A Convex Characterization of Gain-Scheduled $\mathcal{H}_\infty$ Controllers

Pierre Apkarian *   *Pascal Gahinet* †

Abstract

An important class of linear time-varying systems consists of plants where the state-space matrices are fixed functions of some time-varying physical parameters θ. Small Gain techniques can be applied to such systems to derive robust time-invariant controllers. Yet, this approach is often overly conservative when the parameters θ undergo large variations during system operation. In general, much higher performance can be achieved by control laws that incorporate available measurements of θ and therefore "adjust" to the current plant dynamics.

This paper discusses extensions of $\mathcal{H}_\infty$ synthesis techniques to allow for controller dependence on time-varying but measured parameters. When this dependence is linear fractional, the existence of such gain-scheduled $\mathcal{H}_\infty$ controllers is fully characterized in terms of linear matrix inequalities (LMIs). The underlying synthesis problem is therefore a convex program for which efficient optimization techniques are available.

The formalism and derivation techniques developed here apply to both the continuous- and discrete-time problems. Existence conditions for robust time-invariant controllers are recovered as a special case, and extensions to gain-scheduling in the face of parametric uncertainty are discussed. In particular, simple heuristics are proposed to compute such controllers.

Key words: Gain-scheduling, $\mathcal{H}_\infty$ synthesis, Linear parameter-varying systems, Linear matrix inequalities.

1 Introduction

In most linear control problems, the real challenge is to raise and maintain performance in the presence of uncertainty. System uncertainty is essentially of two types:

- dynamical uncertainty, which corresponds to neglected plant dynamics (high-frequency behavior, non-linearities, etc);

*CERT/DERA, 2 Av Ed. Belin, 31055 Toulouse, France. Email: **apkarian@saturne.cert.fr**
†INRIA-Rocquencourt, BP 105, 78153 Le Chesnay cedex, France. Email: **gahinet@colorado.inria.fr**

- parametric uncertainty, which results either from inaccurate knowledge of the value of physical parameters, or from variations of this value during operation.

When the nominal model can be taken linear time-invariant, the $\mathcal{H}_\infty$ theory and its ramifications offer powerful synthesis tools for achieving robust performance (see, e.g., [10, 12, 24] and references thererin). Even though parametric uncertainty remains delicate to handle, the μ-synthesis [9, 5] and Lyapunov-based techniques [7, 27, 36, 17] give satisfactory results in many applications.

In comparison, available design techniques for uncertain linear time-varying (LTV) plants are relatively immature. Recall that a LTV plant is any linear system governed by state equations of the form:

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t) \\ y(t) &= C(t)\,x(t) + D(t)\,u(t) \end{aligned} \tag{1.1}$$

where the state-space matrices $A(t), B(t), C(t), D(t)$ vary in time. A solution of the corresponding $\mathcal{H}_\infty$ control problem has been proposed in [34, 28]. Yet, its implementation is often impractical since it requires real-time integration of Riccati differential equations. Alternatively, Small Gain LTI techniques can be applied to LTV plants whose time dependence assumes the form

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))\,x(t) + B(\theta(t))\,u(t) \\ y(t) &= C(\theta(t))\,x(t) + D(\theta(t))\,u(t) \end{aligned} \tag{1.2}$$

where $\theta(t)$ is a vector of time-varying plant parameters and $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are fixed functions of θ. Such plants will be referred to as linear parameter-varying (LPV) following the terminology of [31]. For tractability reasons, we further restrict our attention to LPV systems where $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are linear fractional functions of θ. Though not fully general, this class of LPV plants encompasses many relevant systems such as aiplanes, missiles, robots, etc. In the approach discussed below, no *a priori* information on $\theta(t)$ is required other than its range of variations.

For LPV plants, the Small Gain approach proceeds by treating the parameter variations as uncertainty and by designing a single robust controller for the resulting family of systems [9]. This is generally very conservative and can result in poor performance when the physical parameters undergo large deviations during system operation. In fact, plant stabilization by a single LTI controller may not even be feasible. When the parameter value is measured during operation, one way of reducing conservatism is to design robust controllers around each operating point and to switch between controllers according to some gain-scheduling policy [3]. This often provides a reasonable compromise between performance and robustness, yet at the expense of higher complexity and delicate stability questions in the switching zone (see [31, 32] and references therein).

The present paper discusses an alternative approach based on the concept of parameter-dependent $\mathcal{H}_\infty$ controllers. Such controllers depend on the varying parameters $\theta(t)$ through

$$\begin{aligned} \dot{\zeta}(t) &= A_K(\theta(t))\,\zeta(t) + B_K(\theta(t))\,y(t) \\ u(t) &= C_K(\theta(t))\,\zeta(t) + D_K(\theta(t))\,y(t). \end{aligned} \tag{1.3}$$

where $A_K(\cdot), B_K(\cdot), C_K(\cdot), D_K(\cdot)$ are linear fractional functions of θ. Introduced by Packard *et al.* in [26], this new and promising control structure is applicable whenever the value of $\theta(t)$ is measured at each time t. The resulting controller is time-varying and smoothly "scheduled" by the measurements of $\theta(t)$. Thanks to their adaptive nature, such controllers can achieve higher performance than classical robust LTI controllers. Moreover, they can be implemented at little or no extra cost. In the sequel, the terms gain-scheduled, parameter-dependent, and LPV will be used interchangeably to refer to control laws of structure (1.3).

The synthesis of gain-scheduled $\mathcal{H}_\infty$ controllers relies on the Small Gain Theorem [35, 9]. Tractability of the discrete-time problem by means of linear matrix inequalities (LMI) was established in [26, 18]. In this paper, we present a direct and complete solution for the continuous-time output feedback problem and propose a unifying LMI approach for both continuous- and discrete-time contexts. Simpler feasibility conditions are obtained as well as a characterization of all adequate controllers. Our derivation technique parallels that of [14] and involves only straightforward linear algebra manipulations on positive definite matrices. Finally, note that this LMI-based approach is practical since efficient interior-point algorithms are now available to numerically solve LMI problems [20, 21, 6, 13, 22].

The paper is organized as follows. Section 2 gives a precise statement of the linear parameter-dependent $\mathcal{H}_\infty$ control problem. Invoking standard results from Small Gain Theory, this original problem is recast as one of robust performance in the face of structured uncertainty. In Section 4, the general robust performance problem is solved using LMI techniques and standard tools such as the scaled Bounded Real Lemma, In Section 5, these results are specialized to the $\mathcal{H}_\infty$ gain-scheduling problem to derive the main result of the paper. That is, a complete and tractable characterization of all LPV controllers achieving some prescribed performance level γ. The computation of LPV controllers is discussed in Section 6 with an emphasis on the well-posedness issue. Finally, Section 7 compares gain-scheduled LPV and robust LTI controllers and Section 8 discusses mixed problems where only a few of the varying parameters are measured while the others must be considered as uncertain.

Linear fractional transformations (LFT) are used extensively in the sequel. For appropriately dimensioned matrices K and $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ and assuming the inverses exist, the lower LFT is defined as

$$F_l(M,K) = M_{11} + M_{12}K(I - M_{22}K)^{-1}M_{21} \tag{1.4}$$

and the upper LFT is defined as

$$F_u(M,K) = M_{22} + M_{21}K(I - M_{11}K)^{-1}M_{12}. \tag{1.5}$$

For a stable real-rational transfer function matrix G, the $\mathcal{H}_\infty$ norm is defined in the usual way

- $\|G(s)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega))$ for continuous-time systems

- $\|G(z)\|_\infty = \sup_{\theta\in[0,2\pi]} \bar{\sigma}(G(e^{j\theta}))$ for discrete-time systems

where $\bar{\sigma}(M)$ stands for the largest singular value of a matrix M. With a slight abuse, $\|T\|_\infty$ will also denote the L_2-induced norm of a general operator T. That is,

$$\|T\|_\infty = \sup_{u\in L_2^*} \frac{\|Tu\|}{\|u\|}$$

where L_2 is the space of square-integrable signals.

For real symmetric matrices M, the notation $M > 0$ stands for "positive definite" and means that all the eigenvalues of M are positive. Similarly, $M < 0$ means negative definite, that is, all eigenvalues of M are negative. For $M > 0$, $M^{1/2}$ will denote the unique positive definite square root. Finally, for an arbitrary matrix P, $Ker(P)$ stands for the null space of the linear operator associated with P.

2 Gain-Scheduled $\mathcal{H}_\infty$ Control

In the sequel, σ stands for the Laplace variable s in the continuous-time context and for the Z transform variable z in the discrete-time context. Similarly, τ stands for the time $t \in \mathbb{R}$ in the continuous-time case and for the sample $k \in \mathbb{Z}$ in the discrete-time case. The parameter vector is denoted by

$$\theta_\tau = (\theta_1, \ldots, \theta_K) \in \mathbb{R}^K$$

to mark its time dependence. The material in this section is essentially borrowed from [23].

2.1 LPV control structure

LPV plants with a linear fractional dependence on θ_τ can be represented by the upper LFT interconnection

$$\begin{pmatrix} q \\ y \end{pmatrix} = F_u(P(\sigma), \Theta_\tau) \begin{pmatrix} w \\ u \end{pmatrix} \tag{2.1}$$

where $P(\sigma)$ is a known LTI plant and Θ_τ is some block diagonal operator specifying how θ_τ enters the plant dynamics. Specifically,

$$\Theta_\tau = \text{blockdiag}\,(\theta_1 I_{r_1}, \ldots, \theta_K I_{r_K}) \tag{2.2}$$

where $r_i > 1$ whenever the parameter θ_i is repeated [9]. The set of operators with structure (2.2) will be denoted by

$$\Delta := \{ \text{blockdiag}\,(\theta_1 I_{r_1}, \ldots, \theta_K I_{r_K}) \;:\; \theta_i(\tau) \in \mathbb{R} \}. \tag{2.3}$$

Note that Δ is traditionally referred to as the uncertainty structure.

The feedback equations associated with the LFT interconnection (2.1) read:

$$\begin{pmatrix} q_\theta(\sigma) \\ q(\sigma) \\ y(\sigma) \end{pmatrix} = \overbrace{\begin{pmatrix} P_{\theta\theta}(\sigma) & P_{\theta 1}(\sigma) & P_{\theta 2}(\sigma) \\ P_{1\theta}(\sigma) & P_{11}(\sigma) & P_{12}(\sigma) \\ P_{2\theta}(\sigma) & P_{21}(\sigma) & P_{22}(\sigma) \end{pmatrix}}^{P(\sigma)} \begin{pmatrix} w_\theta(\sigma) \\ w(\sigma) \\ u(\sigma) \end{pmatrix} \tag{2.4}$$

$$w_\theta = \Theta_\tau \, q_\theta. \tag{2.5}$$

Note that w_θ, q_θ can be interpreted as the inputs/outputs of the time-varying operator Θ_τ. At each time τ, the LPV plant defines a tangent LTI plant of transfer function:

$$\begin{pmatrix} q \\ y \end{pmatrix} = F_u(P, \theta_\tau) \begin{pmatrix} w \\ u \end{pmatrix} = \left\{ \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} + \begin{pmatrix} P_{1\theta} \\ P_{2\theta} \end{pmatrix} \theta_\tau (I - P_{\theta\theta}\theta_\tau)^{-1} (P_{\theta 1} \; , \; P_{\theta 2}) \right\} \begin{pmatrix} w \\ u \end{pmatrix}. \tag{2.6}$$

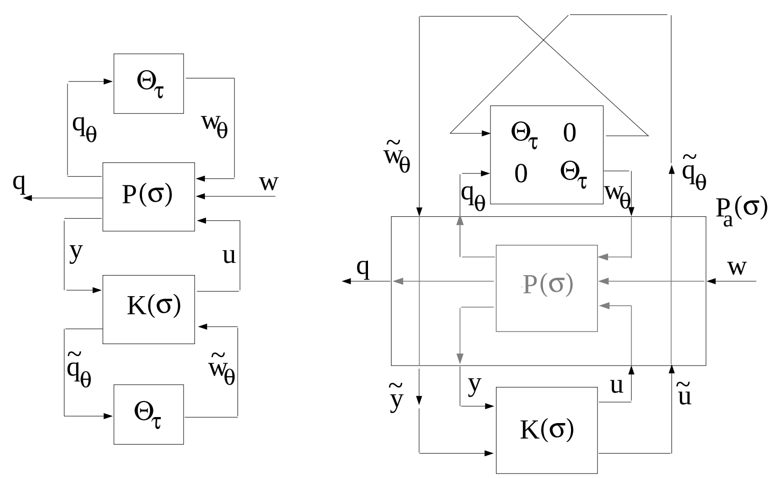

Figure 2.1: a) LPV control structure – b) Transformed structure

Consistently with (2.1), we seek LPV controllers of the form

$$u = F_l(K(\sigma), \Theta_\tau) y \tag{2.7}$$

where the LTI system

$$K(\sigma) = \begin{pmatrix} K_{11}(\sigma) & K_{1\theta}(\sigma) \\ K_{\theta 1}(\sigma) & K_{\theta\theta}(\sigma) \end{pmatrix} \tag{2.8}$$

specifies the LFT dependence of the controller on the measurements of θ_τ. Recalling that θ_τ plays the role of scheduling variable, (2.7) gives the rule for updating the controller state-space matrices based on the measurements of θ_τ. For more details on the implementation of LPV controllers see Section 6.

The overall LFT interconnection is depicted in Figure 2.1a). Note that the closed-loop operator from disturbance w to controlled output q is given by:

$$T(P,K,\Theta_\tau) = F_l(\, F_u(P,\Theta_\tau),\; F_l(K,\Theta_\tau)\,). \tag{2.9}$$

2.2 $\mathcal{H}_\infty$ control of LPV systems

Given some LTI plant $P(\sigma)$ mapping exogenous inputs w and control inputs u to controlled outputs q and measured outputs y, the usual $\mathcal{H}_\infty$ control problem is concerned with finding an internally stabilizing LTI controller $K(\sigma)$ such that

$$\|F_l(P,K)\|_\infty < \gamma$$

where γ is some prescribed performance level.

The gain-scheduled version of this problem has a similar statement, except that both the plant and the controller are now LPV instead of LTI. Here the objective is to guarantee some closed-loop performance $\gamma > 0$ from w to q for all admissible parameter trajectories θ_τ. Assuming that any such trajectory is bounded, the $\mathcal{H}_\infty$ control problem for LPV systems can be formulated as follows:

Find a control structure $K(\sigma)$ such that the LPV controller $F_l(K(\sigma),\Theta_\tau)$ satisfies:

- the closed-loop system (2.9) is internally stable for all parameter trajectories θ_τ such that $\gamma^2\theta_\tau^T\theta_\tau \le 1$,
- the induced $\mathcal{L}_2$-norm of the operator $T(P,K,\Theta_\tau)$ satisfies:

$$\max_{\|\Theta_\tau\|_\infty \le 1/\gamma} \|T(P,K,\Theta_\tau)\|_\infty < \gamma. \tag{2.10}$$

In this statement, the parameter range is restricted to the ball of radius $1/\gamma$ for practical reasons. Note that this implies no loss of generality since θ_τ can always be scaled to comply with this requirement. Such rescaling merely amounts to redefining the inputs w_θ of $P(\sigma)$. Solutions to this $\mathcal{H}_\infty$ problem (if any) will be called γ-suboptimal gain-scheduled controllers.

A particularity of the $\mathcal{H}_\infty$ gain-scheduling problem is that the varying parameters enter both the plant and the controller. To apprehend this problem with Small Gain Theory, we must first gather all parameter-dependent components into a single uncertainty block. Introducing the augmented plant

$$\begin{pmatrix} \tilde{q}_\theta \\ q_\theta \\ q \\ y \\ \tilde{w} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 0 & I_r \\ 0 & P(\sigma) & 0 \\ I_r & 0 & 0 \end{pmatrix}}^{P_a(\sigma)} \begin{pmatrix} \tilde{w}_\theta \\ w_\theta \\ w \\ u \\ \tilde{u} \end{pmatrix}, \tag{2.11}$$

it is readily verified that the closed-loop mapping from exogenous inputs w to controlled outputs q can be expressed as:

$$T(P,K,\Theta_\tau) = F_u\left(F_l(P_a(\sigma),\, K(\sigma))\,,\, \begin{pmatrix} \Theta_\tau & 0 \\ 0 & \Theta_\tau \end{pmatrix}\right). \tag{2.12}$$

This alternative expression amounts to redrawing the interconnection of Figure 2.1 a) as in Figure 2.1 b). By inspection of this second figure, we see that the original LPV problem can be viewed as a more classical robust performance problem in the face of the block-repeated uncertainty structure $\begin{pmatrix} \Theta_\tau & 0 \\ 0 & \Theta_\tau \end{pmatrix}$. With Δ defined by (2.3), this repeated structure will be denoted $\Delta \oplus \Delta$ throughout the paper.

In the light of the reformulation (2.12), the LPV problem can be interpreted as a robust performance problem for the nominal LTI plant $T(P,K,0) = P_a$ in the face of the norm-bounded uncertainty $\Delta \oplus \Delta$. Sufficient conditions for solvability are then provided by Small Gain Theory [35, 9]. Specifically, consider the set of positive definite similarity scalings associated with the structure Δ in (2.3):

$$L_\Delta = \{L > 0 \;:\; L\Theta = \Theta L,\ \forall \Theta \in \Delta\} \subset \mathbb{R}^{r\times r} \ \text{ with } \ r = \sum_{i=1}^{K} r_i. \tag{2.13}$$

This set enjoys the following immediate properties:

(P1) $I_r \in L_\Delta$

(P2) $L \in L_\Delta \Rightarrow L^T \in L_\Delta$

(P3) $L \in L_\Delta \Rightarrow L^{-1} \in L_\Delta$

(P4) $L_1 \in L_\Delta,\ L_2 \in L_\Delta \Rightarrow L_1L_2\Theta = \Theta L_1L_2,\ \forall \Theta \in \Delta$

(P5) L_Δ is a convex subset of $\mathbb{R}^{r\times r}$.

Given L_Δ, the set of scalings commuting with the repeated structure $\Delta \oplus \Delta$ is readily deduced as:

$$L_{\Delta\oplus\Delta} = \left\{ \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix} > 0 \;:\; L_1, L_3 \in L_\Delta \text{ and } L_2\Theta = \Theta L_2, \forall \Theta \in \Delta \right\}. \tag{2.14}$$

From Small Gain Theory, a sufficient condition for robust performance in the face of the uncertainty $\Delta \oplus \Delta$, or equivalently for the existence of gain-scheduled controllers, is as follows.

Theorem 2.1 *Consider an uncertainty structure Δ and the associated set of similarity scalings $L_{\Delta\oplus\Delta}$ defined in (2.14). If there exists a scaling matrix $L \in L_{\Delta\oplus\Delta}$ and an LTI control structure $K(\sigma)$ such that the nominal closed-loop system $F_l(P_a(\sigma), K((\sigma))$ is internally stable and satisfies*

$$\left\| \begin{pmatrix} L^{1/2} & 0 \\ 0 & I \end{pmatrix} F_l(P_a(\sigma),\, K(\sigma)) \begin{pmatrix} L^{-1/2} & 0 \\ 0 & I \end{pmatrix} \right\|_\infty < \gamma, \tag{2.15}$$

then $F_l(K(\sigma), \Theta_\tau)$ is a γ-suboptimal gain-scheduled $\mathcal{H}_\infty$ controller.

Proof: The proof is a straightforward application of the Small Gain Theorem. See [8, 35, 30] for more details. □

The problem stated in Theorem 2.1 is a particular case of the general scaled $\mathcal{H}_\infty$ problem considered in Section 4. Recall that the parameter vector θ_τ is assumed to range in the ball of radius $1/\gamma$. Note also that the operator $F_l(K(\sigma), \Theta_\tau)$ may not be causal. This issue is discussed in more details in Section 6. The problem discussed in the remainder of this paper is that of computing, for a given $\gamma > 0$, an adequate scaling matrix L and a control structure $K(\sigma)$ such that (2.15) holds.

2.3 State-space set-up

The LMI approach discussed in this paper is state-space-based. To set it up, consider some minimal realization of the LTI plant $P(\sigma)$:

$$P(\sigma) = \begin{pmatrix} D_{\theta\theta} & D_{\theta 1} & D_{\theta 2} \\ D_{1\theta} & D_{11} & D_{12} \\ D_{2\theta} & D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_\theta \\ C_1 \\ C_2 \end{pmatrix} (\sigma I - A)^{-1} (B_\theta \quad B_1 \quad B_2) \tag{2.16}$$

where the partitioning is conformable to (2.4). The problem dimensions are given by:

$$A \in \mathbb{R}^{n\times n},\ D_{\theta\theta} \in \mathbb{R}^{r\times r},\ D_{11} \in \mathbb{R}^{p_1\times p_1},\ D_{22} \in \mathbb{R}^{p_2\times m_2}. \tag{2.17}$$

Throughout the paper, we will only assume that

(A1) (A, B_2, C_2) is stabilizable and detectable,

(A2) $D_{22} = 0$.

The first assumption is necessary and sufficient to allow stabilization of the plant by dynamic output feedback and **(A2)** incurs no loss of generality while considerably simplifying the calculations. Note also that the parameter matrix θ as well as the transfer function from the disturbance w to the controlled outputs q have been considered to be square in (2.17). This can be always fulfilled by augmenting the problem with columns and/or rows of zeros. This operation greatly simplifies the notation and the manipulations of the similarity scaling matrices defined in (2.13)

From (2.16), a state-space realization of $P_a(\sigma)$ is readily derived as

$$P_a(\sigma) = \begin{pmatrix} 0 & 0 & 0 & 0 & I_r \\ 0 & D_{\theta\theta} & D_{\theta 1} & D_{\theta 2} & 0 \\ 0 & D_{1\theta} & D_{11} & D_{12} & 0 \\ 0 & D_{2\theta} & D_{21} & 0 & 0 \\ I_r & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ C_\theta \\ C_1 \\ C_2 \\ 0 \end{pmatrix} (\sigma I - A)^{-1} (0\ ,\ B_\theta\ ,\ B_1\ ,\ B_2\ ,\ 0). \tag{2.18}$$

Similarly, realizations of the control structure $K(\sigma)$ will be denoted by

$$K(\sigma) = \begin{pmatrix} D_{K11} & D_{K1\theta} \\ D_{K\theta 1} & D_{K\theta\theta} \end{pmatrix} + \begin{pmatrix} C_{K1} \\ C_{K\theta} \end{pmatrix} (\sigma I - A_K)^{-1} (B_{K1}\ ,\ B_{K\theta}), \qquad A_K \in \mathbb{R}^{k\times k} \tag{2.19}$$

Note that the order k of $K(\sigma)$ is arbitrary at this point.

3 Scaled Bounded Real Lemmas

This LMI approach developed in this paper relies heavily on the Bounded Real Lemma as a means of turning $\mathcal{H}_\infty$ constraints into LMIs. This section states the Bounded Real Lemma for the scaled $\mathcal{H}_\infty$ problem in both continuous- and discrete-time contexts.

Lemma 3.1 (Continuous-time case)
Consider a parameter structure Δ, the associated scaling set L_Δ defined in (2.13), and a square continuous-time transfer function $T(s)$ of realization $T(s) = D + C(sI - A)^{-1}B$. The following statement are equivalent:

(i) *A is stable and there exists $L \in L_\Delta$ such that $\|L^{1/2}(D + C(sI - A)^{-1}B)L^{-1/2}\|_\infty < \gamma$*

(ii) *there exist positive definite solutions X and $L \in L_\Delta$ to the matrix inequality:*

$$\begin{pmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma L & D^T \\ C & D & -\gamma L^{-1} \end{pmatrix} < 0. \tag{3.1}$$

Proof: Omitted for brevity, see e.g., [14] and references therein. □

Lemma 3.2 (Discrete-time case)
Consider a parameter structure Δ, the associated scaling set L_Δ defined in (2.13), and a square discrete-time transfer function $T(z)$ of realization $T(z) = D + C(zI - A)^{-1}B$. The following statements are equivalent:

(i) *A is stable and there exists $L \in L_\Delta$ such that $\|L^{1/2}(D + C(zI - A)^{-1}B)L^{-1/2}\|_\infty < \gamma$*

(ii) *there exist positive definite solutions X and $L \in L_\Delta$ to the matrix inequality:*

$$\begin{pmatrix} -X^{-1} & A & B & 0 \\ A^T & -X & 0 & C^T \\ B^T & 0 & -\gamma L & D^T \\ 0 & C & D & -\gamma L^{-1} \end{pmatrix} < 0. \tag{3.2}$$

Proof: Omitted for brevity, see e.g., [14] and references therein. □

4 Solution of General Scaled $\mathcal{H}_\infty$ Problems

By means of Theorem 2.1, we have replaced our original LPV problem by a scaled $\mathcal{H}_\infty$ problem, admittedly with some degree of conservatism. In this section, all solutions of general scaled $\mathcal{H}_\infty$ problems are characterized for LTI plants and arbitrary uncertainty structures Δ. Consider a proper continuous- or discrete-time LTI plant $G(\sigma)$ mapping exogenous inputs w and control inputs u to controlled outputs q and measured outputs y as:

$$\begin{pmatrix} q(\sigma) \\ y(\sigma) \end{pmatrix} = \begin{pmatrix} G_{11}(\sigma) & G_{12}(\sigma) \\ G_{21}(\sigma) & G_{22}(\sigma) \end{pmatrix} \begin{pmatrix} w(\sigma) \\ u(\sigma) \end{pmatrix}, \quad \sigma = s, z. \tag{4.1}$$

The general statement of scaled $\mathcal{H}_\infty$ problems is as follows:

> Given $\gamma > 0$, an uncertainty structure Δ, and the associated scaling set L_Δ defined in (2.13), find $L \in L_\Delta$ and a LTI controller $K(\sigma)$ such that the closed-loop system is internally stable and
>
> $$\|L^{1/2}\, F_l(G(\sigma),\, K(\sigma))\, L^{-1/2}\|_\infty < \gamma. \tag{4.2}$$

This problem was solved for the state-feedback case in [25] and a generalization to the output feedback case is derived next.

Consider a state-space realization

$$G(\sigma) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (\sigma I - A)^{-1} (B_1 \quad B_2) \tag{4.3}$$

of the plant G where the dimensions of the problem are described in (2.17) and assumptions **(A1)** and **(A2)** hold. Combining the main result of [14] and the scaled Bounded Real Lemmas of Section 3, solvability of the general scaled $\mathcal{H}_\infty$ problem can be characterized as follows.

Theorem 4.1 (Continuous-time case)
With $G(s)$, Δ, and L_Δ defined as above and the realization (4.3) of $G(s)$, let $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null spaces of $(B_2^T, D_{12}^T, 0)$ and $(C_2, D_{21}, 0)$, respectively. With this notation, the suboptimal scaled $\mathcal{H}_\infty$ problem is solvable if and only if there exist pairs of symmetric matrices (R, S) in $\mathbb{R}^{n\times n}$ and (L, J) in $\mathbb{R}^{p_1\times p_1}$ such that

$$\mathcal{N}_R^T \begin{pmatrix} AR + RA^T & RC_1^T & B_1 \\ C_1 R & -\gamma J & D_{11} \\ B_1^T & D_{11}^T & -\gamma L \end{pmatrix} \mathcal{N}_R < 0 \tag{4.4}$$

$$\mathcal{N}_S^T \begin{pmatrix} A^T S + SA & SB_1 & C_1^T \\ B_1^T S & -\gamma L & D_{11}^T \\ C_1 & D_{11} & -\gamma J \end{pmatrix} \mathcal{N}_S < 0 \tag{4.5}$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0 \tag{4.6}$$

$$L \in L_\Delta,\ J \in L_\Delta,\ LJ = I. \tag{4.7}$$

Moreover, there exist suboptimal controllers of order k if and only if (4.4)–(4.7) hold for some quadruple (R, S, L, J) where R, S further satisfy the rank constraint

$$\operatorname{rank}(I - RS) \leq k. \tag{4.8}$$

Proof: See Appendix A. □

The discrete-time version of the Bounded Real Lemma leads to similar solvability conditions for discrete-time systems. These conditions are summarized in the next Theorem and included for the sake of completeness.

Theorem 4.2 (Discrete-time case)
Given a discrete-time plant $G(z)$ and with the notation of Theorem 4.1, the suboptimal scaled $\mathcal{H}_\infty$ problem is solvable if and only if there exist pairs of symmetric matrices (R, S) in $\mathbb{R}^{n\times n}$ and (L, J) in $\mathbb{R}^{p_1\times p_1}$ such that

$$\mathcal{N}_R^T \begin{pmatrix} ARA^T - R & ARC_1^T & B_1 \\ C_1RA^T & -\gamma J + C_1RC_1^T & D_{11} \\ B_1^T & D_{11}^T & -\gamma L \end{pmatrix} \mathcal{N}_R < 0 \tag{4.9}$$

$$\mathcal{N}_S^T \begin{pmatrix} A^TSA - S & A^TSB_1 & C_1^T \\ B_1^TSA & -\gamma L + B_1^TSB_1 & D_{11}^T \\ C_1 & D_{11} & -\gamma J \end{pmatrix} \mathcal{N}_S < 0 \tag{4.10}$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0 \tag{4.11}$$

$$L \in L_\Delta,\ J \in L_\Delta,\ LJ = I. \tag{4.12}$$

Moreover, there exist suboptimal controllers of order k if and only if (4.9)–(4.12) hold for some quadruple (R, S, L, J) where R, S further satisfy the rank constraint

$$\operatorname{rank}(I - RS) \leq k. \tag{4.13}$$

□

Comments on numerical tractability: The inequalities (4.4)–(4.6) for the continuous-time case and (4.9)–(4.11) for the discrete-time case are LMIs in R, S, L, J and the structure constraints $L \in L_\Delta$ and $J \in L_\Delta$ are convex constraints. In the full order case $k \geq n$ moreover, the rank constraints (4.8) and (4.13) are trivially satisfied. However, the constraints (4.7) and (4.12) are strongly non-convex and numerical tractability of these conditions is therefore unclear. Devising algorithms that generate solutions to the problems (4.4)–(4.7) or (4.9)–(4.12) remains a challenging issue because of the importance of output feedback and structured uncertainty in control applications.

Convex solvability conditions for $\mathcal{H}_\infty$ synthesis: The classical $\mathcal{H}_\infty$ control problem corresponds to $\Delta = \mathbb{C}^{p_1 \times p_1}$. The corresponding similarity scaling set is

$$L_\Delta = \{\lambda\, I_{p1} : \lambda \in \mathbb{R},\ \lambda > 0\}$$

and L, J can be set to the identity matrix without loss of generality. In the full order case, the conditions of Theorems 4.1-4.2 then reduce to a system of three LMIs and define a convex program. This LMI-based solvability conditions are numerically tractable in an efficient way and have the merit of eliminating all difficulties related to imaginary axis zeros and rank deficiencies in D_{12} and D_{21}. See [14] for a complete discussion.

5 Gain-Scheduled $\mathcal{H}_\infty$ Synthesis

From Section 2, gain-scheduled $\mathcal{H}_\infty$ synthesis can be recast as a scaled $\mathcal{H}_\infty$ control problem. In this section, we specialize the general results of Section 4 to derive sufficient conditions for feasibility of the LPV design. Thanks to the particular structure of the LPV problem, the difficulty with the non-convex constraint

$$LJ = I$$

in Theorem 4.1 entirely disappears. As a result, we obtain sufficient conditions that are pure LMIs and are therefore numerically tractable.

A complete treatment of the discrete-time case can be found in [23, 18] and we therefore restrict our attention to the continuous-time case. In order to simplify the presentation, the following shorthands will be used hereafter:

$$\hat{B}_1 = (B_\theta, B_1), \quad \hat{C}_1 = \begin{pmatrix} C_\theta \\ C_1 \end{pmatrix}, \quad \hat{D}_{11} = \begin{pmatrix} D_{\theta\theta} & D_{\theta 1} \\ D_{1\theta} & D_{11} \end{pmatrix}. \tag{5.1}$$

Theorem 5.1 *Consider an LPV plant given by the LFT interconnection (2.1) where $P(s)$ is a proper continuous-time LTI plant with minimal realization (2.16), and Θ_τ is the parameter operator given by (2.2). Let Δ denote the structure set associated with Θ_τ and L_Δ denote the corresponding set of scaling matrices defined by (2.13). Finally, assume* **(A1)–(A2)** *and let $\mathcal{N}_R$ and $\mathcal{N}_S$ be arbitrary bases of the null spaces of $(B_2^T, D_{\theta 2}^T, D_{12}^T, 0)$ and $(C_2, D_{2\theta}, D_{21}, 0)$, respectively.*

With this notation and assumptions, the gain-scheduled $\mathcal{H}_\infty$ control problem of Section 2 is solvable if there exist pairs of symmetric matrices (R, S) in $\mathbb{R}^{n\times n}$ and (L_3, J_3) in $\mathbb{R}^{r\times r}$ such that

$$\mathcal{N}_R^T \begin{pmatrix} AR + RA^T & R\hat{C}_1^T & \hat{B}_1 \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \\ \hat{C}_1 R & -\gamma \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} & \hat{D}_{11} \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \hat{B}_1^T & \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \hat{D}_{11}^T & -\gamma \begin{pmatrix} J_3 & 0 \\ 0 & I \end{pmatrix} \end{pmatrix} \mathcal{N}_R < 0 \tag{5.2}$$

$$\mathcal{N}_S^T \begin{pmatrix} A^TS+SA & S\hat{B}_1 & \hat{C}_1^T \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \\ \hat{B}_1^T S & -\gamma \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} & \hat{D}_{11}^T \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \hat{C}_1 & \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \hat{D}_{11} & -\gamma \begin{pmatrix} L_3 & 0 \\ 0 & I \end{pmatrix} \end{pmatrix} \mathcal{N}_S < 0 \tag{5.3}$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0 \tag{5.4}$$

$$L_3 \in L_\Delta, \quad J_3 \in L_\Delta, \quad \begin{pmatrix} L_3 & I \\ I & J_3 \end{pmatrix} \geq 0. \tag{5.5}$$

Moreover, there exist γ-suboptimal controllers of order k if (5.2)–(5.5) hold for some quadruple (R, S, L_3, J_3) where R, S further satisfy the rank constraint

$$\operatorname{rank}(I - RS) \leq k. \tag{5.6}$$

Proof: The proof is a straightforward specialization of Theorem 4.1 to the problem (2.15). The conditions (5.2)–(5.4) follow from (4.3)–(4.5) when performing the following substitutions in the state-space realization of $G(s)$:

$$C_1 \to \begin{pmatrix} 0 \\ C_\theta \\ C_1 \end{pmatrix}, \ B_1 \to (0 \ B_\theta \ B_1), \ C_2 \to \begin{pmatrix} C_2 \\ 0 \end{pmatrix}, \ B_2 \to (B_2 \ 0) \tag{5.7}$$

$$\begin{array}{ll} D_{11} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_{\theta\theta} & D_{\theta 1} \\ 0 & D_{1\theta} & D_{11} \end{pmatrix}, & D_{12} \to \begin{pmatrix} 0 & I \\ D_{\theta 2} & 0 \\ D_{12} & 0 \end{pmatrix} \\ D_{21} \to \begin{pmatrix} 0 & D_{2\theta} & D_{21} \\ I & 0 & 0 \end{pmatrix}, & D_{22} \to \begin{pmatrix} D_{22} & 0 \\ 0 & 0 \end{pmatrix} \end{array} \tag{5.8}$$

Note that the A matrix remains unchanged and that $\mathcal{N}_R$ and $\mathcal{N}_S$ become the null spaces of $(B_2^T, D_{\theta 2}^T, D_{12}^T)$ and $(C_2, D_{2\theta}, D_{21})$, respectively.

Meanwhile, the condition (5.5) follows from (4.7) of Theorem 4.1. To see this, first note that the similarity scalings are modified to

$$L \to \begin{pmatrix} L_1 & L_2 & 0 \\ L_2^T & L_3 & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad J \to \begin{pmatrix} J_1 & J_2 & 0 \\ J_2^T & J_3 & 0 \\ 0 & 0 & I \end{pmatrix}$$

with

$$\begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix} \in L_{\Delta\oplus\Delta}, \qquad \begin{pmatrix} J_1 & J_2 \\ J_2^T & J_3 \end{pmatrix} \in L_{\Delta\oplus\Delta}.$$

Since L_1, L_2, J_1, and J_2 are not involved in (5.2)-(5.4), the constraint $LJ = I$ reduces to the spectral condition $\lambda_{\min}(J_3 L_3) \geq 1$ (see [26] for details). This together with the positive definiteness of L and J yields

$$\begin{pmatrix} L_3 & I \\ I & J_3 \end{pmatrix} \geq 0.$$

As for the constraints $L_3 \in L_\Delta$ and $J_3 \in L_\Delta$, they are direct consequences of the definition (2.14) of $L_{\Delta\oplus\Delta}$. □

Note that the discrete-time counterpart of this result can be derived from Theorem 4.2 by exactly the same manipulations.

Convexity of the solvability conditions In the full-order case $k \geq n$, the sufficient solvability conditions of Theorem 5.1 form a system of LMIs with variables R, S, L_3, J_3. Testing the feasibility of these conditions is therefore a convex program which is readily solve using interior-point LMI solvers [33, 20, 21, 6, 22] and software like LMI-LAB [13, 16]. Note that minimizing γ subject to the feasibility of these constraints is also an LMI problem [14]. Hence it is possible to directly optimize the closed-loop performance γ.

6 Computation of the Gain-Scheduled Controller

The results of Section 5 provide sufficient conditions for the existence of gain-scheduled $\mathcal{H}_\infty$ controllers but do not address the actual computation of such controllers. We now turn to this issue and discuss various aspects of the controller computation and implementation.

Recall from Section 2 that the gain-scheduled controller is specified by the LTI system $K(\sigma)$ of (2.8). Denote by

$$K(\sigma) = \begin{pmatrix} D_{K11} & D_{K1\theta} \\ D_{K\theta 1} & D_{K\theta\theta} \end{pmatrix} + \begin{pmatrix} C_{K1} \\ C_{K\theta} \end{pmatrix} (\sigma I - A_K)^{-1} (B_{K1}\ ,\ B_{K\theta})\ , \qquad A_K \in \mathbb{R}^{k\times k} \tag{6.1}$$

any realization of $K(\sigma)$ and let

$$\Omega := \begin{pmatrix} A_K & B_{K1} & B_{K\theta} \\ C_{K1} & D_{K11} & D_{K1\theta} \\ C_{K\theta} & D_{K\theta 1} & D_{K\theta\theta} \end{pmatrix} \in \mathbb{R}^{(k+m_2+r)\times(k+p_2+r)}. \tag{6.2}$$

For notational simplicity, the following shorthands are used in the sequel:

$$\begin{gathered} A_0 = \begin{pmatrix} A & 0 \\ 0 & 0_{k\times k} \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & B_\theta & B_1 \\ 0_{k\times r} & 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & B_2 & 0 \\ I_k & 0 & 0_{k\times r} \end{pmatrix} \\ C_0 = \begin{pmatrix} 0 & 0_{r\times k} \\ C_\theta & 0 \\ C_1 & 0 \end{pmatrix}, \quad \mathcal{D}_{11} = \begin{pmatrix} 0_{r\times r} & 0 & 0 \\ 0 & D_{\theta\theta} & D_{\theta 1} \\ 0 & D_{1\theta} & D_{11} \end{pmatrix}, \quad \mathcal{D}_{12} = \begin{pmatrix} 0_{r\times k} & 0 & I_r \\ 0 & D_{\theta 2} & 0 \\ 0 & D_{12} & 0 \end{pmatrix} \\ \mathcal{C} = \begin{pmatrix} 0 & I_k \\ C_2 & 0 \\ 0 & 0_{r\times k} \end{pmatrix}, \quad \mathcal{D}_{21} = \begin{pmatrix} 0_{k\times r} & 0 & 0 \\ 0 & D_{2\theta} & D_{21} \\ I_r & 0 & 0 \end{pmatrix}. \end{gathered} \tag{6.3}$$

6.1 Computation of the controller matrices

Solving the characteristic LMI conditions of Theorem 5.1 for some feasible performance γ provides a quadruple of symmetric matrices (R, S, L_3, J_3). A systematic procedure for deriving the state-space data Ω of some gain-scheduled controller from (R, S, L_3, J_3) is outlined next. This algorithm involves solving one extra LMI and parallels the algorithm described in [14] for pure $\mathcal{H}_\infty$ control. Note that the feasible set of (5.2)–(5.5) parametrizes all solutions $K(\sigma)$ of the scaled $\mathcal{H}_\infty$ problem stated in Theorem 2.1 [14].

Algorithm 6.1 Given any solution (R, S, L_3, J_3) of the LMI system (5.2)–(5.5), the state-space data Ω of some γ-suboptimal $K(\sigma)$ can be computed as follows:

- From R, S derive the Bounded Real Lemma matrix $X_{cl} > 0$ by
  1. computing via SVD two full-column-rank matrices $M, N \in \mathbb{R}^{n\times k}$ such that
  $$MN^T = I - RS \tag{6.4}$$
  2. computing X_{cl} as the unique solution of the linear matrix equation (see [14]):
  $$X_{cl}\begin{pmatrix} I & R \\ 0 & M^T \end{pmatrix} = \begin{pmatrix} S & I \\ N^T & 0 \end{pmatrix}. \tag{6.5}$$
- Compute two matrices $L_1 \in L_\Delta$ and L_2 commuting with the structure Δ such that
  $$L := \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix} > 0, \qquad L^{-1} = \begin{pmatrix} * & * \\ * & J_3 \end{pmatrix}.$$
  The matrix L can be computed in the same way as X_{cl} above, except that the SVD (6.4) must now be performed on each diagonal block of $I - L_3 J_3$ separately (see [11] for more details).
- Set
  $$\mathcal{L} := \begin{pmatrix} L & 0 \\ 0 & I_{p_1} \end{pmatrix}, \qquad \mathcal{J} := \mathcal{L}^{-1}.$$
- Solve for Ω the LMI
  $$\Psi + \begin{pmatrix} X_{cl} & 0 \\ 0 & I \end{pmatrix} P^T \Omega Q + Q^T \Omega^T P \begin{pmatrix} X_{cl} & 0 \\ 0 & I \end{pmatrix} < 0 \tag{6.6}$$
  where
  $$\Psi := \begin{pmatrix} A_0^T X_{cl} + X_{cl} A_0 & X_{cl} B_0 & C_0^T \\ B_0^T X_{cl} & -\gamma \mathcal{L} & \mathcal{D}_{11}^T \\ C_0 & \mathcal{D}_{11} & -\gamma \mathcal{J} \end{pmatrix} \tag{6.7}$$
  $$P := (\mathcal{B}^T, 0, \mathcal{D}_{12}^T), \qquad Q := (\mathcal{C}, \mathcal{D}_{21}, 0). \tag{6.8}$$

□

Note that the LMI (6.6) is nothing else than the scaled Bounded Real Lemma of Section 3. Interestingly, the same algorithm applies to discrete-time LPV synthesis upon replacing (6.6) by

$$\Psi + Q^T \Omega^T P + P^T \Omega Q < 0 \tag{6.9}$$

where Ψ, P, Q are now given by

$$\Psi := \begin{pmatrix} -X_{cl}^{-1} & A_0 & B_0 & 0 \\ A_0^T & -X_{cl} & 0 & C_0^T \\ B_0^T & 0 & -\gamma\mathcal{L} & \mathcal{D}_{11}^T \\ 0 & C_0 & \mathcal{D}_{11} & -\gamma\mathcal{J} \end{pmatrix} \tag{6.10}$$

$$P := (\mathcal{B}^T,\ 0,\ 0,\ \mathcal{D}_{12}^T), \quad Q := (0,\ \mathcal{C},\ \mathcal{D}_{21},\ 0). \tag{6.11}$$

Finally, the state-space matrices of $K(\sigma)$ can also be computed using the explicit formulas of [14] or [15]. Involving only standard linear algebra, these formulas are appealing from a computational efficiency viewpoint. In comparison, the LMI approach is more costly but also offers more flexibility. In particular, additional LMI constraints on Ω can be handled by the algorithm given above (see next subsection and Section 7 for illustrations).

6.2 Well-posedness of the gain-scheduled controller

Once the state-space data Ω of $K(\sigma)$ has been computed as indicated above, the time-varying gain-scheduled controller is given by

$$F_l(K(\sigma), \Theta_\tau) \tag{6.12}$$

conformably to Figure 2.1a). One question arises when inspecting this expression: is this linear fractional interconnection well-posed for all admissible values θ_τ of the parameter vector? In other words, does the resulting time-varying controller remain causal along all parameter trajectories?

Given the realization (6.1) of $K(\sigma)$, well-posedness is equivalent to the invertibility of the matrix

$$I - D_{K\theta\theta}\Theta_\tau \tag{6.13}$$

at all times τ and for all Θ_τ in the closed ball

$$\|\Theta_\tau\|_\infty \leq 1/\gamma\ . \tag{6.14}$$

Recalling that the LPV synthesis is performed via the scaled $\mathcal{H}_\infty$ problem (2.15), we are guaranteed that the overall closed-loop interconnection

$$F_u\left(F_l(P_a(\sigma),\ K(\sigma)),\ \begin{pmatrix} \Theta_\tau & 0 \\ 0 & \Theta_\tau \end{pmatrix}\right)$$

is well-posed for all Θ_τ satisfying (6.14). Equivalently, the matrix

$$I - \begin{pmatrix} D_{K\theta\theta} & D_{K\theta 1} D_{2\theta} \\ D_{\theta 2} D_{K1\theta} & D_{\theta\theta} + D_{\theta 2} D_{K11} D_{2\theta} \end{pmatrix} \begin{pmatrix} \Theta_\tau & 0 \\ 0 & \Theta_\tau \end{pmatrix} \tag{6.15}$$

is invertible at each time for all Θ_τ satisfying (6.14). Unfortunately, this is not sufficient to guarantee well-posedness of the LPV controller as seen when comparing (6.13) and (6.15). In other words, there may be parameter values for which the control input u is not causally determined by the output measurement y and the parameter measurement θ_τ.

This difficulty can be alleviated in a number of ways. First of all, observe that the invertibility of (6.15) implies that of (6.13) whenever $D_{2\theta} = 0$ or $D_{\theta 2} = 0$. This corresponds to LPV plants where either

- the measurement equation is
$$y = C_2 x + D_{21} w$$
independently of θ, or
- the matrices $B_2(\theta)$ and $D_{12}(\theta)$ specifying how the control input u enters the state equation and the controlled output equation are both independent of θ.

For such LPV plants the gain-scheduled controller $F_l(K, \Theta_\tau)$ is guaranteed to be causal. While many problems already fall in this category, it is possible to systematically enforce $D_{2\theta} = 0$ or $D_{\theta 2} = 0$ by inserting low-pass filters in the control or measurement channels (see [2]). Note that the introduction of such filters is very reasonable from a practical standpoint.

If this first remedy is not applicable, we can work with the Small Gain condition for well-posedness. From standard μ theory, $I - D_{K\theta\theta}\Theta$ is invertible for all Θ of structure (2.3) satisfying (6.14) if

$$\inf_{L \in L_\Delta} \bar{\sigma}(L^{1/2} D_{K\theta\theta} L^{-1/2}) < \gamma. \tag{6.16}$$

Recalling that the state-space data Ω of $K(\sigma)$ can be chosen as any solution of the LMI (6.6) or (6.9), we should seek solutions of this LMI which further satisfy (6.16). This can be formulated as the following feasibility problem:

$$\text{Find } L \in L_\Delta \text{ and } \Omega \text{ satisfying (6.6) and } D_{K\theta\theta}^T L D_{K\theta\theta} < \gamma^2 L. \tag{6.17}$$

This problem is not jointly convex in Ω and L due to the last constraint. Nevertheless, it can be attacked by the following "Ω–L iteration" scheme:

1. *Start with $L = I$,*

2. *Solve for fixed L the LMI problem:*

$$\textit{Minimize } t \textit{ over all } \Omega \textit{ such that (6.6) and } \begin{pmatrix} tL & D_{K\theta\theta}^T \\ D_{K\theta\theta} & tL^{-1} \end{pmatrix} \geq 0,$$

3. *Solve for fixed Ω the LMI problem:*

$$\textit{Minimize } t \textit{ over } L \in L_\Delta \textit{ such that } L > I \textit{ and } D_{K\theta\theta}^T L D_{K\theta\theta} < t^2 L.$$

4. Repeat steps 2 and 3 until $t < \gamma$.

Note that this scheme is not guaranteed to find a solution to the problem (6.17) even if one exists.

Finally, another possible approach to well-posedness is to add extra uncertainty on the parameter measurements θ_τ. Specifically, instead of assuming exact measurement of θ_i, we consider the case where only an approximate value

$$\bar{\theta}_i = \theta_i(1 + \delta_i)$$

is available for control purposes. Here δ_i can be seen as a multiplicative norm-bounded uncertainty that models measurement errors on θ_i. In addition, this extra uncertainty leaves room for adjusting $\bar{\theta}$ so as to prevent singularity of $I - D_{K\theta\theta}\bar{\Theta}$.

This modified problem is more realistic and was also considered in [23]. Gathering all δ_i in a block-diagonal operator δ_τ, the gain-scheduled controller is now given by

$$F_l(K(\sigma), \Theta_\tau(I + \delta_\tau)).$$

Equivalently, this controller is represented by the feedback equations:

$$\begin{aligned}
\begin{pmatrix} u \\ \tilde{q}_\theta \end{pmatrix} &= K(\sigma) \begin{pmatrix} y \\ \tilde{w}_\theta \end{pmatrix} \\
\bar{q} &= \tilde{q}_\theta + \bar{w} \\
\begin{pmatrix} \tilde{w}_\theta \\ \bar{w} \end{pmatrix} &= \begin{pmatrix} \Theta_\tau & 0 \\ 0 & \delta_\tau \end{pmatrix} \begin{pmatrix} \bar{q} \\ \tilde{q}_\theta \end{pmatrix}
\end{aligned}$$

Note that δ and Θ share the same structure Δ.

The corresponding control structure is represented in Figure 6.2. As before, this interconnection can be rearranged to isolate $K(\sigma)$ at the bottom and the uncertainty operator

$$\begin{pmatrix} \delta_\tau & 0 & 0 \\ 0 & \Theta_\tau & 0 \\ 0 & 0 & \Theta_\tau \end{pmatrix}$$

at the top. This leads to another scaled $\mathcal{H}_\infty$ problem, this time with both measured and uncertain parameters. For a discussion of such mixed problems, see Section 8.

6.3 Practical implementation

As mentioned above, the LPV controller is given in LFT form as $F_l(K(\sigma), \Theta_\tau)$. Equivalently, its state-space equations read

$$\begin{aligned}
\dot{x}_K &= A_K(\theta)x_K + B_K(\theta)y \\
u &= C_K(\theta)x_K + D_K(\theta)y
\end{aligned} \tag{6.18}$$

where

$$\begin{aligned}
A_K(\theta) &:= A_K + B_{K\theta}\, \Lambda_\theta\, C_{K\theta} \\
B_K(\theta) &:= B_{K1} + B_{K\theta}\, \Lambda_\theta\, D_{K\theta 1} \\
C_K(\theta) &:= C_{K1} + D_{K1\theta}\, \Lambda_\theta\, C_{K\theta} \\
D_K(\theta) &:= D_{K11} + D_{K1\theta}\, \Lambda_\theta\, D_{K\theta 1} \\
\Lambda_\theta &:= \Theta_\tau (I - D_{K\theta\theta}\Theta_\tau)^{-1}
\end{aligned}$$

Given the measurement θ_τ of the parameter vector at time τ, the matrix Λ_θ is readily computed and the control input u is obtained at the flight by real-time integration of (6.18).

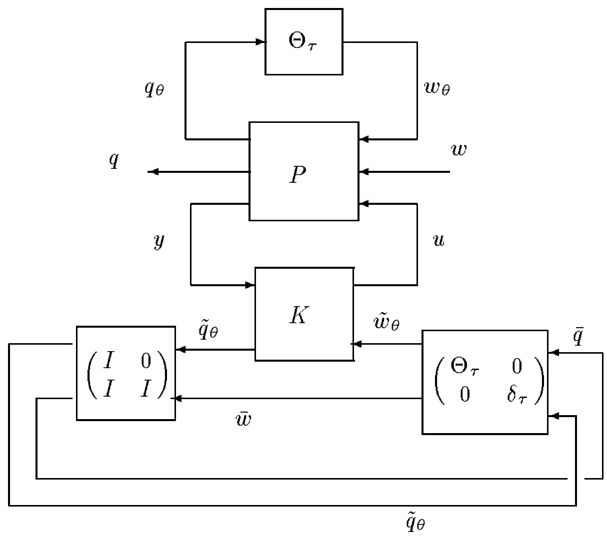

Figure 6.2: Modified LPV control structure

7 Comparison of LPV and Robust LTI controllers

Compared to classical Small-Gain-based robust LTI control, the LPV synthesis technique proposed above yields higher performance controllers in general. To see this, consider the LTI counterpart of problem (2.10) which seeks a parameter-independent LTI controller $K(s)$ such that

$$\max_{\|\Theta_\tau\|_\infty \le 1/\gamma} \|F_u(F_l(P(s), K(s)), \Theta_\tau)\|_\infty < \gamma \tag{7.1}$$

where the plant $P(s)$ is given by (2.4). Applying Small Gain theory to this standard problem of robust performance in the face of time-varying parametric uncertainty, a sufficient condition for solvability is the existence of a scaling matrix $L \in L_\Delta$ and of a LTI controller $K(s)$ such that

$$\left\| \begin{pmatrix} L^{1/2} & 0 \\ 0 & I_{p_1} \end{pmatrix} F_l(P(s), K(s)) \begin{pmatrix} L^{-1/2} & 0 \\ 0 & I_{p_1} \end{pmatrix} \right\|_\infty < \gamma. \tag{7.2}$$

In turn, this problem is addressed by Theorem 4.1 which provides the following solvability conditions in the continuous-time case.

Theorem 7.1 (Robust LTI Controllers) *Assuming* **(A1)-(A2)***, the problem (7.2) is solvable if and only if there exists a quadruple* (R, S, L_3, J_3) *satisfying (5.2)–(5.5) together with*

$$L_3 J_3 = I_r. \tag{7.3}$$

Proof: This is a straightforward application of Theorem 4.1 where the scalings L and J are replaced by $\begin{pmatrix} L_3 & 0 \\ 0 & I_{p_1} \end{pmatrix}$ and $\begin{pmatrix} J_3 & 0 \\ 0 & I_{p_1} \end{pmatrix}$, respectively. □

Comparing Theorems 5.1 and 7.1, the only difference between the characterizations of LPV and LTI controllers is the extra condition (7.3) on the scaling matrices L_3, J_3. Recalling that the positivity condition (5.5) is equivalent to

$$L_3 > 0, \qquad J_3 > 0, \qquad \lambda_{\min}(L_3 J_3) \geq 1,$$

this additional constraint has a simple interpretation in terms of admissible scaling matrices. While any positive scalings L_3, J_3 such that

$$\lambda_i(L_3 J_3) \geq 1, \quad i = 1, \ldots, r \tag{7.4}$$

can be used for LPV synthesis, the set of admissible scalings for robust LTI synthesis is restricted to positive matrices L_3, J_3 satisfying:

$$\lambda_i(L_3 J_3) = 1, \quad i = 1, \ldots, r. \tag{7.5}$$

Clearly (7.5) corresponds to a small subset of the convex set of scalings defined by (7.4). In particular, all scalings satisfying (7.5) lie on the boundary of this convex set. Since it works with a larger set of scalings, gain-scheduled synthesis will typically achieve higher $\mathcal{H}_\infty$ performance than robust LTI synthesis based on (7.2).

Interestingly, there is a complete analogy between the robust LTI problem and the static output feedback problem. Specifically, the robust LTI problem amounts to finding scaling matrices L_3, J_3 satisfying (5.2)–(5.3) and such that

$$L_3 > 0, \qquad J_3 > 0, \qquad L_3 J_3 = I.$$

Meanwhile, the static output feedback amounts to finding R, S satisfying (5.2)–(5.4) and such that $\operatorname{rank}(I - RS) = 0$. That is, such that

$$R > 0, \qquad S > 0, \qquad RS = I.$$

Observing that the gain-scheduled controller can be written as

$$\begin{aligned} F_l(K(s),\Theta_\tau) &= F_l(F_u(\Omega, s^{-1}I_k),\, \Theta_\tau) \\ &= D_{K11} + (C_{K1}\ D_{K1\theta})\left\{\begin{pmatrix} sI_k & 0 \\ 0 & \Theta_\tau^{-1}I_r \end{pmatrix} - \begin{pmatrix} A_K & B_{K\theta} \\ C_{K\theta} & D_{K\theta\theta} \end{pmatrix}\right\}^{-1}\begin{pmatrix} B_{K1} \\ D_{K\theta 1} \end{pmatrix}, \end{aligned}$$

this analogy has the following simple interpretation. While static output feedback corresponds to zero-order controllers with respect to the Laplace variable s, robust LTI control seeks controllers with zero-order dependence on the parameter variable Θ. In this formal analogy, both s and Θ^{-1} play the role of dynamical variables.

8 LPV Systems with Uncertain Parameters

In many practical situations, only some physical parameters are measured while the others must be regarded as uncertain. If gain scheduling is applicable to the measured parameters, parametric uncertainty must be handled by classical robust synthesis techniques. This mixed problem will be referred to as the synthesis of robust LPV controllers.

Let θ_u and θ_m denote the vectors of uncertain and measured parameters, respectively, and let Δ_u and Δ_m denote the corresponding uncertainty structures. The robust LPV problem again falls within the scope of the results of Section 4. Here the total uncertainty structure assumes the form

$$\begin{pmatrix} \Theta_u & 0 & 0 \\ 0 & \Theta_m & 0 \\ 0 & 0 & \Theta_m \end{pmatrix},$$

where the operators Θ_u and Θ_m are independent. It follows that the corresponding set of scalings reads

$$\left\{\begin{pmatrix} L_u & 0 \\ 0 & L_m \end{pmatrix} > 0 \ :\ L_u \in L_{\Delta_u},\ \ L_m \in L_{\Delta_m \oplus \Delta_m}\right\}.$$

The solvability conditions of Theorem 4.1 can be applied to the resulting scaled $\mathcal{H}_\infty$ control problem. Unfortunately, convexity is destroyed by the extra uncertainty block Θ_u and D–K iterations are needed to compute adequate LPV controllers. Since the problem is convex as soon as the upper block L_u of the scaling matrix is fixed, it is advisable to alternatively

- fix L_u and compute a controller by solving the resulting convex LPV problem,
- fix K and L_m and optimize the performance γ with respect to L_u.

Like μ synthesis algorithms, such a scheme is only guaranteed to find a local minimum [1, 19, 4, 29].

Finally, we mention a simple heuristics which may prove useful for robust LPV synthesis. From Section 7, robust LPV controllers can be viewed as LPV controllers with a zero-order dependence on the parametric uncertainty Θ_u. This suggests applying the following heuristics:

- solve the LPV problem where θ_u is assumed to be measured. Let R, S, L_3, J_3 be a solution of the LMIs (5.2)–(5.5).
- when computing the LPV controller data Ω as indicated in Section 6, minimize the norm of the matrices $\begin{pmatrix} B_{K\theta_u} \\ D_{K1\theta_u} \end{pmatrix}$ or $(C_{K\theta_u} \;\; D_{K\theta_u 1})$ characterizing the interconnection with Θ_u.

This minimization problem is still an LMI problem in Ω and the resulting LPV controller will be independent of θ_u if the global minimum is zero. Note that this scheme is not guaranteed to find adequate controllers even when there exists some.

9 Conclusions

We have presented an LMI approach to the synthesis of gain-scheduled $\mathcal{H}_\infty$ controllers. Using a Small Gain argument, the original problem has been reformulated as one of robust performance in the face of structured uncertainty. Thanks to the special structure of the plant/uncertainty interconnection, necessary and sufficient LMI-based conditions for solvability of this robust performance problem have been derived.

Using this technique, the synthesis of gain-scheduled $\mathcal{H}_\infty$ controllers proceeds in two stages. First, the closed-loop $\mathcal{H}_\infty$ performance is optimized by solving the underlying LMI problem. Efficient numerical algorithms are available for this purpose. Secondly, an LPV control structure is computed from the quadruple R, S, L_3, J_3 of matrices produced by this LMI optimization. Up to taking care of well-posedness issues, the corresponding time-varying parameter-dependent controller is readily implemented. Specifically, its state-space matrices are "self-scheduled" by the measurements of the varying parameter values according to simple explicit formulas.

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Appendix A

Proof of Theorem 4.1: Given the realization (4.3) of the plant G, a realization

$$K(\sigma) = D_K + C_K(sI - A_K)^{-1}B_K, \qquad A_K \in \mathbb{R}^{k\times k} \tag{A.1}$$

of the controller K and with assumption **(A2)** in force, a realization

$$F_l(G(s), K(s)) = D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl} \tag{A.2}$$

of the closed-loop transfer function is given by:

$$\begin{pmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{pmatrix} = F_l\left\{ \begin{pmatrix} A_0 & B_0 & \mathcal{B} \\ C_0 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C} & \mathcal{D}_{21} & 0 \end{pmatrix}, \Omega \right\} = \begin{pmatrix} A_0 + \mathcal{B}\Omega\mathcal{C} & B_0 + \mathcal{B}\Omega\mathcal{D}_{21} \\ C_0 + \mathcal{D}_{12}\Omega\mathcal{C} & \mathcal{D}_{11} + \mathcal{D}_{12}\Omega\mathcal{D}_{21} \end{pmatrix} \quad (A.3)$$

where

$$A_0 = \begin{pmatrix} A & 0 \\ 0 & 0_{k\times k} \end{pmatrix}, \quad B_0 = \begin{pmatrix} B_1 \\ 0_{k\times m_1} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & B_2 \\ I_k & 0 \end{pmatrix},$$

$$C_0 = (C_1\ ,\ 0_{p_1\times k}), \quad \mathcal{D}_{12} = (0_{p_1\times k}\ D_{12}), \quad (A.4)$$

$$\mathcal{C} = \begin{pmatrix} 0 & I_k \\ C_2 & 0 \end{pmatrix}, \quad \mathcal{D}_{21} = \begin{pmatrix} 0_{k\times m_1} \\ D_{21} \end{pmatrix}$$

and

$$\Omega = \begin{pmatrix} A_K & B_K \\ C_k & D_k \end{pmatrix}. \quad (A.5)$$

Using the Bounded Real Lemma 3.1, the k-th order controller $K(s)$ and the similarity scaling $L \in L_\Delta$ solve the scaled $\mathcal{H}_\infty$ problem (4.2) if and only if the LMI

$$\begin{pmatrix} A_{cl}^T + X_{cl}A_{cl} & X_{cl}B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma L & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma L^{-1} \end{pmatrix} < 0 \quad (A.6)$$

holds for some $X_{cl} > 0$ in $\mathbb{R}^{(n+k)\times(n+k)}$ and for the closed-loop state-space parameters $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ defined by (A.3)–(A.5). Using the notations (A.4), the inequality (A.6) can be rewritten as

$$\Psi + P_{X_{cl}}^T \Omega Q + Q^T \Omega^T P_{X_{cl}} < 0 \quad (A.7)$$

where

$$\Psi := \begin{pmatrix} A_0^T X_{cl} + X_{cl}A_0 & X_{cl}B_0 & C_0^T \\ B_0^T X_{cl} & -\gamma L & D_{11}^T \\ C_0 & D_{11} & -\gamma L^{-1} \end{pmatrix} \quad (A.8)$$

$$P_X = (\mathcal{B}^T X_{cl}, 0, \mathcal{D}_{12}^T), \qquad Q = (\mathcal{C}, \mathcal{D}_{21}, 0). \quad (A.9)$$

In turn, the LMI (A.7) has a solution Ω if and only if (see [14]):

$$W_{P_X}^T \Psi W_{P_X} < 0, \qquad W_Q^T \Psi W_Q < 0 \quad (A.10)$$

where W_{P_X} and W_Q denote any bases of the null spaces of P_X and Q respectively. Observing that

$$P_X = P \begin{pmatrix} X_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad P = (\mathcal{B}^T, 0, \mathcal{D}_{12}^T), \quad (A.11)$$

a basis for the null space of P_X is given by

$$W_{P_X} = \begin{pmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} W_P \tag{A.12}$$

where W_P denotes any basis of the null space of P. Consequently, the first inequality in (A.10) can be written as $W_P^T \Phi W_P < 0$ where

$$\Phi := \begin{pmatrix} A_0 X_{cl}^{-1} + X_{cl}^{-1} A_0^T & B_0 & X_{cl}^{-1} C_0^T \\ B_0^T & -\gamma L & D_{11}^T \\ C_0 X_{cl}^{-1} & D_{11} & -\gamma L^{-1} \end{pmatrix} \tag{A.13}$$

Partition X_{cl} as

$$X_{cl} = \begin{pmatrix} S & N \\ N^T & E \end{pmatrix}, \quad X_{cl}^{-1} = \begin{pmatrix} R & M \\ M^T & F \end{pmatrix} \tag{A.14}$$

where $N, M \in \mathbb{R}^{n \times k}$ and $E, F \in \mathbb{R}^{k \times k}$, and introduce bases $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ and $\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ of the null spaces of (B_2^T, D_2^T) and (C_2, D_{21}), respectively. Then bases for the null spaces of P and Q are given by:

$$W_P = \begin{pmatrix} 0 & P_1 \\ 0 & 0 \\ P_3 & 0 \\ 0 & P_2 \end{pmatrix}, \quad W_Q = \begin{pmatrix} 0 & Q_1 \\ 0 & 0 \\ 0 & Q_2 \\ Q_3 & 0 \end{pmatrix}. \tag{A.15}$$

Observing that the second row is identically zero in the expressions (A.15), evaluation of the block matrix products $W_P^T \Phi W_P$ and $W_Q^T \Psi W_Q$ yields the solvability conditions:

$$\begin{pmatrix} 0 & P_1 \\ P_3 & 0 \\ 0 & P_2 \end{pmatrix}^T \begin{pmatrix} AR + RA^T & B_1 & RC_1^T \\ B_1^T & -\gamma L & D_{11}^T \\ C_1 R & D_{11} & -\gamma L^{-1} \end{pmatrix} \begin{pmatrix} 0 & P_1 \\ P_3 & 0 \\ 0 & P_2 \end{pmatrix} < 0 \tag{A.16}$$

and

$$\begin{pmatrix} 0 & Q_1 \\ 0 & Q_2 \\ Q_3 & 0 \end{pmatrix}^T \begin{pmatrix} A^T S + SA & SB_1 & C_1^T \\ B_1^T S & -\gamma L & D_{11}^T \\ C_1 & D_{11} & -\gamma L^{-1} \end{pmatrix} \begin{pmatrix} 0 & Q_1 \\ 0 & Q_2 \\ Q_3 & 0 \end{pmatrix} < 0. \tag{A.17}$$

Defining $J := L^{-1} \in L_\Delta$, (A.16) and (A.17) are exactly the conditions (4.4) and (4.5). Meanwhile, the condition $X_{cl} > 0$ is equivalent to (4.6) and (4.8) given the block partitioning (A.14). Consequently, the problem (4.2) has a solution if and only if there exist R, S, L, J for which (4.4)-(4.8) hold. □

References

[1] Apkarian, P., J. Chretien, P. Gahinet, and J. M. Biannic, "μ Synthesis by $D-K$ Iterations with Constant Scaling," *Proc. European Contr. Conf.*, 1993, pp. 1819-1825.

[2] Apkarian, P., P. Gahinet, and G. Becker, "Self-Scheduled $\mathcal{H}_\infty$ Control of Linear Parameter-Varying Systems," submitted to *Automatica*, 1993.

[3] Astrom, K. J., and B. Wittenmark, *Adaptive Control*, Addison-Wesley, 1989.

[4] Balas, G. J., and J. C. Doyle, "Robustness and Performance Tradeoffs in Control Design of Flexible Structures," in *Proc. Conf. Dec. Contr.*, Honolulu, Hawaii, 1990, pp. 2999-3010.

[5] Balas, G. J., J. C. Doyle, K. Glover, A. Packard, and R. Smith, *μ-Analysis and Synthesis Toolbox: User's Guide*, The MathWorks Inc., 1991.

[6] Boyd, S.P., and L. El Ghaoui, "Method of Centers for Minimizing Generalized Eigenvalues," *Lin. Alg. & Applic.*, 188 (1993), pp. 63-111.

[7] Boyd, S., and Q. Yang, "Structured and Simultaneous Lyapunov Functions for System Stability Problems," *Int. J. Contr.*, 49 (1989), pp. 2215-2240.

[8] Desoer C. A., and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.

[9] Doyle, J. C., "Structured Uncertainty in Control System Design," *Proc. Conf. Dec. Contr.*, 1985, pp. 260-265.

[10] Doyle, J.C., B.A. Francis, and A.R. Tannenbaum, *Feedback Control Theory*, Macmillan Publishing Company, New York, 1992.

[11] Doyle, J.C., A. Packard, and K. Zhou, "Review of LFTs, LMIs, and μ," *Proc. Conf. Dec. Contr.*, 1991, pp. 1227-1232.

[12] Doyle, J. C., J. Wall, and G. Stein, "Performance and Robustness Analysis for Structured Uncertainty," in *Proc. IEEE Conf. Dec. Contr.*, Orlando, FL, 1982, pp. 629-636.

[13] Gahinet, P., and A. Nemirovskii, "General-Purpose LMI Solvers with Benchmarks," *Proc. Conf. Dec. Contr.*, 1993, pp. 3162-3165.

[14] Gahinet, P., and P. Apkarian, "A Linear Matrix Inequality Approach to $\mathcal{H}_\infty$ Control," *Int. J. Robust and Nonlinear Contr.*, July 1994.

[15] Gahinet, P., "Explicit Controller Formulas for LMI-based $\mathcal{H}_\infty$ Synthesis," submitted to *Automatica.* Also in *Proc. Amer. Contr. Conf.*, 1994, pp. 2396-2400.

[16] Gahinet, P., A. Nemirovskii, and A.J. Laub, "A Toolbox for LMI Control Design," submitted to '94 CDC.

[17] Haddad, W. M., and D. S. Bernstein, "Explicit Construction of Quadratic Lyapunov Functions for the Small Gain, Positivity, Circle, and Popov Theorems and their Applications to Robust Stability," *Proc. Conf. Dec. Contr.*, Brighton, UK, 1991, pp. 2618-2623.

[18] Lu, W. M., and J. C. Doyle, "H_∞ Control of LFT Systems: An LMI Approach," *Proc. Conf. Dec. Contr.*, Tucson, AR, 1992, pp. 1997-2001.

[19] Morris, J. C., P. Apkarian, and J. C. Doyle, "Synthesizing Robust Mode Shapes with μ and Implicit Model Following," *Proc. IEEE Conf. on Control Applications*, Dayton, Ohio, September 1992.

[20] Nesterov, Y. E., and A. S. Nemirovskii, *Self-concordant functions and polynomial-time methods in convex programming*, Technical report, Centr. Econ. Math. Instr., Moscow, USSR, 1990.

[21] Nesterov, Y. E., and A. S. Nemirovskii, *Interior-Point Polynomial Methods in Convex Programming: Theory and Applications*, SIAM publications, 1994.

[22] Nemirovskii, A., and P. Gahinet, "The Projective Method for Solving Linear Matrix Inequalities," *Proc. Amer. Contr. Conf.*, 1993, pp. 840-844.

[23] Packard, A., "Gain Scheduling via Linear Fractional Transformations," *Syst. Contr. Letters*, 22 (1994), pp. 79-92.

[24] Packard, A., G. J. Balas, J. C. Doyle, K. Glover, and R. Smith, *Control Design with μ-tools*, in preparation, 1991.

[25] Packard, A., P. Pandey, J. Leonhardson, and G. Balas, "Optimal, Constant I/O Similarity Scaling for Full-Information and State-Feedback Control Problems," *Syst. Contr. Letters*, 19 (1992), pp. 271-280.

[26] Packard, A., K. Zhou, P. Pandey, and G. Becker, "A Collection of Robust Control Problems Leading to LMIs," *Proc. Conf. Dec. Contr.*, Brighton,UK, 1991, pp. 1245-1250.

[27] Petersen, I. R., "Notions of Stabilizability and Controllability for a Class of Uncertain Linear Systems," *Int. J. Contr.*, 46 (1987), pp. 409-422.

[28] Ravi, R., R. M. Nagpal, and P. P. Khargonekar, "H_∞ Control of Linear Time-Varying Systems: A State-Space Approach," *SIAM J. Contr. Opt.*, 29 (1991), pp. 1394-1413.

[29] Reichert, R. T., "Robust Autopilot Design Using μ-Synthesis," *Proc. Amer. Contr. Conf.*, 1990, pp. 2368-2373.

[30] Safonov, M. G., and M. Athans, "A Multiloop Generalization of the Circle Criterion for Stability Margin Analysis," *IEEE Trans. Aut. Contr.*, AC-28 (1981), pp. 415-422.

[31] Shamma, J. F., and M. Athans, "Guaranteed Properties of Gain Scheduled Control for Linear Parameter-varying Plants," Automatica, 27 (1991), pp. 559-564.

[32] Shamma, J. F., and J. R. Cloutier, "A Linear Parameter-Varying Approach to Gain Scheduled Missile Autopilot Design," *Proc. Amer. Contr. Conf.*, 1992, pp. 1317-1321.

[33] Shor, N. Z., *Minimization Methods for Non-Differentiable Functions*, Springer Series in Computational Mathematics, Springer Verlag, Berlin, 1985.

[34] Tadmor, G., "Time Domain Optimal Control and Worst Case Linear System Design," *Proc. Conf. Dec. Contr.*, Tampa, FL, 1989, pp. 403-406.

[35] Zames, G., "On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems–Part I: Conditions Derived Using Concepts of Loop Gain, Conicity, and Positivity," *IEEE Trans. Aut. Contr.*, AC-11 (1966), pp. 228-238.

[36] Zhou, K., and P. P. Khargonekar, "On the Stabilization of Uncertain Linear Systems via Bound Invariant Lyapunov Functions," *SIAM J. Contr. Opt.*, 26 (1988), pp. 1265-1273.