Nonsmooth H_∞ Synthesis

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Outline

- ◇ context
- ◇ nonsmooth formulations
- ◇ nonsmooth descent algorithms
- ◇ applications
- ◇ concluding remarks

hard non-LMI problems

many synthesis problems do not reduce to LMI/SDP !

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- reduced- and fixed-order synthesis (PID H_∞, etc.)
- structured and decentralized synthesis problems
- general robust control with uncertain and/or nonlinear components
- simultaneous model/controller design, multimodel control
- unrelaxed LTI and LPV multi-objective
- combinations of the above, etc

hard non-LMI problems

many synthesis problems do not reduce to LMI/SDP !

- problems are nonconvex and/or nonsmooth
- even large size LMIs/SDPs still difficult to solve

hard non-LMI problems

many synthesis problems do not reduce to LMI/SDP !

$\Rightarrow$ need new algorithms for hard problems

current potential techniques

- coordinate descent schemes (D-K iterations,...) ⇒ slow and lack of convergence certificate
- solutions based on relaxations ⇒ unknown conservatism

- other proposals
  - TR method by Leibfritz
  - nonsmooth gradient sampling by Burke, Lewis & Overton
  - non-quadratic penalty method for SDP casts by Kocvara, Zibulesky,
  - ...

fundamental limitation of SDP casts

benchmark : Boeing 767 at flutter condition

- static stabilization pb.

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad u = Ky$$

$$A = 55 \times 55 \qquad B = 55 \times 2 \qquad C = 2 \times 55 \qquad K = 2 \times 2$$

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accept nonsmoothness and design appropriate algorithms

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$$\underset{x}{\text{minimize}} \, \lambda_1 \left(A_0 + \sum_{i=1}^{r} x_i A_i + \sum_{\ell=1}^{r} \sum_{k=1}^{s} x_\ell x_k B_{\ell k} \right)$$

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- H_∞ synthesis

$$\underset{K}{\text{minimize}} \ \sup_{\omega} \overline{\sigma}\left(T_{w\to z}(K, j\omega)\right)$$

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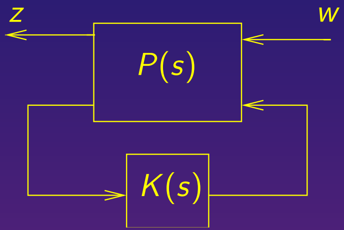
- H_∞ synthesis

$$\underset{K}{\text{minimize}}\ \sup_{\omega} \overline{\sigma}\left(T_{w \to z}(K, j\omega) \right)$$

- many others

closed-loop H_∞ norm



$$T_{w\to z}(K(s)) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

- composite $\|\cdot\|_\infty \circ T_{w\to z}$ is Clarke regular
- $\Rightarrow$ exhaustive description of subdifferential

$$\partial(\|\cdot\|_\infty \circ T_{w\to z})(K)$$

- $\Rightarrow$ variety of algorithms

Clarke subdifferential of H_∞ norm

with standard notations define closed-loop data

$$\begin{array}{rclrcl} \mathcal{A}(K) & := & A + B_2 K C_2, & \mathcal{B}(K) & := & B_1 + B_2 K D_{21}, \\ \mathcal{C}(K) & := & C_1 + D_{12} K C_2, & \mathcal{D}(K) & := & D_{11} + D_{12} K D_{21}\,, \end{array}$$

introduce notation

$$\begin{bmatrix} T_{w\to z}(K,s) & G_{12}(K,s) \\ G_{21}(K,s) & \star \end{bmatrix} :=$$

$$\begin{bmatrix} \mathcal{C}(K) \\ C_2 \end{bmatrix} (sI - \mathcal{A}(K))^{-1} \begin{bmatrix} \mathcal{B}(K) & B_2 \end{bmatrix} + \begin{bmatrix} \mathcal{D}(K) & D_{12} \\ D_{21} & \star \end{bmatrix}.$$

Clarke subdifferential of H_∞ norm

subdif. is convex set of subgradients Φ_Y's

$$\frac{\sum_{\nu=1}^{p} \mathrm{Re}\,\left\{G_{21}(K, j\omega_\nu)\, T_{w\to z}(K, j\omega_\nu)^H Q_\nu Y_\nu Q_\nu^H G_{12}(K, j\omega_\nu)\right\}^T}{\|T_{w\to z}(K)\|_\infty^{-1}}$$

- ω_ν are active (peak) frequencies at K
- $Y := (Y_1, \ldots, Y_p)$ ranges over (convex)

$$\{Y = (Y_1, \ldots, Y_p) : \; Y_i = Y_i^H, \succeq 0, \sum_{\nu=1}^{p} \mathrm{Tr}\,(Y_\nu) = 1\}$$

convergent nonsmooth descent method

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- define descent function (1st-order model)

$$\theta(K) := \inf_{H \in R^{m_2 \times p_2}} \sup_{\omega \in \Omega_e(K)} \sup_{\mathrm{Tr}\, Y_\omega = 1, Y_\omega \succeq 0} \{ \ldots$$
$$-\|T_{w \to z}(K)\|_\infty + \overline{\sigma}(T_{w \to z}(K, j\omega)) + \langle \Phi_{Y_\omega}, H \rangle + \frac{1}{2}\delta \|H\|_F^2 \ \}$$

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- H is (controller) search direction computed explicitly

convergent nonsmooth descent method

- Fenchel duality $\Longrightarrow$ descent direction at K

$$H(K) := -\frac{1}{\delta} \sum_{\omega \in \Omega_e(K)} \tau_\omega \Phi_{Y_\omega} \,.$$

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- and SDP dual form

$$\begin{aligned} \theta(K) \quad := & \sup_{\sum_{\Omega_e(K)} \tau_\omega = 1, \tau_\omega \geq 0} \; \sup_{\mathrm{Tr}\, Y_\omega = 1, Y_\omega \succeq 0} \{ -\| T_{w \to z}(K) \|_\infty \ldots \\ + & \sum_{\Omega_e(K)} \tau_\omega \overline{\sigma}(T_{w \to z}(K, j\omega)) - \tfrac{1}{2\delta} \| \sum_{\Omega_e(K)} \tau_\omega \Phi_{Y_\omega} \|_F^2 \; \} \end{aligned}$$

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- $\theta(K) \leq 0, \forall K$ and $\theta(K) = 0 \Leftrightarrow 0 \in \partial(\| \cdot \|_\infty \circ T_{w \to z})(K)$

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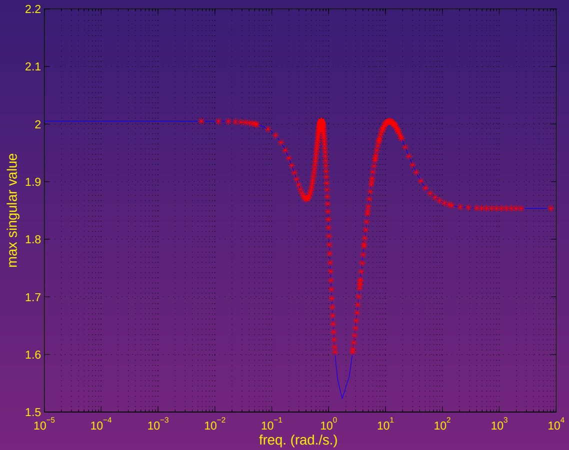
- $\theta(K) \leq 0, \forall K$ and $\theta(K) = 0 \Leftrightarrow 0 \in \partial(\| \cdot \|_\infty \circ T_{w\to z})(K)$
- algorithm : line search using $H(K)$ with $\theta(K)$ as stopping test

example of enriched set $\Omega_e(K)$

- $\Omega_e(K)$ captures extra info. on $\overline{\sigma}(T_{w\to z}(K, j\omega))$ $\Longrightarrow$ better steps



fixed-order output-feedback H_∞ synthesis

examples from Leibfritz's collection

problem	(n, m, p)	order	iter	cpu (sec.)	nonsmooth H_∞	H_∞ AL	FW	H_∞ full
AC8	$(9, 1, 5)$	0	20	45	2.005	2.02	2.612	1.62
HE1	$(4, 2, 1)$	0	4	7	0.154	0.157	0.215	0.073
REA2	$(4, 2, 2)$	0	31	51	1.192	1.155	1.263	1.141
AC10	$(55, 2, 2)$	0	15	294	13.11	$*$	$*$	3.23
AC10	$(55, 2, 2)$	1	46	408	10.21	$*$	$*$	3.23
BDT2	$(82, 4, 4)$	0	44	1501	0.8364	$*$	$*$	0.2340
HF1	$(130, 1, 2)$	0	11	1112	0.447	$*$	$*$	0.447
CM4	$(240, 1, 2)$	0	2	3052	0.816	$*$	$*$	$*$
CM5	$(480, 1, 2)$	0	2	4785	0.816	$*$	$*$	$*$

H_∞ synthesis with nonsmooth algorithm
low-level SUNW, Sun-Blade-1500

extensions

- pure stabilization problems $A + B_2KC_2$ is Hurtwitz iff

$$\|C_2(sI - (A + B_2KC_2))^{-1}B_2\|_\infty < \infty$$

- multidisk problems

$$\underset{K}{\text{minimize}}\, f(K) := \max_{i=1,\ldots,N} \|T_{w^i \to z^i}(K)\|_\infty$$

- structured feedback design

$$K = K_0 + L\,\text{diag}(\kappa)R$$

- etc

concluding remarks

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- second-order version in progress