$ (see proof in Appendix A). The variable P also accounts for the non uniqueness of the factorization $M = M_1^T M_2$. Specifically, replacing M_1, M_2 by $Q^{-T}M_1, QM_2$ is equivalent to replacing P by $Q^{-T}PQ^{-1}$. Note that the size of P is typically small since for most useful LMI regions, the matrix M in (8) has rank less than three.

It is insightful to explicitate the LMI (10) for well-known regions such as the left half-plane and the disk:

- The open left-half plane corresponds to $L = 0$ and $M = 1$ (i.e., $z + \bar{z} < 0$). Taking $M_1 = M_2 = 1$, (10) reduces to

$$\begin{bmatrix} A^T X + XA & XB & pC^T \\ B^T X & -\gamma p & pD^T \\ pC & pD & -\gamma p \end{bmatrix} < 0$$

  which coincides with the Bounded Real Lemma inequality (7) up to dividing by the scalar $p > 0$ and redefining X as X/p.

- The disk $D(q,r)$ with center $(-q,0)$ and radius r corresponds to

$$L = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{M_1^T} \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{M_2}$$

  Since M has rank one, P is again scalar and we can take $P = 1$ without loss of generality. The LMI constraint (10) then reads

$$\left[\begin{array}{cc|c|c} -rX & qX + XA & XB & 0 \\ qX + A^T X & -rX & 0 & C^T \\ \hline BX^T & 0 & -\gamma I & D^T \\ \hline 0 & C & D & -\gamma I \end{array}\right] < 0 .$$

  By a Schur complement with respect to the block (1,1), this is equivalent to

$$\begin{bmatrix} \hat{A}^T X \hat{A} - X & \hat{A}^T X \hat{B} & \hat{C}^T \\ \hat{B}^T X \hat{A} & -\gamma I + \hat{B}^T X \hat{B} & D^T \\ \hat{C} C & D & -\gamma I \end{bmatrix} < 0, \quad \hat{A} := \frac{A + qI}{r}, \quad \hat{B} := r^{-\frac{1}{2}} B, \quad \hat{C} := r^{-\frac{1}{2}} C .$$

which is simply the discrete-time version of the Bounded Real Lemma applied to the system $(\hat{A}, \hat{B}, \hat{C}, D)$. This stems from the fact that $A(\Delta)$ has all its eigenvalues in $D(q, r)$ if and only if $(A(\Delta) + qI)/r$ is stable in the discrete-time sense, i.e., has all its eigenvalues in the unit disk.

3.3 Intersections of LMI regions

In practical applications, LMI regions are often specified as the intersection of elementary regions such as conic sectors, disks, or vertical half-planes. Given LMI regions $\mathcal{D}_1, \ldots, \mathcal{D}_N$, the intersection

$$\mathcal{D} = \mathcal{D}_1 \cap \ldots \cap \mathcal{D}_N$$

has characteristic function

$$f_{\mathcal{D}}(z) = \operatorname*{diag}_{i=1}^{N} f_{\mathcal{D}_i}(z) \; .$$

If quadratic $\mathcal{D}$-stability is of interest, then Theorem 3.3 should be applied to the overall characteristic function $f_{\mathcal{D}}(z)$. However, when robust $\mathcal{D}$-stability is the primary concern, it is more efficient and less conservative to test quadratic stability for each elementary region $\mathcal{D}_i$ *independently.* Indeed, this guarantees robust stability with respect to each region $\mathcal{D}_i$, which in turn establishes robust $\mathcal{D}$-stability.

More specifically, if $\mathcal{D}$ is the intersection of N elementary LMI regions with characteristic functions

$$f_{\mathcal{D}_i}(z) = L_i + zM_i + \bar{z}M_i^T, \quad M_i = M_{1i}^T M_{2i} \; ,$$

a sufficient condition for robust $\mathcal{D}$-stability against norm-bounded uncertainty

$$\Delta \in \mathbf{C}^{m \times m}, \quad \sigma_{max}(\Delta) \leq \gamma^{-1}$$

is the existence, for each region $\mathcal{D}_i$, of a pair of matrices (X_i, P_i) such that

$$\begin{bmatrix} M_{\mathcal{D}_i}(A, X_i) & M_{1i}^T \otimes (X_i B) & (M_{2i}^T P_i) \otimes C^T \\ M_{1i} \otimes (B^T X_i) & -\gamma P_i \otimes I & P_i \otimes D^T \\ (P_i M_{2i}) \otimes C & P_i \otimes D & -\gamma P_i \otimes I \end{bmatrix} < 0, \quad P_i > 0, \quad X_i > 0 \, . \tag{12}$$

Note that the LMI feasibility problems (12) $(i = 1, \ldots, N)$ should be solved *independently* for each region $\mathcal{D}_i$ since there is no coupling between the constraints for each region. By contrast, applying Theorem 3.3 directly to $f_{\mathcal{D}}(z)$ amounts to *jointly* solving all the LMIs (12) with $X = X_1 = \ldots = X_N$ and $P_i, i = 1, \ldots, N$ as variables. This is clearly more costly, and also more conservative due to the requirement that a single X satisfy (12) for all regions $\mathcal{D}_i$

3.4 Comments on quadratic $\mathcal{D}$-stability

Theorem 3.3 gives a sufficient condition for quadratic $\mathcal{D}$-stability in the face of complex and unstructured uncertainty. As mentioned earlier, the uncertainty Δ must be real for the uncertain model

$$\dot{x} = A(\Delta)x, \qquad A(\Delta) := A + B(I - \Delta D)^{-1}\Delta C \tag{13}$$

to be physically meaningful. When $\mathcal{D}$ is the open left-half plane and robust stability is of interest, the quadratic stability test is known to be conservative for real uncertainty Δ. It is therefore legitimate to question the value of Theorem 3.3 as a tool for assessing robust $\mathcal{D}$-stability.

While acknowledging conservatism for this particular uncertainty model, we now briefly review other benefits of quadratic $\mathcal{D}$-stability that strengthen its practical appeal. Rewrite (13) as

$$\begin{array}{rcl} \dot{x} &=& Ax + Bw_\Delta \\ z_\Delta &=& Cx + Dw_\Delta \\ w_\Delta &=& \Delta z_\Delta \end{array} \tag{14}$$

and let $G(s) := D + C(sI - A)^{-1}B$. Then $A(\Delta)$ is simply the closed-loop matrix for the feedback loop of Figure 1, and robust $\mathcal{D}$-stability is therefore equivalent to requiring that the closed-loop poles remain in $\mathcal{D}$ for all Δ satisfying $\sigma_{max}(\Delta) \le \gamma^{-1}$.

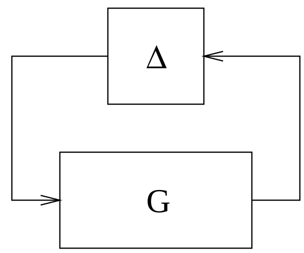

Figure 1: Robustness analysis interconnection

When $\mathcal{D}$ is the open left-half plane, there is equivalence between [24]:

- quadratic stability
- robust stability against complex Δ with $\sigma_{max}(\Delta) \le \gamma^{-1}$
- $\|G(s)\|_\infty < \gamma$ where $\|G\|_\infty := \sup_\omega \sigma_{max}(G(j\omega))$
- robust stability against stable dynamic uncertainty $\Delta(s)$ satisfying $\|\Delta(s)\|_\infty < \gamma^{-1}$
- feasibility of the Bounded Real Lemma LMI (7)

Similar connections between quadratic $\mathcal{D}$-stability, robust $\mathcal{D}$-stability against dynamic uncertainty, and Theorem 3.3 can be established for general LMI regions. Specifically, for $G(s)$ analytic in $\mathbf{C}\backslash\mathcal{D}$ (i.e., $\mathcal{D}$-stable), define the H_∞ norm with respect to $\mathcal{D}$ as

$$\|G\|_\infty^{\mathcal{D}} = \sup_{s \notin \mathcal{D}} \sigma_{max}\left(G(s)\right) = \sup_{s \in \partial\mathcal{D}} \sigma_{max}\left(G(s)\right) .$$

Straightforward adaptations of the small gain and generalized Nyquist theorems [25] lead to the following results.

Theorem 3.4 *The following properties are equivalent:*

- $A(\Delta)$ *is robustly* $\mathcal{D}$*-stable for static complex uncertainty* Δ *satisfying* $\sigma_{max}(\Delta) \le \gamma^{-1}$
- $\|G(s)\|_\infty^{\mathcal{D}} < \gamma$
- *The closed-loop system* $G_\Delta(s) = (I - G(s)\Delta)^{-1}G(s)$ *is robustly* $\mathcal{D}$*-stable against dynamic uncertainties* $\Delta(s)$ *that are* $\mathcal{D}$*-stable and satisfy* $\|\Delta(s)\|_\infty^{\mathcal{D}} < \gamma^{-1}$

- *If $\Delta(s)$ has no poles on the boundary of $\mathcal{D}$ and $\sup_{s\in\partial\mathcal{D}} \sigma_{max}(\Delta(s)) \leq \gamma^{-1}$, then the nominal poles of $G(s)$ remain in $\mathcal{D}$. More precisely, the number of poles of $G_\Delta(s)$ in $\mathcal{D}$ is always equal to the number of nominal poles (all in $\mathcal{D}$) plus the number of poles of $\Delta(s)$ in $\mathcal{D}$.*

These results indicate that quadratic $\mathcal{D}$-stability (and the related test in Theorem 3.3) also provides some robustness against dynamic uncertainty, which is desirable in practice. It is also worth noticing that for general $\mathcal{D}$ regions, $\mathcal{D}$-stability is difficult to handle numerically as it requires an exhaustive sweep of $\partial\mathcal{D}$, the boundary of $\mathcal{D}$. In this respect, quadratic $\mathcal{D}$-stability provides tractable, though possibly conservative, means for checking robust $\mathcal{D}$-stability.

3.5 Time-varying uncertainties

The proof of Theorem 3.3 remains valid when the uncertainty is time-varying, i.e.,

$$A(t) = A(\Delta(t)) := A + B(I - D\Delta(t))^{-1}\Delta(t)C, \quad \sigma_{max}(\Delta(t)) \leq 1 .$$

While the notion of "pole" disappears for linear time-varying systems, the generalized Bounded Real Lemma of Theorem 3.3 still provides the following guarantees:

- $\mathcal{D}$-stability of the matrix $A(\Delta(t))$ at all time t
- Exponential decay of the transients whenever $\mathcal{D}$ is contained in some stable half-plane $z + \bar{z} < -2\alpha$ with $\alpha > 0$.

The second property is a consequence of the following lemma.

Lemma 3.5 *Consider an LMI region $\mathcal{D}$ with characteristic function $f_\mathcal{D}(z) = L + zM + \bar{z}M^T$ and suppose the dynamical system*

$$\dot{x} = A(t)x$$

is quadratically $\mathcal{D}$-stable, i.e., there exists $X > 0$ such that

$$M_\mathcal{D}(A(t), X) = L \otimes X + M \otimes XA(t) + M^T \otimes A(t)^T X < 0 \tag{15}$$

for all time t. Then the quadratic function $V(x(t)) = x(t)^T X x(t)$ satisfies, for all $x(t) \neq 0$,

$$\frac{1}{2}\frac{\dot{V}(x)}{V(x)} \in \mathcal{D} \cap \mathbf{R} .$$

Proof

Multiplying (15) left and right by $I \otimes x^T$ and $I \otimes x$, respectively, we get for all $x \neq 0$,

$$L \otimes x^T X x + M \otimes x^T XA(t)x + M^T \otimes x^T A(t)^T X x < 0 .$$

Recalling that $1/2\,\dot{V}(x) = x^T XA(t)x = x^T A(t)^T X x$, and dividing by $V(x)$, this leads to

$$L \otimes 1 + M \otimes \frac{1}{2}\frac{\dot{V}(x)}{V(x)} + M^T \otimes \frac{1}{2}\frac{\dot{V}(x)}{V(x)} < 0.$$

which, by definition of $\mathcal{D}$, ensures that $\frac{1}{2}\frac{\dot{V}(x)}{V(x)} \in \mathcal{D}$. ∎

This lemma shows that for quadratic $\mathcal{D}$-stable time-varying systems, stability and decay rate are essentially determined by time-invariant considerations, i.e., whether $\mathcal{D} \subset \mathbf{C}^-$. However, it says little more about transient behaviors. Can we also expect well-damped responses by choosing an adequate conic sector? Does a disk prevent fast dynamics? Such extensions to the time-varying case remain open for future research.

4 Parameter Uncertainty

This section discusses refinements of the previous robustness analysis results when the uncertainty is structured. The main motivation is the assessment of robust $\mathcal{D}$-stability in the face of parameter uncertainties. As is usual when dealing with structured uncertainty, the resulting tests are only sufficient conditions for robust pole clustering in a given LMI region $\mathcal{D}$. Our analysis technique relies on the use of a parameter-dependent matrix $X(\delta)$ similar to the parameter-dependent Lyapunov functions used in [19, 13, 11] for regular robust stability analysis. Such approaches have proved to be significantly less conservative than quadratic stability for time-invariant parameter uncertainty.

The analysis below deals with the same basic uncertain model (3) but now assumes that Δ is real and structured, i.e.,

$$\Delta = \Delta(\delta) := \operatorname*{diag}_{i=1}^{q} \delta_i I_{r_i}, \quad \delta_i \in \mathbf{R}, \quad |\delta_i| \leq 1 \tag{16}$$

where the δ_i's denote the (normalized) uncertain parameters. We denote by $\mathcal{H} \subset \mathbf{R}^q$ the hypercube in which $\delta = (\delta_1, \ldots, \delta_q)$ ranges according to (16), and by $\mathcal{V}$ the set of vertices of this hypercube, that is,

$$\mathcal{V} = \{(\delta_1, \ldots, \delta_q) : \quad \delta_i = \pm 1\} .$$

To stress the dependence on the parameter vector δ, the uncertain state matrix is written as

$$A(\delta) = A + B(I - \Delta(\delta)D)^{-1}\Delta(\delta)C . \tag{17}$$

The relevant dimensionality parameters are defined by

$$A \in \mathbf{R}^{n\times n}, \quad L \in \mathbf{R}^{l\times l}, \quad k := \operatorname{rank}(M), \quad r := \sum_{i=1}^{q} r_i, \quad s := kr + qnl . \tag{18}$$

For such parameter uncertainty, robust pole clustering in the LMI region

$$\mathcal{D} = \left\{ z \in \mathbf{C} : \quad L + zM + \overline{z}M^T < 0 \right\} \tag{19}$$

is equivalent to the existence of symmetric matrices $X(\delta) > 0$ parametrized by δ such that

$$L \otimes X(\delta) + M \otimes X(\delta)A(\delta) + M^T \otimes A(\delta)^T X(\delta) < 0, \quad \forall \delta \in \mathcal{H} . \tag{20}$$

To enforce tractability of (20), we restrict the search of functions $X(\delta)$ to matrices with affine dependence on δ:

$$X(\delta) = X_0 + \sum_{i=1}^{q} \delta_i X_i = X_0 + J^T \widehat{\Delta}(\delta) \hat{X} J,$$

where

$$J = [\underbrace{I_n, \ldots, I_n}_{q \text{ times}}]^T, \qquad \widehat{\Delta}(\delta) = \operatorname*{diag}_{i=1}^{q} \delta_i I_n, \qquad \hat{X} = \operatorname*{diag}_{i=1}^{q} X_i .$$

Two robust $\mathcal{D}$-stability tests are derived next using such affine $X(\delta)$'s. The first test applies to general linear-fractional dependence of $A(\delta)$ on δ whereas the second test is restricted to affine dependence ($D = 0$). These results are strongly related to the general Integral Quadratic Constraint framework developed by Megretski and Rantzer in [27] (see pp. 825–826). For simplicity, our results are stated for a single Lyapunov matrix $X(\delta)$ regardless of the complexity of the LMI region. For LMI regions that are intersections of N elementary LMI regions $\mathcal{D}_i$, sharper tests can be obtained by using independent Lyapunov matrices $X_i(\delta)$ for each $\mathcal{D}_i$ as indicated in Section 3.3.

4.1 General parameter dependence

Theorem 4.1 *Given the parameter uncertainty δ specified by (16), the LMI region $\mathcal{D}$ in (19), a full-rank factorization $M = M_1^T M_2$ of M, and the notation (18), the uncertain matrix $A(\delta)$ in (17) is robustly $\mathcal{D}$-stable if there exist*

- *symmetric matrices $X_0 > 0$ and $\hat{X} = \operatorname{diag}_{i=1}^{q} X_i$ (with $X_i \in \mathbf{R}^{n\times n}$),*
- *scaling matrices $S = S^T$, $R < 0$, and Q (all in $\mathbf{R}^{s\times s}$)*

such that

$$\begin{bmatrix} L\otimes X_0 + M\otimes X_0 A + M^T \otimes A^T X_0 & M_1^T \otimes X_0 B & \star \\ M_1 \otimes B^T X_0 & 0 & \star \\ L\otimes \hat{X}J + M\otimes \hat{X}JA & M_1^T \otimes \hat{X}JB & 0 \end{bmatrix} +$$
$$\begin{bmatrix} M_2\otimes C & I_k\otimes D & 0 \\ I_l\otimes J & 0 & 0 \\ \hline 0 & I_{kr} & 0 \\ 0 & 0 & I_{qnl} \end{bmatrix}^T \left[\begin{array}{c|c} S & Q \\ \hline Q^T & R \end{array}\right] \begin{bmatrix} M_2\otimes C & I_k\otimes D & 0 \\ I_l\otimes J & 0 & 0 \\ \hline 0 & I_{kr} & 0 \\ 0 & 0 & I_{qnl} \end{bmatrix} < 0 \qquad (21)$$

and, for all vertices $\bar{\delta}$ of the uncertainty hypercube $\mathcal{H}$,

$$\left[I_s, \operatorname{diag}\left(I_k\otimes\Delta(\bar{\delta}), I_l\otimes\widehat{\Delta}(\bar{\delta})\right)\right] \begin{bmatrix} S & Q \\ Q^T & R\end{bmatrix} \begin{bmatrix} I_s \\ \operatorname{diag}\left(I_k\otimes\Delta(\bar{\delta}), I_l\otimes\widehat{\Delta}(\bar{\delta})\right)\end{bmatrix} > 0\,. \qquad (22)$$

Proof
See Appendix B. ■

This theorem provides a test for robust $\mathcal{D}$-stability that involves solving a finite set of LMIs and is therefore tractable. Applications to some aeronautics systems suggest that it can be quite sharp. In its most general form, this test can be computationally demanding for high-order systems with multiple uncertainties. With additional conservatism, the computational cost can be reduced by

- Using symmetric and skew-symmetric scalings in place of the general and unconstrained scalings S, Q and R. Specifically, impose $S = -R > 0$ and $Q = -Q^T$ and make the structure of R, Q consistent with the uncertainty structure

$$\begin{bmatrix} I_k\otimes\Delta & 0 \\ 0 & I_l\otimes\widehat{\Delta}\end{bmatrix}, \qquad (23)$$

  That is, require $S\Xi = \Xi S$ and $Q\Xi = \Xi Q$ for all Ξ matrices of the form (23).

- Setting some of the matrices X_i to zero.

4.2 Affine parameter dependence

When $D = 0$ in (17), the uncertain state matrix depends affinely on the parameters δ_i:

$$A(\delta) = A_0 + \sum_{i=1}^{q} \delta_i A_i\,. \qquad (24)$$

For such uncertain systems, the following robust $\mathcal{D}$-stability test is easily derived using the multi-convexity technique developed in [13]. Recall that a function $f(x_1, \ldots, x_q)$ is multi-convex when it is convex with respect to each of its variables separately. For differentiable functions this property is equivalent to requiring that the Hessian of f has non-negative diagonal entries.

Theorem 4.2 *Given the parameter uncertainty δ specified by (16), the uncertain system with state matrix (24) is robustly $\mathcal{D}$-stable if there exist symmetric matrices $X_0, X_1, \ldots, X_q$ and scalars m_i, $i = 1, \ldots, q$ such that*

$$\begin{aligned} L \otimes X(\delta) + M \otimes X(\delta)A(\delta) + M^T \otimes A(\delta)^T X(\delta) + \sum_{i=1}^{q} \delta_i^2 m_i I &< 0 \qquad (25) \\ M \otimes (X_i A_i) + M^T \otimes (A_i^T X_i) + m_i I &\geq 0 \qquad (26) \\ X_0 > 0, \qquad m_i &\geq 0 \qquad (27) \end{aligned}$$

hold at all vertices δ of $\mathcal{H}$ and for $i = 1, \ldots, q$, with

$$X(\delta) := X_0 + \sum_{i=1}^{q} \delta_i X_i \, .$$

Proof
Condition (26) ensures that for any η, the quadratic function of δ

$$\eta^T \left(L \otimes X(\delta) + M^T \otimes X(\delta)A(\delta) + M^T \otimes A(\delta)^T X(\delta) + \sum_{i=1}^{q} \delta_i^2 m_i I \right) \eta$$

is multi-convex in the δ_i's. Using the same argument as in [13], it follows that (25) holds over the entire hypercube if it holds at the vertices. ■

4.3 Robust analysis application

The analysis techniques developed in this section are applied to a realistic missile example (see [34] for additional details and insights). The purpose is to determine admissible uncertainty levels for which stability and adequate damping are preserved.

The dynamics of the controlled missile roll axis are described by

$$\begin{aligned} \dot{x} &= Ax + \delta_1 A_1 x + Bu + \delta_2 B_2 u \\ y &= Cx \\ u &= Ky \end{aligned}$$

where

$$x = [\, \delta_r \quad \delta_p \quad r \quad n_y \quad p \,]^T \,, \quad u = [\, \delta_{rc} \quad \delta_{pc} \,] \,,$$

and the meaning of the different variables is given in table 1.

Symbols	*Meaning*
δ_r	yaw control surface deflection
δ_p	roll control surface deflection
r	yaw rate
n_y	yaw acceleration
p	roll rate
δ_{rc}	yaw control command
δ_{pc}	roll control command
δ_1, δ_2	uncertainties

Table 1: Variable description

The output-feedback gain matrix K is given and has been designed using eigenspace techniques. The parameters δ_1 and δ_2 represent uncertainties whose effects on the missile dynamics are reflected in the matrices A_1 and B_2. Numerical values for these matrices can be found in Appendix C.

The objective is to estimate, in the parameter space (δ_1, δ_2),

- the largest square $|\delta_i| < \rho$ where closed-loop stability is maintained,
- the largest square where closed-loop damping is adequate, that is, $\zeta > 0.6$ for the missile roll axis.

Note that the uncertain parameters δ_1 and δ_2 enter affinely in the state-space matrices, so the techniques of Theorems 4.1 and 4.2 are both applicable. The closed-loop pole locations for parameter values in the set

$$(\delta_1,\, \delta_2) \in \{-1,\, -0.5,\, 0,\, 0.5,\, 1\} \times \{-1,\, -0.5,\, 0,\, 0.5,\, 1\}$$

are plotted in Figure 2. Clearly, both stability and damping constraints ($\zeta > 0.6$) are violated for some $(\delta_1,\, \delta_2)$ pairs in this uncertainty set.

The shaded area in Figure 3 shows the region in the parameter space (δ_1, δ_2) where closed-loop stability is maintained. This area has been computed using an exhaustive search over a fine grid in the parameter space. Based on the results of Theorem 4.1, we estimate the largest stability square using either fixed or parameter-dependent X matrices. As expected, a fixed X leads to a conservative answer (dashed square in Figure 3). In contrast, using a parameter-dependent X provides a sharp estimate of the largest allowable uncertainty (solid square).

Similarly, Figure 4 shows the uncertainty region where adequate damping $\zeta = 0.6$ is maintained, and the dashed and solid lines delimit the estimated safe regions using Theorem 4.2 with fixed or parameter-dependent X matrices. Note that the damping constraint $\zeta > 0.6$ is captured by the conic LMI region

$$L = 0, \qquad M = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad \theta = \arcsin 0.6\,.$$

Again the estimate based on parameter-dependent X matrices provides a sharp answer.

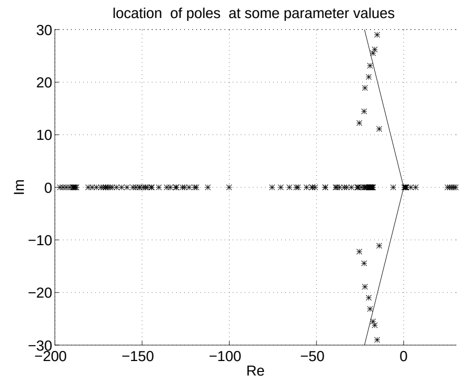

Figure 2: Closed-loop poles of $A(\delta) + B(\delta)KC$ for some parameter values $(\delta_1,\, \delta_2) \in \{-1,\, -0.5,\, 0,\, 0.5,\, 1\}^2$

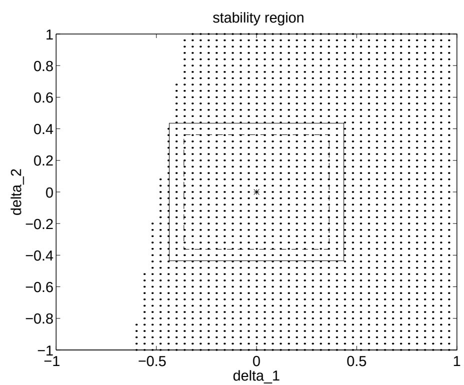

Figure 3: Stability region estimates
with fixed (dotted) and parameter-dependent (solid) X matrices.

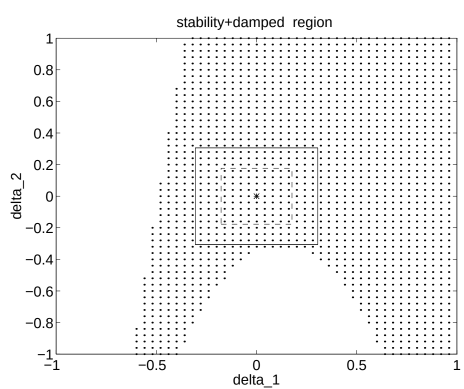

Figure 4: Adequate damping region estimates
with fixed (dotted) and parameter-dependent (solid) X matrices.

5 Output-Feedback Synthesis

This last section shows how to use our main analysis result (Theorem 3.3) for synthesis purposes. Specifically, we consider the problem of computing an output-feedback controller that robustly assigns the closed-loop poles in a prescribed LMI region $\mathcal{D}$. For tractability reasons, the discussion is restricted to unstructured uncertainty.

The problem statement is as follows. Consider the uncertain state-space model

$$\left\{\begin{array}{rcl} \dot{x} & = & Ax + B_\Delta w_\Delta + B_u u \\ z_\Delta & = & C_\Delta x + D_{\Delta\Delta} w_\Delta + D_{\Delta u} u \\ y & = & C_y x + D_{y\Delta} w_\Delta + D_{yu} u \\ w_\Delta & = & \Delta\, z_\Delta \end{array}\right. \tag{28}$$

where $A \in \mathbf{R}^{n\times n}$ and the static uncertainty Δ satisfies $\sigma_{max}(\Delta) \le \gamma^{-1}$. Given the LMI region

$$\mathcal{D} = \left\{ z \in \mathbf{C} : \quad L + zM + \overline{z}M^T < 0 \right\}$$

we are interested in designing a full-order dynamic controller

$$K(s) \quad \left\{\begin{array}{rcl} \dot{x}_K & = & A_K\, x_K + B_K\, y \\ u & = & C_K\, x_K + D_K\, y \end{array}\right. \tag{29}$$

that robustly assigns the closed-loop poles in $\mathcal{D}$.

Without loss of generality, assume that $D_{yu} = 0$ since this amounts to a mere change of variable in the controller matrices and considerably simplifies the formulae. The closed-loop matrix is

$$A_{c\ell}(\Delta) = A_{c\ell} + B_{c\ell}(I - \Delta D_{c\ell})^{-1}\Delta C_{c\ell}$$

where

$$A_{cl} := \begin{bmatrix} A + B_u D_K C_y & B_u C_K \\ B_K C_y & A_K \end{bmatrix}, \quad B_{cl} := \begin{bmatrix} B_\Delta + B_u D_K D_{y\Delta} \\ B_K D_{y\Delta} \end{bmatrix}$$
$$C_{cl} := [\, C_\Delta + D_{\Delta u} D_K C_y \;,\; D_{\Delta u} C_K \,], \quad D_{cl} := D_{\Delta\Delta} + D_{\Delta u} D_K D_{y\Delta}\;.$$

From Theorem 3.3 with $P = I$, a sufficient condition for quadratic (hence robust) $\mathcal{D}$-stability of $A_{c\ell}(\Delta)$ is the existence of $X > 0$ such that

$$\begin{bmatrix} M_{\mathcal{D}}(A_{cl}, X) & M_1^T \otimes (XB_{cl}) & M_2^T \otimes C_{cl}^T \\ M_1 \otimes (B_{cl}^T X) & -\gamma I & I \otimes D_{cl}^T \\ M_2 \otimes C_{cl} & I \otimes D_{cl} & -\gamma I \end{bmatrix} < 0 \tag{30}$$

where $M_1^T M_2 = M$ is a full-rank factorization of M. This matrix inequality is not jointly convex in X and the controller matrices. However, it can be reduced to a convex LMI problem by using the linearizing change of controller variables introduced in [26, 33, 9]. This leads to the following synthesis result.

Theorem 5.1 *There exists a full-order output-feedback controller $K(s)$ and a matrix $X > 0$ such that (30) holds if and only if there exist two $n \times n$ symmetric matrices R and S and matrices $\mathcal{A}_K$, $\mathcal{B}_K$, $\mathcal{C}_K$ and D_K such that*

$$\Lambda(R, S) := \begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0 \tag{31}$$

$$\begin{bmatrix} L \otimes \Lambda(R,S) + M \otimes \Phi_A + M^T \otimes \Phi_A^T & M_1^T \otimes \Phi_B & M_2^T \otimes \Phi_C^T \\ M_1 \otimes \Phi_B^T & -\gamma I & I \otimes \Phi_D^T \\ M_2 \otimes \Phi_C & I \otimes \Phi_D & -\gamma I \end{bmatrix} < 0 \qquad (32)$$

where

$$\Phi_A = \begin{bmatrix} AR + B_u \mathcal{C}_K & A + B_u D_K C_y \\ \mathcal{A}_K & SA + \mathcal{B}_K C_y \end{bmatrix}, \quad \Phi_B = \begin{bmatrix} B_\Delta + B_u D_K D_{y\Delta} \\ SB_\Delta + \mathcal{B}_K D_{y\Delta} \end{bmatrix}$$
$$\Phi_C = [\, C_\Delta R + D_{\Delta u}\mathcal{C}_K \;,\; C_\Delta + D_{\Delta u} D_K C_y \,], \quad \Phi_D = D_{\Delta\Delta} + D_{\Delta u} D_K D_{y\Delta} \,.$$

If these LMIs are feasible, then a nth-order controller that robustly assigns the closed-loop poles in $\mathcal{D}$ is

$$K(s) = D_K + C_K(sI - A_K)^{-1} B_K$$

where the matrices A_K, B_K, C_K are derived as follows:

- *Compute any square matrices $\hat{M}$ and $\hat{N}$ such that $\hat{M}\hat{N}^T = I - RS$*
- *Solve the following linear equations for B_K, C_K, and A_K:*

$$\left\{ \begin{array}{lcl} \mathcal{B}_K &=& \hat{N} B_K + S B_u D_K \\ \mathcal{C}_K &=& C_K \hat{M}^T + D_K C_y R \\ \mathcal{A}_K &=& \hat{N} A_K \hat{M}^T + \hat{N} B_K C_y R + S B_u C_K \hat{M}^T + S(A + B_u D_K C_y) R \end{array} \right. \qquad (33)$$

Proof

The proof involves the changes of variable introduced in [26, 33, 9] and is omitted for brevity. ■

Inequalities (31)–(32) are LMIs in the variables R, S, $\mathcal{A}_k$, $\mathcal{B}_K$, $\mathcal{C}_K$, D_K that can be solved numerically using LMI optimization software [14]. Theorem 5.1 therefore provides a tractable (but somewhat conservative) approach to robust pole assignment in LMI regions.

Remark 5.2 When $\mathcal{D}$ is the intersection of several elementary LMI regions $\mathcal{D}_i$ as discussed in Subsection 3.3, the synthesis LMIs (31)–(32) must be written for each region using the *same* R, S variables, and the resulting set of LMIs must be solved jointly. Indeed, the synthesis problem is no longer convex when a different X_i is used for each $\mathcal{D}_i$ (this prevents using the linearizing change of variable). Note that the extra conservatism introduced by this additional restriction is modest in most applications.

5.1 Mixed design specifications

From a practical viewpoint, enforcing quadratic $\mathcal{D}$-stability is rarely sufficient since most design problems are essentially multi-objective. For instance, realistic designs are likely to include H_2 or H_∞ (loop shaping) objectives in addition to robust pole assignment for transient tuning. Fortunately, LMI-based synthesis can accomodate a rich variety of closed-loop specifications within a single LMI optimization problem as shown in [32]. This is achieved with some conservatism but has proved effective in a number of applications. The basic requirement is that a single closed-loop Lyapunov function $V(x) = x_{cl}^T X x_{cl}$ should account for all design specifications.

As an immediate extension of the results in [26, 33, 32], it is easy to mix quadratic $\mathcal{D}$-stability with other objectives such as H_∞ or H_2 performance, passivity constraints, bounds on the impulse response, etc. As an example, one can combine a quadratic $\mathcal{D}$-stability objective as captured by

Theorem 5.1 with an H_∞-norm bound on some input/output channels of the closed-loop system. For instance, if the nominal plant is described by

$$\begin{aligned}\dot{x} &= Ax + B_w w + B_u u \\ z &= C_z x + D_{zw} w + D_{zu} u \\ y &= C_y x + D_{yw} w\end{aligned}$$

the (nominal) closed-loop transfer function $T(s)$ from w to z can be further constrained to

$$\|T(s)\|_\infty < \beta$$

by combining the LMI conditions (31)–(32) for robust pole assignment with the additional LMI constraint

$$\begin{bmatrix} AR + RA^T + B_u \mathcal{C}_K + (B_u \mathcal{C}_K)^T & \mathcal{A}_K^T + (A + B_u D_K C_y) & * & * \\ \mathcal{A}_K + (A + B_u D_K C_y)^T & A^T S + SA + \mathcal{B}_K C_y + (\mathcal{B}_K C_y)^T & * & * \\ (B_w + B_u D_K D_{yw})^T & (SB_w + \mathcal{B}_K D_{yw})^T & -\beta I & * \\ C_z R + D_{zu} \mathcal{C}_K & C_z + D_{zu} D_K C_y & D_{zw} + D_{zu} D_K D_{yw} & -\beta I \end{bmatrix} < 0\,, \tag{34}$$

6 Design Example

This section illustrates the benefits of robust pole placement in LMI regions through a missile autopilot design example. The problem setup comes from [31, 21] where additional motivations and details can be found. It has been slightly simplified to focus on aspects relevant to the technique proposed in this paper.

The linearized longitudinal dynamics of the missile are described by

$$\begin{aligned}\begin{bmatrix}\dot{\alpha} \\ \dot{q}\end{bmatrix} &= \begin{bmatrix}-0.89 & 1 \\ -142.6 & 0\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} + \begin{bmatrix}0 \\ 178.25\end{bmatrix} w_\Delta + \begin{bmatrix}-0.119 \\ -130.8\end{bmatrix}\delta_{\text{fin}} \\ \begin{bmatrix}\eta_z \\ q\end{bmatrix} &= \begin{bmatrix}-1.52 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} - \begin{bmatrix}0.203 \\ 0\end{bmatrix}\delta_{\text{fin}} \\ z_\Delta &= \begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} \\ w_\Delta &= \Delta z_\Delta\,,\end{aligned}$$

where α, q, η_z and δ_{fin} denote the angle of attack, pitch rate, vertical accelerometer measurement, and fin deflection, respectively. The variables w_Δ and z_Δ are auxiliary signals used to model variations of the aerodynamic coefficients for α ranging between 0 to 20 degrees. The parameter uncertainty Δ has been normalized, that is, $\Delta \in [-1, 1]$.

We need to design a dynamic compensator $K(s)$ that meets the following specifications:

- Settling time of 0.2 second with minimal overshoot and zero steady-state error for the vertical acceleration η_z in response to a step command
- Adequate high-frequency roll-off for noise attenuation and to withstand neglected dynamics and flexible modes
- Maximum deflection of 2 (in normalized units) imposed on the control signal δ_{fin}
- Time-domain specifications must be met over the uncertainty range $|\Delta| \leq 1$.

To attack this problem, we use the feedback structure sketched in Figure 5 below. Here η_c denotes the reference acceleration signal, and e, u denote the weighted tracking error and control input, respectively. An integrator is introduced in the acceleration channel to enforce zero steady-state error. The full compensator is therefore given by

$$K(s) = K_0(s) \begin{bmatrix} 1/s & 0 \\ 0 & 1 \end{bmatrix} .$$

To incorporate bounds on the size of unmodeled dynamics and penalize tracking error, we use the weighting filters (see [31]):

$$W_u = \frac{0.001s^3 + 0.03s^2 + 0.3s + 1}{1\text{e-}5s^3 + 3\text{e-}2s^2 + 30s + 10000}, \quad W_e = 0.8 \; .$$

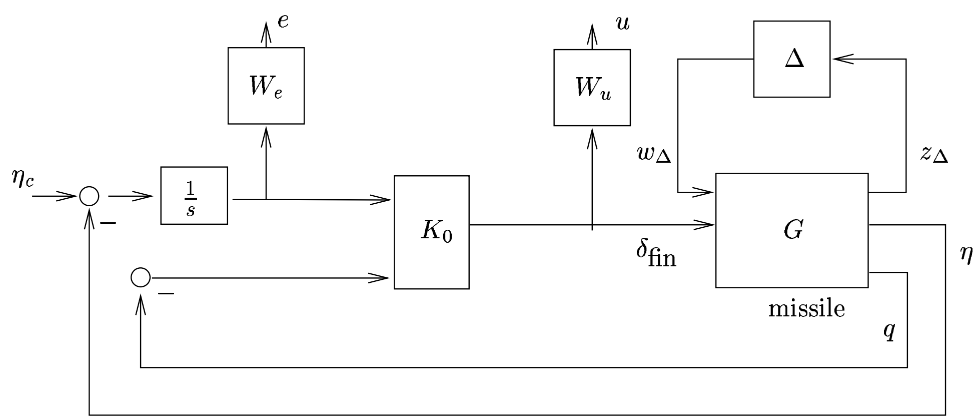

Figure 5: Synthesis structure

First, we perform a standard H_∞-optimal design where we minimize the closed-loop L_2 gain between the inputs (w_Δ, η_c) and the outputs (z_Δ, e, u). This is meant to enforce high-frequency roll-off as well as stability and performance for all admissible uncertainties Δ. The step responses of the resulting (pure) H_∞ controller are depicted in Figure 7 for $\Delta = 0$ (nominal) and $\Delta = \pm 1$ (perturbed). While this first design could be deemed acceptable, it suffers from up to 30% overshoot in the perturbed transient responses.

To improve transient behavior, we add a robust pole clustering constraint to achieve better damping across the uncertainty range. Specifically, we require robust pole clustering in the LMI region represented in Figure 6. This region is defined as the intersection of

- the disk with center 0 and radius 1500 (to prevent fast dynamics)
- the shifted conic sector with apex at $x = \beta = 1$ and angle $\theta = 30$ degrees. Its characteristic function is

$$-2\beta \cos\theta + z e^{i\theta} + \bar{z} e^{-i\theta} < 0$$

and the corresponding L and M matrices are:

$$L = \begin{bmatrix} -2\beta\cos\theta & 0 \\ 0 & -2\beta\cos\theta \end{bmatrix}; \qquad M = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

This particular region is chosen in order to achieve differential damping at low and high frequency (the damping constraint takes effect for $\omega > 1.73$). Because the H_∞ constraint already enforces closed-loop stability, it is inconsequential that this LMI region intersects the right half-plane.

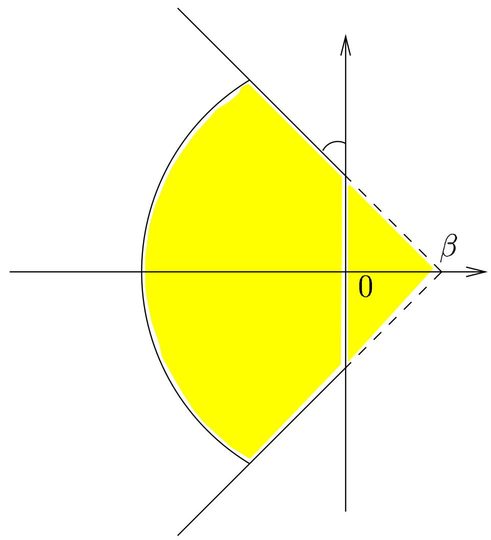

Figure 6: LMI region

The resulting synthesis problem is multi-objective since it involves minimizing the closed-loop H_∞ norm subject to robust pole clustering in the selected region. In the LMI framework, this problem is attacked by minimizing the closed-loop gain γ subject to

- the LMI constraints of Theorem 5.1 for robust pole clustering
- the LMI constraint (34) for the constraint "closed-loop gain $< \gamma$"

(see Section 5.1 for details).

This LMI optimization problem was solved with [14] and produced the compensator

$$K(s) = [\, K_\nu(s) \quad K_q(s) \,]$$

with zero-pole-gain description in Table 2. The corresponding step responses are shown in Figure 8. The transients are smoother than those obtained with pure H_∞ control for both nominal and perturbed plants. More importantly, thanks to the disk constraint, this is achieved with significantly slower controller dynamics. Indeed, the fastest mode in the pure H_∞ controller is -7×10^3, whereas it is only -1.3×10^3 in the multi-objective controller. Note that these improvements are secured without tangible degradation of the H_∞ performance since both designs have nearly optimal performance $\gamma = 0.8$. Finally, Figure 9 shows that the final controller has adequate roll-off properties.

	$K_\nu(s)$	$K_q(s)$
Gains	2.5676	-6.4909×10^{-3}
Zeros	-1.0143×10^3 $-9.9297 \times 10^2 \pm 1.3204 \times 10^1 j$ -9.1054×10^2 -6.7379×10^2 -8.9000×10^{-1}	0 -1.4824×10^3 -1.0433×10^3 $-1.0063 \times 10^3 \pm 1.7801 \times 10^1 j$ -9.8382×10^2 -2.0589×10^1
Poles	0 -1.3050×10^3 -7.9192×10^2 -5.6938×10^2 $-1.3172 \times 10^2 \pm 1.0787 \times 10^2 j$ -2.2752	

Table 2: Controller zero-pole-gain description

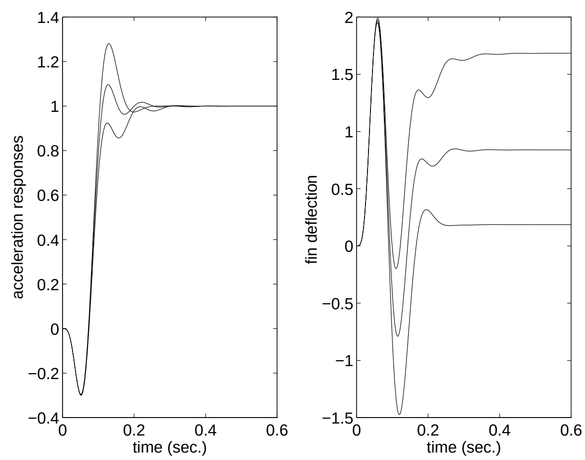

Figure 7: Pure H_∞ design - Nominal and perturbed ($\Delta = \pm 1$) step responses

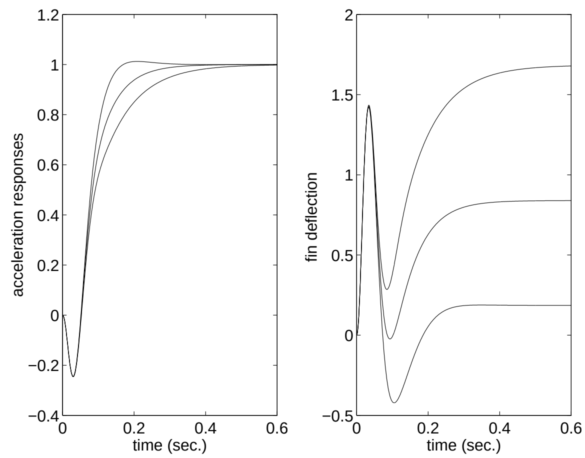

Figure 8: Final design - Nominal and perturbed ($\Delta = \pm 1$) step responses

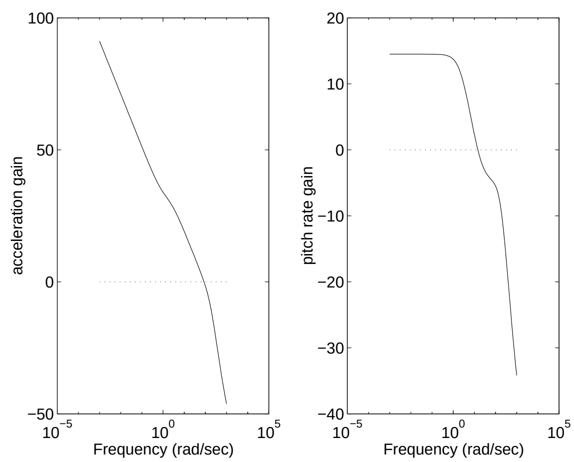

Figure 9: Roll-off in final design

7 Conclusion

Tractable analysis and synthesis techniques have been derived for robust pole placement in LMI regions. For analysis with unstructured uncertainties, the Bounded Real Lemma characterization of quadratic stability has been generalized to pole clustering in arbitrary LMI regions. For parameter uncertainty, two robust $\mathcal{D}$-stability tests have been derived that rely on scaling and multi-convexity techniques. While both provide only sufficient conditions, they have proven quite sharp in a number of applications. Finally, we have proposed a tractable LMI-based approach to the synthesis of output-feedback controllers that robustly assign the closed-loop poles in a prescribed LMI region. Combination of robust pole assignment with other closed-loop design specifications have also been discussed.

Appendix A

Proof of Theorem 3.3 : Assume $\gamma = 1$ without loss of generality. First observe that $I - \Delta D$ is always invertible since (10) implies that

$$\begin{bmatrix} -I & D \\ D^T & -I \end{bmatrix} \otimes P < 0 ,$$

which, from $P > 0$ and the properties of eigenvalues of Kronecker products, secures $\sigma_{max}(D) < 1$ and hence $\sigma_{max}(\Delta D) < 1$ for all admissible Δ.

By definition, quadratic $\mathcal{D}$-stability holds if and only if there exists $X > 0$ such that, for all Δ satisfying $\sigma_{max}(\Delta) \leq 1$,

$$\begin{aligned} 0 > M_{\mathcal{D}}(A(\Delta), X) &= L \otimes X + \text{Herm}\left\{M \otimes \left(X(A + B(I - \Delta D)^{-1}\Delta C)\right)\right\} \\ &= M_{\mathcal{D}}(A, X) + \text{Herm}\left\{(M_1^T M_2) \otimes \left(XB(I - \Delta D)^{-1}\Delta C\right)\right\} \\ &= M_{\mathcal{D}}(A, X) + \text{Herm}\left\{\left(M_1^T \otimes XB\right)\left(M_2 \otimes (I - \Delta D)^{-1}\Delta C\right)\right\} . \end{aligned}$$

Equivalently, the inequality

$$\nu^H M_{\mathcal{D}}(A, X)\nu + 2\nu^H \left(M_1^T \otimes XB\right)\left(M_2 \otimes (I - \Delta D)^{-1}\Delta C\right)\nu < 0 \tag{35}$$

should hold for any nonzero vector ν and admissible Δ.

For fixed $\nu \neq 0$, this amounts to requiring that

$$\begin{aligned} &\nu^H M_{\mathcal{D}}(A, X)\nu + 2\nu^H \left(M_1^T \otimes XB\right) p < 0 \quad \text{whenever} \\ &\qquad p \in S_\nu := \left\{(M_2 \otimes (I - \Delta D)^{-1}\Delta C)\,\nu \ : \ \sigma_{max}(\Delta) \leq 1\right\} . \end{aligned} \tag{36}$$

Observing that $p = (M_2 \otimes (I - \Delta D)^{-1}\Delta C)\nu$ is the unique solution of the equation $p = (I_k \otimes \Delta)q_{p,\nu}$ where

$$q_{p,\nu} := (I_k \otimes D)p + (M_2 \otimes C)\nu , \qquad k := \text{rank}(M) , \tag{37}$$

an equivalent and simpler characterization of S_ν is:

$$S_\nu = \{p \ : \ p = (I_k \otimes \Delta)q_{p,\nu} \ , \ \sigma_{max}(\Delta) \leq 1\} .$$

Now, $p = (I_k \otimes \Delta)q$ together with $\sigma_{max}(\Delta) \leq 1$ ensures that, for any $k \times k$ matrix $P > 0$,

$$q^H(P \otimes I)q - p^H(P \otimes I)p = q^H(P \otimes (I - \Delta^H\Delta))q \geq 0 .$$

Consequently, a sufficient condition for (36) to hold is that

$$\begin{aligned} &\nu^H M_{\mathcal{D}}(A, X)\nu + 2\nu^H \left(M_1^T \otimes XB\right) p < 0 \quad \text{whenever} \\ &\qquad q_{p,\nu}^H(P \otimes I)q_{p,\nu} - p^H(P \otimes I)p \geq 0 , \end{aligned} \tag{38}$$

or equivalently from the expression (37) for $q_{p,\nu}$,

$$\begin{aligned} &\begin{bmatrix} \nu \\ p \end{bmatrix}^H \begin{bmatrix} M_{\mathcal{D}}(A, X) & M_1^T \otimes (XB) \\ M_1 \otimes (B^T X) & 0 \end{bmatrix} \begin{bmatrix} \nu \\ p \end{bmatrix} < 0 \qquad \text{whenever} \\ &\qquad \begin{bmatrix} \nu \\ p \end{bmatrix}^H \left(\begin{bmatrix} M_2^T \otimes C^T \\ I \otimes D^T \end{bmatrix} (P \otimes I)\,[\, M_2 \otimes C \ , \ I \otimes D\,] + \begin{bmatrix} 0 & 0 \\ 0 & -P \otimes I \end{bmatrix} \right) \begin{bmatrix} \nu \\ p \end{bmatrix} \geq 0 . \end{aligned}$$

Using a standard **S**-procedure argument [36, 12], this condition is in turn equivalent (up to rescaling X) to the single LMI constraint:

$$\begin{bmatrix} M_{\mathcal{D}}(A, X) & M_1^T \otimes (XB) \\ M_1 \otimes (B^T X) & -P \otimes I \end{bmatrix} + \begin{bmatrix} M_2^T \otimes C^T \\ I \otimes D^T \end{bmatrix} (P \otimes I)\,[\, M_2 \otimes C \ , \ I \otimes D\,] < 0 . \tag{39}$$

Finally, a Schur complement with respect to the block (3,3) of (10) shows the equivalence between (39) and (10).

Remark A.1 Note that Theorem 3.3 is non conservative when M has rank one ($k = 1$). Indeed, S_ν is then characterized by the relation $p = \Delta\, q_{p,\nu}$ and coincides with the set of vectors p such that $q_{p,\nu}^H P q_{p,\nu} - p^H P p \geq 0$ for some $P > 0$. Since the gap between these two sets is the only source of conservatism in the proof, the LMI constraints (10)–(11) then become necessary and sufficient for quadratic $\mathcal{D}$-stability against complex unstructured uncertainty. Note that no scaling is needed in this case, i.e., setting $P = 1$ incurs no conservatism.

Appendix B

Proof of Theorem 4.1 : Fisrt observe that for $R < 0$, the matrix-valued function

$$F(\delta) = S + QD(\delta) + D(\delta)Q^T + D(\delta)RD(\delta), \quad D(\delta) := \mathrm{diag}\left(I \otimes \Delta(\delta), I \otimes \widehat{\Delta}(\delta)\right)$$

is concave in δ. As a result, (22) holds for all $\delta \in \mathcal{H}$ if it holds at the vertices $\bar{\delta}$ of $\mathcal{H}$.

To establish well-posedness, i.e, invertibility of $I - D\Delta(\delta)$ for all $\delta \in \mathcal{H}$, suppose that $I - D\Delta(\delta)$ is singular for some δ, and consider $y \neq 0$ such that $y = D\Delta(\delta)y$. Then

- Pre- and post-multiplying (21) by the full-rank matrix $[0, I \otimes (\Delta(\delta)y)^T, 0]$ and its transpose shows that the matrix

$$\begin{bmatrix} I \otimes y \\ 0 \\ \hline I \otimes \Delta(\delta)y \\ 0 \end{bmatrix}^T \left[\begin{array}{c|c} S & Q \\ \hline Q^T & R \end{array}\right] \begin{bmatrix} I \otimes y \\ 0 \\ \hline I \otimes \Delta(\delta)y \\ 0 \end{bmatrix}$$

  is negative definite

- Pre- and post-multiplying (22) by $[I \otimes y^T, 0]$ and its transpose shows that the same matrix is positive definite, a contradiction.

The remainder of the proof parallels that of Theorem 3.3. $\mathcal{D}$-stability for the uncertain system (17) is guaranteed if $X(\delta) = X_0 + J^T \widehat{\Delta}(\delta)\hat{X}J$ satisfies, for all $\eta \neq 0$ and $\delta \in \mathcal{H}$,

$$\eta^T \left(L \otimes X(\delta) + M \otimes X(\delta)A(\delta) + M^T \otimes A(\delta)^T X(\delta)\right)\eta \; < \; 0 \qquad (40)$$

$$X(\delta) \; > \; 0\,. \qquad (41)$$

Using the expressions of $A(\delta), X(\delta)$ and $M = M_1^T M_2$, direct calculations show that for fixed $\eta \neq 0$, the δ-dependent inequalities (40) are collectively equivalent to

$$\begin{bmatrix} \eta \\ p \\ \pi \end{bmatrix}^T \begin{bmatrix} L \otimes X_0 + M \otimes X_0 A + M^T \otimes A^T X_0 & M_1^T \otimes X_0 B & \star \\ M_1 \otimes B^T X_0 & 0 & \star \\ L \otimes \hat{X}J + M \otimes \hat{X}JA & M_1^T \otimes \hat{X}JB & 0 \end{bmatrix} \begin{bmatrix} \eta \\ p \\ \pi \end{bmatrix} < 0 \quad \text{whenever}$$

$$\begin{bmatrix} p \\ \pi \end{bmatrix} \in S_\eta := \left\{ \begin{bmatrix} M_2 \otimes \Delta(\delta)(I_r - D\Delta(\delta))^{-1}C \\ I_l \otimes \widehat{\Delta}(\delta)J \end{bmatrix} \eta \; : \; \delta \in \mathcal{H} \right\}. \qquad (42)$$

Now, it is easily shown that

$$S_\eta := \left\{ \begin{bmatrix} p \\ \pi \end{bmatrix} = \begin{bmatrix} I \otimes \Delta(\delta) & 0 \\ 0 & I \otimes \widehat{\Delta}(\delta) \end{bmatrix} \begin{bmatrix} q \\ \rho \end{bmatrix} \; : \; \delta \in \mathcal{H} \right\}$$

with the notation

$$\begin{bmatrix} q \\ \rho \end{bmatrix} (\nu, p, \pi) := \begin{bmatrix} M_2 \otimes C & I_k \otimes D & 0 \\ I_l \otimes J & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ p \\ \pi \end{bmatrix}.$$

Using this simpler characterization, we can show that any $\begin{bmatrix} p \\ \pi \end{bmatrix} \in S_\eta$ satisfies

$$\begin{bmatrix} q \\ \rho \\ \hline p \\ \pi \end{bmatrix}^T \left[\begin{array}{c|c} S & Q \\ \hline Q^T & R \end{array}\right] \begin{bmatrix} q \\ \rho \\ \hline p \\ \pi \end{bmatrix} =$$

$$\begin{bmatrix} \eta \\ p \\ \pi \end{bmatrix}^T \left[\begin{array}{ccc} M_2 \otimes C & I \otimes D & 0 \\ I \otimes J & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^T \left[\begin{array}{c|c} S & Q \\ \hline Q^T & R \end{array}\right] \left[\begin{array}{ccc} M_2 \otimes C & I \otimes D & 0 \\ I \otimes J & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \begin{bmatrix} \eta \\ p \\ \pi \end{bmatrix} \geq 0 \qquad (43)$$

(Simply pre- and post-multiply (22) by $[\, q^T \quad \rho^T \,]$ and its transpose). Consequently,

$$S_\eta \subset \bar{S}_\eta = \left\{ \begin{bmatrix} p \\ \pi \end{bmatrix} \; : \; (43) \text{ holds} \right\}$$

and a sufficient condition for (40) is (42) with S_η replaced by $\bar{S}_\eta$, the latter being equivalent to (21) by a standard **S**-procedure argument [36, 12].

To complete the proof, we need to show that (41) holds for all $\delta \in \mathcal{H}$. Suppose that $X(\delta)w = 0$ for some $\delta \in \mathcal{H}$ and nonzero w. Then the left-hand side in (40) evaluates to zero when setting $\eta = [1, ..., 1]^T \otimes w$, a contradiction. Hence $X(\delta)$ cannot be singular over $\mathcal{H}$, which together with $X(0) = X_0 > 0$ guarantees (41).

Appendix C

$$A = \begin{bmatrix} -180 & 0 & 0 & 0 & 0 \\ 0 & -180 & 0 & 0 & 0 \\ -21.23 & 0 & -0.6888 & -14.7 & 0 \\ 256.7 & 0 & 122.6 & -1.793 & 0 \\ -52.33 & 304.7 & 0 & 36.7 & -9.661 \end{bmatrix}, \quad B = \begin{bmatrix} 180 & 0 \\ 0 & 180 \\ 0 & 0 \\ 256.7 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad D = 0$$

$$A_1 = \begin{bmatrix} 27 & 0 & 0 & 0 & 0 \\ 0 & 27 & 0 & 0 & 0 \\ 21.2 & 0 & 0.688 & 14.96 & 0 \\ 38.6 & 0 & 122.6 & 0 & 0 \\ 52.4 & 304.8 & 0 & 36.8 & 9.66 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 40.5 & 0 \\ 0 & 40.5 \\ 0 & 0 \\ 57.9 & 0 \\ 0 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.12090 & -0.06350 & 0.00000 \\ -0.06730 & -0.10380 & -0.03020 \end{bmatrix}$$

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