Nonlinear $\mathcal{H}_\infty$ Control for an Integrated Suspension System via Parameterized Linear Matrix Inequality Characterizations

H.D. Tuan
S. Hosoe
Dept. of Electronic-Mechanical Eng.
Nagoya University
Furo-cho, Chikusa-ku
Nagoya 464-01, Japan
tuan,hosoe@nuem.nagoya-u.ac.jp

E. Ono
Toyota Central R&D Labs., Inc.
Nagakute
Aichi 480-01
Japan
e-ono@mosk.tytlabs.co.jp

P. Apkarian
ONERA-CERT
Control Systems Dept.
2 av. Edouard Belin
31055 Toulouse, France
apkarian@cert.fr

Abstract

The automotive hydro-pneumatic integrated suspension model is nonlinear with large dimensions. As a consequence, the nonlinear $\mathcal{H}_\infty$ control methodology based on the traditional Hamilton-Jacoby-Isaacs equation is impractical in this application. An alternative so-called Parameterized Linear Matrix Inequality (PLMI) approach is proposed for solving this hard nonlinear $\mathcal{H}_\infty$ control problem. The validity of the proposed approach is confirmed not only by detailed and realistic simulations but also by extensive experiments. Specifically, the proposed nonlinear control method outperforms the more classical feedback linearization control technique.

1 Introduction

The central target of nonlinear $\mathcal{H}_\infty$ control is to internally stabilize the nonlinear plant while minimizing the effect of disturbances such as measurement noise, input disturbances and other exogeneous signals which invariably occur in most applications because of plant interactions with the environment. However, in deep contrast with linear $\mathcal{H}_\infty$ control methods which are flexible, efficient and allow to solve a broad class of linear control problems, there are few practical methods in nonlinear $\mathcal{H}_\infty$ control which can handle real engineering problems with similar comfort. For such hard nonlinear problems, our opinion is that it is of extreme importance to expoits the specific characteristics. It is not doubtful that special structures and properties of a given class of systems will play a crucial role for developing adequate solution methods.

The purpose of the automotive hydro-pneumatic integrated suspension is to improve the ride comfort by oil flow control to cylinder despite bad road environment or vibrations in the human sensitivity band. The ride comfort can be enhanced by attenuating vibration in the human sensitivity band, and therefore, $\mathcal{H}_\infty$ control with loop-shaping specifications is an effective methodology. The integrated suspension is different from the pure active suspension system [13] by the additional presence of the semi-active valve which exhibits nonlinear characteristics. Thus the control design for the integrated suspension system becomes inevitably difficult as the resulting model is nonlinear with a large dimension. Therefore the control design problem here is a very challenging one in nonlinear $\mathcal{H}_\infty$ control. Perhaps, the most essential characterization of this control system is that its nonlinearity is caused by the semi-active input. Considering this semi-active input as a parameter, the system can be viewed as a family of parameter-dependent linear systems. It is well known that the linear matrix inequality (LMI) approach is a very efficient and powerful tool to solve various problems for linear systems including linear $\mathcal{H}_\infty$ problems [5] thanks to the availability of efficient interior-point polynomial-time algorithms for solving semidefinite programming problems [6]. In [2, 12], we have extended the LMI approach to so-called parameterized LMIs (PLMIs) in order to solve various challenging problems of linear robust control. The purpose of the present paper is to take advantage of these results to solve the nonlinear $\mathcal{H}_\infty$ control associated with the integrated suspension system. Note that many systems like the integrated suspension system with few state variables responsible of the nonlinearity very frequently arise in practical nonlinear models. This was our main motivation for proposing an alternative and practical approach to solve nonlinear $\mathcal{H}_\infty$ control for such class of systems. The power and efficiency of the proposed method are confirmed by realistic simulations but also by experiments on the physical plant. Particularly, the proposed control is shown to outperforms feedback linearization control and linear control techniques.

The organization of the paper is as follows. Section 2 deals with the model of the integrated suspension system with some preliminary structural analysis. Useful theoretical characterizations involving PLMIs which will constitute our constructive tools are detailed in Section 3. Justification and validation of the approach are shown through simulations and experiments in Section 4. We conclude the paper in Section 5 with some remarks and recommendations for future work.

The notation in the paper is quite standard. Namely, $M > 0$ or $M < 0$ for a symmetric matrix M, means it is negative definite or positive definite. In symmetric block matrices we use $*$ as an ellipsis for the terms that are induced by symmetry, e.g

$$\begin{bmatrix} S+(K+*) & * \\ M & Q \end{bmatrix} = \begin{bmatrix} S+(K+K') & M' \\ M & Q \end{bmatrix}.$$

2 Modeling of controlled integrated suspension system

A quarter-car test bench with two degrees of freedom is shown in Fig.1. This system has two control valves. The first one is the active control valve which controls the oil flow from hydraulic pump to suspension cylinder. The second one is the semi-active control valve which controls the cross sectional area of the pipe between cylinder and accumulator. The semi-active valve avails to reduce energy consumption.

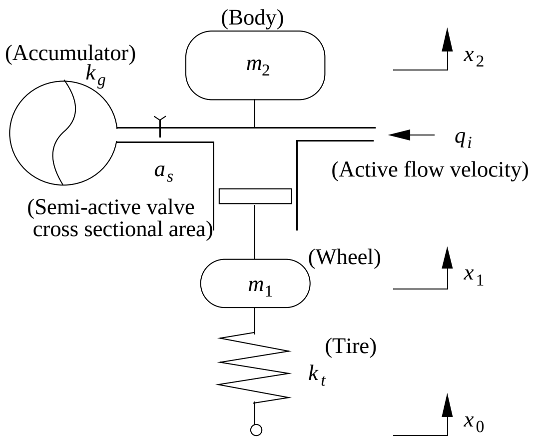

Figure 1: The active suspension system

The sensitivity band is the limited frequency band within which human is most sensitive and it is assumed to range from 3 to 8Hz. [13]. With the assumptions that the oil is incompressible, the active control valve and the gas spring characteristics can be linearized, the active suspension model can be represented as

$$\begin{aligned} \dot{x}_p &= A_p(\phi)x_p + B_{p1}w + B_{p2}(\phi)u, \\ z_p &= C_p(\phi)x_p + D_p(\phi)u, \end{aligned} \qquad (1)$$

where

$$\begin{aligned} x_p &= [x_{01} \quad x_{12} \quad \nu_i \quad \dot{x}_1 \quad \dot{x}_2 \quad q_i \quad \dot{q}_i]' \\ u &= [u_a \quad u_s]' \\ z_p &= \ddot{x}_2 \end{aligned}$$

$$A_p(\phi) = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ 0 & a_7 & a_8 & a_9 & a_{10} & a_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_{12} & a_{13} \end{bmatrix}$$

$$B_{p1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad B_{p2}(\phi) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & b_1 \\ 0 & b_2 \\ 0 & 0 \\ b_3 & 0 \end{bmatrix}$$

$$\begin{aligned} C_p(\phi) &= [0 \quad a_7 \quad a_8 \quad a_9 \quad a_{10} \quad a_{11} \quad 0] \\ D_p(\phi) &= [0 \quad b_2] \end{aligned}$$

and

$$\begin{aligned}
&a_1 = \frac{k_t}{m_1},\ a_2 = -\frac{k_g}{m_1},\ a_3 = -\frac{k_g}{m_1 a_p}, \\
&a_4 = -\frac{\rho a_p^2}{2m_1 c_s^2 a_{s0}^2}|\phi(x)|,\ a_5 = \frac{\rho a_p^2}{2m_1 c_s^2 a_{s0}^2}|\phi(x)|, \\
&a_6 = -\frac{\rho a_p}{2m_1 c_s^2 a_{s0}^2}|\phi(x)|,\ a_7 = \frac{k_g}{m_2},\ a_8 = \frac{k_g}{m_2 a_p}, \\
&a_9 = \frac{\rho a_p^2}{2m_2 c_s^2 a_{s0}^2}|\phi(x)|,\ a_{10} = -\frac{\rho a_p^2}{2m_2 c_s^2 a_{s0}^2}|\phi(x)|, \\
&a_{11} = \frac{\rho a_p}{2m_2 c_s^2 a_{s0}^2}|\phi(x)|,\ a_{12} = -\omega_a^2,\ a_{13} = -2\eta_a\omega_a, \\
&b_1 = -\frac{\rho a_p}{2m_1 c_s^2 a_{s0}^2}|\phi(x)|\phi(x), \\
&b_2 = -\frac{\rho a_p}{2m_2 c_s^2 a_{s0}^2}|\phi(x)|\phi(x),\ b_3 = k_a\omega_a^2.
\end{aligned} \qquad (2)$$

with the oil flow $\phi(x)$ in the semi-active valve defined by

$$\phi(x) = a_p(\dot{x}_1 - \dot{x}_2) + q_i \qquad (3)$$

To achieve improved ride comfort in the human sensitivity band (3 to 8Hz), we introduce the following frequency weighting function which in state-space is described as

$$\begin{aligned} \dot{x}_w &= A_w x_w + B_w z_p \\ z_w &= C_w x_w, \end{aligned} \qquad (4)$$

with

$$A_w = \begin{bmatrix} 0 & 1 \\ -900 & -24 \end{bmatrix},\ B_w = \begin{bmatrix} 0 \\ 2.7 \end{bmatrix},\ C_w = [1 \quad 0].$$

Now, using (1) and (4) and taking the road holding condition and the energy consumption (control input) into account, the generalized plant of our nonlinear problem can be obtained as

$$\begin{aligned} \dot{x} &= A(\phi)x + B_1 w + B_2(\phi)u, \\ z &= \begin{bmatrix} Cx \\ Du \end{bmatrix}, \end{aligned} \qquad (5)$$

where

$$A(\phi) = \begin{bmatrix} A_p(\phi) & 0 \\ B_w C_p(\phi) & A_w \end{bmatrix}, B_1 = \begin{bmatrix} B_{p1} \\ 0 \end{bmatrix},$$

$$B_2(\phi) = \begin{bmatrix} B_{p2}(\phi) \\ B_w D_p(\phi) \end{bmatrix},\ C = \begin{bmatrix} W_c & 0 \\ 0 & C_w \end{bmatrix},$$

$$D = \begin{bmatrix} w_{ac} & 0 \\ 0 & w_{se} \end{bmatrix},\ W_c = \begin{bmatrix} w_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_{12} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

- $w_{01} = 0.2$ is the weighting constant corresponding to road holding,

- $w_{12} = 0.03$ is the weighting constant corresponding to attitude,
- $w_{ac} = 0.0015$ is the weighting constant corresponding to active control input,
- $w_{se} = 0.0015$ is the weighting constant corresponding to semi-active control input.

Now, a feedback control

$$u = u(x) \tag{6}$$

will be called a $\gamma-$gain control if it internally stabilizes the system (5) and the following L_2- gain condition holds true for the closed-loop system (5)-(6)

$$\int_0^T ||z(t)||^2 dt \leq \gamma^2 \int_0^T ||w(t)||^2 dt \;\; \forall T > 0,\; \forall w \in L_2(0,T) \tag{7}$$

with zero state initial conditions (i.e $x(0) = 0$). The $\mathcal{H}_\infty$ control problem for the system (5) is to find a $\gamma-$gain control $u(x)$ with minimal $\gamma > 0$.

Clearly, (5) is a nonlinear system with 9 state variables, so the traditional approach based on HJI equation cannot be applied to solve the $\mathcal{H}_\infty$ control problem. Referring to equations (2) we see that (5) is nonlinear by the presence of the semi-active input $\phi(x)$ defined by (3). For physical reasons, $\phi(x)$ cannot take arbitrary values but is restricted in some predefined bounded set $\mathcal{D}$. Therefore, it is sufficient to design a control such that both internal stability and L_2 gain conditions (7) are practically fulfilled, i.e. they have to hold whenever $\phi(x) \in \mathcal{D}$ only. Therefore, an alternative way to attack the nonlinear $\mathcal{H}_\infty$ for the system (5) is to view the system as a family of linear systems depending on the semi-active input parameter $\phi(x)$. Suppose that for every fixed $\phi \in \mathcal{D}$, a $\gamma-$gain linear control is $K(\phi)x$ associated with some matrix $K(\phi)$ and a quadratic Lyapunov function $x'P(\phi)x$ establishing an L_2-gain condition. Then, as ϕ is varying as a function of x, we must find conditions on $K(\phi(x))$ and on

$$V(x) = x'P(\phi(x))x \tag{8}$$

such that the nonlinear system (5) with control input

$$u = K(\phi(x))x \tag{9}$$

satisfy the L_2-gain condition (7). It turns out in the next section that such conditions admit a tractable formulation in terms of PLMIs.

Note that function $V(x)$ might appear restrictive. However, such form is general enough since the recent max-plus algebra based results [7] show that the value function for a nonlinear system is indeed piecewise quadratic which obviously has a strong connection with the form (8).

3 PLMI characterization

Lemma 3.1 *There is $\gamma-$gain control if the following matrix inequalities hold true for $P(\phi) > 0$,*

$$\begin{bmatrix} -\dot{P}(\phi) + (P(\phi)A'(\phi) + *) & * & * \\ C_1 P(\phi) & -\gamma I & * \\ B_1' & 0 & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} B_2(\phi) \\ D_{12} \\ 0 \end{bmatrix} \begin{bmatrix} B_2'(\phi) & D_{12}' & 0 \end{bmatrix} < 0 \tag{10}$$

A particular K in (7) is

$$K(\phi) = -(D_{12}'D_{12})^{-1}[D_{12}'C_1 + \gamma B_2(\phi)'P(\phi)^{-1}] \tag{11}$$

The reader is referred to [1] and references therein for more details. Thus our focus now becomes to solve the differential inequality (10), which is still a difficult problem. To the aim of simplifying this problem, we shall examine some approximated representations of $P(\phi)$. Looking at the nonlinear system (5), we see that the nonlinear terms $A(\phi)$ and $B_2(\phi)$ can be expressed as

$$A(\phi) = A_0 + |\phi|A_1, \qquad B_2(\phi) = B_{20} + |\phi|\phi B_{21} \tag{12}$$

where A_0, A_1, B_{20}, B_{21} are constant matrices.

The structure (12) suggests to seek a solution $P(\phi)$ of the inequality (10) in the ad-hoc basis

$$P(\phi) = P_0 + |\phi|P_1 + |\phi|\phi P_2. \tag{13}$$

Note that $P(\phi)$ may be not differentiable at $\phi = 0$ but this does not cause any trouble in this application.

Now, restricting $(\dot{\phi}, \phi)$ on the area $M \times [-m_0, m_0]$ with $M \subset R$ bounded and $0 < m_0 \leq 1$ which can be done by changing A_1, B_{21} in (12) if necessary, and with the notation $\mathcal{P} := (P_0, P_1, P_2)$, $\theta := \dot{\phi}$, we can rewrite (10) in the form

$$\begin{aligned} M_0(\mathcal{P}) + \theta M_{01}(\mathcal{P}) + \theta\phi M_{02}(\mathcal{P}) + \phi M_1(\mathcal{P}) \\ +\phi^2 M_2(\mathcal{P}) + \phi^3 M_3(\mathcal{P}) + \phi^4 M_4 < 0, \\ \forall(\theta, \phi) \in M \times [0, m_0], \end{aligned} \tag{14}$$

where $M_0(\mathcal{P}), M_{01}(\mathcal{P}), M_{02}(\mathcal{P}),\ M_1(\mathcal{P}),\ M_3(\mathcal{P})$ are affine matrix-functions in $\mathcal{P}$.
Analogously, the positive definiteness of $P(\phi)$ can rewritten as

$$\begin{bmatrix} P_0 & \\ & P_0 \end{bmatrix} + \phi \begin{bmatrix} P_1 & \\ & P_1 \end{bmatrix} + \phi^2 \begin{bmatrix} P_2 & \\ & -P_2 \end{bmatrix} > 0 \quad \forall \phi \in [0, m_0] \tag{15}$$

Using a methos developed in [2, 12], the solvability of (14)-(15) is guaranteed by the following LMIs

$$\begin{aligned} M_0(\mathcal{P}) + \theta M_{01}(\mathcal{P}) + \theta\phi M_{02}(\mathcal{P}) + \phi M_1(\mathcal{P}) \\ + \max\{\phi^2 M_2(\mathcal{P}), (m_0\phi - 0.25m_0^2)M_2(\mathcal{P})\} \\ + \max\{\phi^3 M_3(\mathcal{P}), (0.75m_0^2\phi - 0.25m_0^3)M_3(\mathcal{P})\} \\ +(0.5m_0^3\phi - 0.1875m_0^4)M_4 < 0 \\ P_0 + \phi P_1 + \min\{\phi^2 P_2, (m_0\phi - 0.25m_0^2)P_2\} > 0 \\ P_0 + \phi P_1 + \min\{-\phi^2 P_2, -(m_0\phi - 0.25m_0^2)P_2\} > 0 \\ \forall(\theta, \phi) \in \text{vert}M \times \{0, m_0\}, \end{aligned} \tag{16}$$

4 Experimental results

The controller developed in Section 3 is implemented with a sample period of 5 ms. A hydraulic shaker simulates road disturbance generated by driving at 50 km/h. For solving the LMIs (16), we use the MATLAB LMI Control Toolbox [6].

The performance of our nonlinear control can be assessed by comparing its performance with other design methods such as

- Control with passive suspension having constant damping coefficient.
- Linear $\mathcal{H}_\infty$ control for the feedback linearized model of (5).

The passive suspension condition can be realized in our apparatus, by adjusting the semi-active valve according to

$$a_s = a_{s0}\sqrt{\frac{a_p|\dot{x}_1 - \dot{x}_2|}{|\phi_0|}}, \tag{17}$$

where ϕ_0 is the coefficient determining the damping coefficient.

On the other hand, if we use the nonlinear transformation

$$\hat{u}_s = \frac{|\phi(x)|\phi(x)}{|\phi_0|}(1 + u_s) - \phi(x), \tag{18}$$

then we can get the following exactly linearized model

$$\begin{aligned} \dot{x} &= A(\phi_0)x + B_1 w + B_2(\phi_0)\hat{u}, \\ z &= \begin{bmatrix} Cx \\ D\hat{u} \end{bmatrix}, \end{aligned} \tag{19}$$

where

$$\hat{u} = [\, u_a \quad \hat{u}_s \,]'.$$

Note that the parameters in (18) are chosen so that matrix A in (19) and for the passive suspension system coincide. Also, the same weighting function is used both for the linear and the nonlinear $\mathcal{H}_\infty$ control. The linear $\mathcal{H}_\infty$ control theory is readily applied to solve the $\mathcal{H}_\infty$ control problem for system (19). The frequency responses in Fig.2 and Fig. 3 represent the ratio of FFT for the road displacement and body accelerations. Fig.2 shows simulation results with impluse road displacement (height=0.03 m) and Fig.3 shows experimental results with random road displacement which expresses actual road surface (driving at 50 km/h). The time responses of the random road displacement and body acceleration with the nonlinear $\mathcal{H}_\infty$ controller are shown in Fig. 4. The control effect at human sensitivity frequency band ($3 \sim 8$ Hz) and lower frequencies is indicated in Fig.2. Clearly, the control effect of the nonlinear $\mathcal{H}_\infty$ control at frequencies lower than 5 Hz is better than that of the linear $\mathcal{H}_\infty$ control for the feedback linearized system (19).

Fig. 4 displays the ride comfort and control inputs with changing weighting constant w_{se} corresponding to semi-active control input from 0.0015 down to 0.0005. The frequency response characteristics in Fig. 5 demonstrate that the ride comfort can be improved by decreasing the weighting constant corresponding to the semi-active control input. The pure active suspension has a constant damping coefficient and the controller is designed by linear $\mathcal{H}_\infty$ control[16]. The integrated suspension and the pure active suspension have almost the same control performance. However, the energy consumption of the integrated suspension is better than pure active suspension as indicated in Fig 7. This shows that the semi-active valve avails to reduce the energy consumption.

5 Conclusion

In this paper, we have considered the nonlinear $\mathcal{H}_\infty$ control problem of an active suspension system. A novel approach has been proposed in this context. It is based on a PLMI characterization which provides sufficient conditions for closed-loop stability and performance of the nonlinear system. The main thrust of this approach, which is seemingly absent in many existing methodologies, is that it allows to solve nonlinear problems with large state dimensions. The only limitation appears to be the number of nonlinearities involved in the model description. When compared to more traditional techniques, it appears that the additional cost required for solving PLMI problems is more than offset by the advantages provided by the technique in terms of augmented stability and improved performance. This has been showed by a fairly complete set of simulations and experiments which finally more than anything else advocate for the use of the proposed method.

References

[1] P. Apkarian, R. J. Adams (1997), Advanced Gain-Scheduling Techniques for Uncertain Systems, *IEEE Trans. on Control System Technology*, vol. 6, 1, pp 21-32

[2] P. Apkarian, H.D. Tuan, Parameterized LMIs in control theory, To appear in *SIAM J. Control and Optimization.*

[3] S. Boyd, L. ElGhaoui, E. Feron, V. Balakrishnan (1994), *Linear Matrix Inequalities in Systems and Control Theory*, SIAM Studies in Applied Mathematics, Philadelphia.

[4] W.H. Fleming, W.M. McEneaney, A max-plus based algorithm for an HJB equation of nonlinear filtering, *Submitted.*

[5] Gahinet, P. and P. Apkarian (1994). A linear matrix inequality approach to $\mathcal{H}_\infty$ control. *Int. J. Robust Nonlinear Contr.* **4**, 421-448.

[6] P. Gahinet, A. Nemirovski, A.J.Laub, M.Chilali (1994), *LMI Control Toolbox,* The MathWorks Inc..

[7] M. Horton, W.L. McEneaney, Max-plus eigenvector representations for nonlinear H_∞ value functions, *To appear in ACC99.*

[8] E. Ono, K. Takanami, N. Iwama, Y. Hayashi, Y. Hirano, Y. Satoh (1994), Vehicle integrated control for steering and traction systems by μ synthesis, *Automatica* 30, 1639-1647.

[9] E.Ono, S.Hosoe, H.D.Tuan and Y.Hayashi (1996) Nonlinear $\mathcal{H}_\infty$ control of active suspension, *Vehicle System Dynamics Supplement* 25, 489-501.

[10] E. Ono, S. Hosoe, H.D. Tuan, S. Doi (1998) Bifurcation in vehicle dynamics and robust front wheel steering control, *IEEE Transactions on Control Systems Technology* 6, 412-420.

[11] H.D. Tuan, S. Hosoe (1996), On linearization technique in robust nonlinear $\mathcal{H}_\infty$ control, *Systems and Control Letters* 26, 21-27.

[12] H.D. Tuan, P. Apkarian, Relaxation of parameterized LMIs with control applications, To appear in *International J. of Nonlinear Robust Controls*.

[13] Yamashita, M., K. Fujimori, C. Uhlik, R. Kawatani and H. Kimura (1990). $\mathcal{H}_\infty$ control of an automotive active suspension. *Proc. of 29-nd CDC*, 2244-2250.

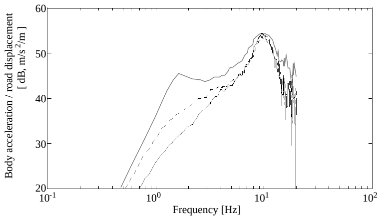

Figure 2: Frequency responses of the experimental results: Nonlinear $\mathcal{H}_\infty$ control (solid), linear $\mathcal{H}_\infty$ control for the feedback linearized model (dashed) and passive (gray)

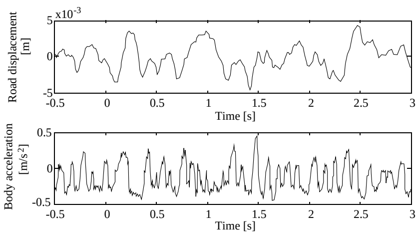

Figure 3: Road displacement and body acceleration with nonlinear $\mathcal{H}_\infty$ control

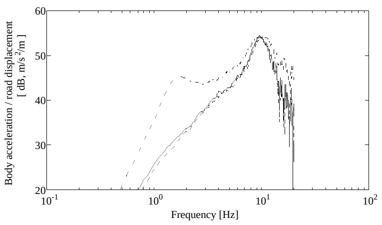

Figure 4: Ride comfort performance of nonlinear $\mathcal{H}_\infty$ control (at $W_{se} = 0.0015$ (solid) and $W_{se} = 0.0005$ (dashed)) and passive control (dash-dot)

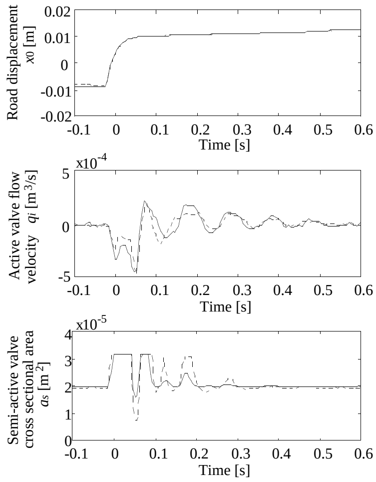

Figure 5: Change of the control parameter $W_{se} = 0.0015$ (solid) and $W_{se} = 0.0005$ (dashed)

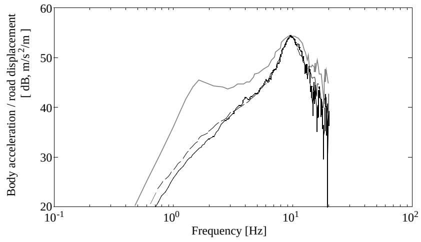

Figure 6: Frequency characteristics of experimental results: the integrated suspension (solid), the pure active suspension (dashed) and passive suspension (gray)

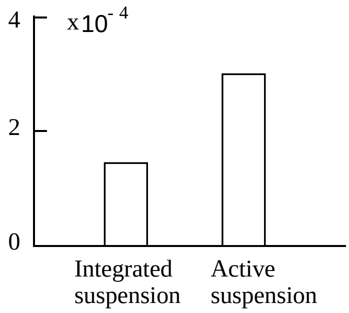

Figure 7: Active oil consumption during 20 seconds