A New Lagrangian Dual Global Optimization Algorithm for Solving Bilinear Matrix Inequalities

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Abstract

A new global optimization algorithm for solving Bilinear Matrix Inequalities (BMI) problems is developed. It is based on a dual Lagrange formulation for computing lower bounds that are used in a branching procedure to eliminate partition sets in the space of nonconvex variables. The advantage of the proposed method is twofold. First, lower bound computations reduce to solving easily tractable Linear Matrix Inequality (LMI) problems. Secondly, the lower bounding procedure guarantees global convergence of the algorithm when combined with an exhaustive partitioning of the space of nonconvex variables. Another important feature is that the branching phase takes place in the space of nonconvex variables only, hence limiting the overall cost of the algorithm. Also, an important point in the method is that separated LMI constraints are encapsulated into an augmented BMI for improving the lower bound computations. Applications of the algorithm to robust structure/controller design are considered.

1 Introduction

A general BMI problem can be formulated as follows

$$\min \langle c, x\rangle + \langle d, y\rangle : \quad (1)$$

$$x \in X := [p^0, q^0] \subset R^n, \quad (2)$$

$$G_0 + \sum_{j=1}^{m} y_j G_j \leq 0, \quad (3)$$

$$L_0 + \sum_{i=1}^{n} x_i L_{i0} + \sum_{j=1}^{m} y_j L_{0j} + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j L_{ij} < 0, \quad (4)$$

where $G_0, G_j, L_0, L_{0i}, L_{j0}, L_{ij}$ are symmetric matrices of appropriate sizes and x and y are the decision variables.

It is widely recognized that such BMI problems arises frequently in the study of control problems and thus they constitute a very important class of optimization problems with a vast array of potential applications. Unfortunately, in contrast to linear matrix inequality (LMI) problems which are convex and can be solved by polynomial-time interior-point methods, BMI problems (1)-(4) are nonconvex and known to be NP-hard [19]. The hardness comes from the BMI constraint (4).

Alternatively, the BMI (4) can be rewritten as

$$L_0 + \sum_{i=1}^{n} x_i L_{i0} + \sum_{j=1}^{m} y_j L_{0j} + \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij} L_{ij} < 0, \quad (5)$$

$$w_{ij} = x_i y_j,\ j = 1, 2, ..., n;\ j = 1, 2, ..., m, \quad (6)$$

where (5) is now a convex LMI constraint in (x, y, w) and (6) is a nonconvex indefinite quadratic constraint. Therefore techniques which can efficiently handle the quadratic constraint (6) are of special interest for solving (1)-(4).

Obviously, the LMI constraint (3) includes as a particular case the box constraint

$$y \in [r, s] \subset R^m \quad (7)$$

and other linear constraints for y.

Goh et al. [10] proposed a simple branch and bound (BB) algorithm for problem (1), (4), (7) with branching in the (x, y)-space of all variables hence of dimension $n + m$, and bounding based upon a relaxation of the nonconvex constraint (6) for $(x_i, y_j) \in [p_i, q_i] \times [r_j, s_j]$ by a coarse approximated constraint

$$w_{ij} \in [p_i r_j, q_i s_j]. \quad (8)$$

It can be assumed without loss of generality that $p_i \geq 0$ and $r_j \geq 0$. Note that the nonconvex quadratic constraint (6) has been well studied in global optimization and it is classically known that a much tighter relaxed constraint than (8) for (6) (see e.g. [15, 2, 12, 18]) is given as

$$\max\{r_j x_i + p_i y_j - p_i r_j, s_j x_i + q_i y_j - q_i s_j\} \leq w_{ij} \leq \min\{s_j x_i + p_i y_j - p_i s_j, r_j x_i + q_i y_j - q_i r_j\} \quad (9)$$

In [3, 22] it is further shown that functions max and min in (9) are in fact concave and involves convex envelopes of the bilinear function $x_i y_j$ in (6). The constraint (9) is nothing else than the following constraint

$$(x_i, y_j, w_{ij}) \in \text{convex hull } \{(p_i, r_j, p_i r_j), (p_i, s_j, p_i s_j), (q_i, r_j, q_i r_j), (q_i, s_j, q_i s_j)\} \quad (10)$$

which is used in [11] to compute a lower bound of (1)-(4). However, the form (9) which is an explicit description of (10), is more convenient for computational implementation. Both (9) and (10) have been essentially established in [15].

On the other hand, the most fundamental convexity property of the BMI problem (1)-(4) is that it becomes convex when either y or x is fixed. Therefore, regarding only x (when $n \leq m$) or y (when $m < n$) as "complicating variables", BB algorithms with branching only on the space of complicating variables (instead of the whole space of all variables as done in [10, 11]) have been proposed in [20, 5, 6]. Such branching techniques are really important as the global search procedure is restricted to operate in the low dimensional space of dimension $\min\{n, m\}$ compared with the dimension $n + m$ of the space of all variables and thus bring to a more reasonable extent the difficulties of "the curse of dimensionality" inherent to most nonconvex problems. An improved algorithm to that of [10, 11] has been proposed in [13]. It requires computing a local optimal solution of problem (1)-(4) at every iteration. Note that even this local optimization problem is itself NP-hard and therefore time consuming. The algorithm of [20] uses the relaxed constraint (9) for obtaining a lower bound and the d.c. (difference of convex functions) structure (see e.g [14, 22]) of the nonconvex constraint (6) to perform the decomposition. In [5, 6], the authors proposed lower bounds based upon $2^{\min\{n,m\}}$ LMI *relaxed* convex problems of the Lagrange dual problem of (1)-(4) by fixing the complicating variables followed by a generalized Bender decomposition. While the convergence (to a global optimal solution) of algorithm [20] has been shown, it remains a very delicate issue in the algorithm of [5, 6].

In this paper, we first convert the BMI problem (1)-(4) into a BMI problem with special structure. Exploiting this structure, the Lagrange dual problem is shown to be an LMI problem and is used to compute a new lower bound. This lower bound is then encapsulated in a general rectangular partitioning scheme to constitute the proposed global algorithm. With the proposed decomposition technique, we provide a rigorous proof of global convergence of our algorithm.

The organization of the paper is as follows. Section 2 investigates the dual BMI problem which is used in Section 3 to compute cheap lower bounds. The new BB algorithm is also established in this section. Finally, an application of the proposed algorithm to a robust structure/control design problem is given in Section 4. The notation in the paper is standard. $\langle ., . \rangle$ is the scalar product in a finite dimensional linear space. $A < 0$ ($A \leq 0$, resp.) for a symmetric matrix A, means that A is negative definite (semi-definite, resp.). Accordingly, $A < B$ for symmetric matrices A and B means that $A - B$ is negative definite. Also, $Tr(A)$ stands for the trace of A. To save a description space we shall use

$$\begin{bmatrix} A \\ B \end{bmatrix}_d \quad \text{for} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Finally, $x \geq p$ for vectors in R^n indicates componentwise inequalities $x_i \geq p_i$ with $i = 1, 2, ..., n$.

2 The BMI Problem and its dual

Without loss of generality, we assume throughout that $n \leq m$ in (1)-(4). By translation if necessary, we can assume that $y \in R^m_+$. For fixed x problem (1)-(4) is a convex LMI problem, so x can be considered as the complicating variable[14, 22]. Based on this observation, our target is to develop a BB method for solving (15) with branching performed upon x. For this purpose, for $M = [p, q] \subset X$, we are interested in the subproblem

$$\min \langle c, x \rangle + \langle d, y \rangle \quad : \quad x \in M = [p, q], \quad (3) - (4)$$

which can be rewritten as

$$\min\ t \quad : \quad x \in M = [p, q], \tag{11}$$

$$A_{00}(x) + \sum_{j=1}^{m} y_j A_{j0}(x) \leq t Q_{00} \tag{12}$$

with

$$\begin{aligned} A_{00}(x) &= \begin{bmatrix} G_0 \\ L_0 + \sum_{i=1}^{n} x_i L_{i0} \\ \langle x, c \rangle \end{bmatrix}_d, \\ A_{j0}(x) &= \begin{bmatrix} G_j \\ L_{0j} + \sum_{i=1}^{n} x_i L_{ij} \\ d_j \end{bmatrix}_d, \quad j = 1, ..., n; \\ Q_{00} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_d. \end{aligned} \tag{13}$$

Note that the LMI structure of (3) is hidden in (12). It is possible to exploit this substructure and the box constraint $x \in M$ to augment the BMI constraint by additional BMI constraints which leaves the optimal value of the program (11)-(12) unchanged but leads to improved lower bounds.

Proposition 1 *LMI (3) with the box constraint (11) in force infer the additional set of BMI constraints*

$$\begin{aligned} (x_i - p_i)G_0 + \sum_{j=1}^{n} y_j (x_i - p_i) G_j &\leq 0; \\ (q_i - x_i)G_0 + \sum_{j=1}^{n} y_j (q_i - x_i) G_j &\leq 0, \\ j = 1, 2, ..., n. \end{aligned} \tag{14}$$

Thus we can convert (11)-(12) into the following BMI problem

$$\begin{aligned} f^*_M := \min\ t : \quad & A_0(x, p, q) + \sum_{j=1}^{m} y_j A_j(x, p, q) \leq t Q, \\ & y \geq 0, \ x \in M, \end{aligned} \tag{15}$$

where

$$A_j(x) = \begin{bmatrix} A_{j0}(x) \\ A_{j1}(x,p,q) \end{bmatrix}_d, \quad Q = \begin{bmatrix} Q_{00} \\ Q_{01} \end{bmatrix}_d, \quad Q_{01} = 0,$$

$$A_{j1}(x,p,q) = \begin{bmatrix} (x_1 - p_1)G_j \\ (q_1 - x_1)G_j \\ .... \\ (x_n - p_n)G_j \\ (q_n - x_n)G_j \end{bmatrix}_d, \quad j = 0, 1, ..., n.$$

The Lagrange dual of (15) is then described as

$$g_M := \max_{Z \geq 0} \min_{t \in R, y \geq 0, x \in M} \{t + \mathrm{Tr}[Z(A_0(x,p,q) + \sum_{j=1}^{m} y_j A_j(x,p,q) - tQ)]\} =$$
$$\max_{Z \geq 0}\{\min_{x \in M} \mathrm{Tr}(ZA_0(x,p,q)) : \mathrm{Tr}(ZQ) = 1,$$
$$\mathrm{Tr}(ZA_j(x,p,q)) \geq 0, \ \forall x \in M, \ j = 1, 2, ..., m\} \quad (16)$$

We note that generally $g_M < f_M^*$, since there is a duality gap in nonconvex problems. However, when M is just a singleton then (15) is convex and $g_M = f_M^*$ under the standard constraint qualification in semidefinite programming. Thus as shown in the next sections, we can reduce the duality gap $f_M^* - g_M$ by partitioning X. As shown hereafter, problem (16) actually is an easily tractable LMI optimization problem. This is in sharp contrast with the generalized Bender's decomposition for (15) in [5, 6] where a nonconvex dual problem has to be considered.

3 A Branch and Bound Method and its convergence

For every rectangle $M \subset X$ denote by $(Q(M))$ problem (16). Since $A_j(x,p,q)$ is affine in x, we infer

$$\min_{x \in M} \mathrm{Tr}(ZA_0(x,p,q)) = \min_{x \in \mathrm{vert}(M)} \mathrm{Tr}(ZA_0(x,p,q)),$$

and

$$\mathrm{Tr}(ZA_j(x,p,q)) \geq 0, \ \forall x \in M \quad \Leftrightarrow$$
$$\mathrm{Tr}(ZA_j(x,p,q)) \geq 0, \ \forall x \in \mathrm{vert}(M),$$

where $\mathrm{vert}(M)$ stand for the set of vertices of M.

We deduce that $(Q(M))$ is equivalent to

$$\max_{Z \geq 0}\{ \min_{x \in \mathrm{vert}(M)} \mathrm{Tr}(ZA_0(x,p,q)) :$$
$$\mathrm{Tr}(ZQ) = 1, \ \mathrm{Tr}(ZA_j(x,p,q)) \geq 0$$
$$\forall x \in \mathrm{vert}(M), \ j = 1, 2, ..., m\} \quad (17)$$

which is actually an *LMI program*, alternatively rewritten as

$$\psi_M^* = \max\{t : \ \mathrm{Tr}(ZA_0(x,p,q)) \geq t,$$
$$\mathrm{Tr}(ZA_j(x,p,q)) \geq 0, \ x \in \mathrm{vert}(M),$$
$$j = 1, 2, ..., m,$$
$$\mathrm{Tr}(ZQ) = 1, \ Z \geq 0\}$$

Clearly, $\psi_M^* \leq f_M^*$, i.e. ψ_M^* is a lower bound for f_M^*. Using this lower bound, we can conceive the following BB algorithm for solving the BMI problem.

Algorithm

- *Initialization.* Let $(\bar{x}^1, \bar{y}^1, \bar{t}^1)$ be an initial feasible solution with $\bar{t}^1 = \lambda_{\max}[A_{00}(\bar{x}^1) + \sum_{j=1}^m \bar{y}_j^1 A_j(\bar{x}^1)]$. Set $M_0 = X, \mathcal{S}_1 = \mathcal{P}_1 = \{M_0\}$. Set $k = 1$.

- *Step* 1. For each rectangle $M = [p,q] \in \mathcal{P}_k$ solve

  $$\max_{Z \geq 0}\{\min_{x \in M} \ \mathrm{Tr}(ZA_0(x,p,q)) : \ \mathrm{Tr}(ZQ) = 1,$$
  $$\mathrm{Tr}(ZA_j(x,p,q)) \geq 0, \ \forall x \in \mathrm{vert}(M)\}$$

  to obtain ψ_M^*.

- *Step* 2. Delete every rectangle $M \in \mathcal{S}_k$ such that $\psi_M^* \geq \bar{t}^k - \varepsilon$. Let $\mathcal{R}_k$ be the collection of remaining rectangles.

- *Step* 3. If $\mathcal{R}_k = \emptyset$ then terminate: $(\bar{x}^k, \bar{y}^k, \bar{t}^k)$ is an ε-optimal solution.

- *Step* 4. Let $M_k \in \mathrm{argmin}\{\psi_M^* | \ M \in \mathcal{R}_k\}$.

  Compute a new feasible solution by local search from the center of M_k (or by solving an LMI program, see Remark below). Let $(\bar{x}^{k+1}, \bar{y}^{k+1}, \bar{t}^{k+1})$ be the new incumbent with $\bar{t}^{k+1} = \lambda_{\max}[A_{00}(\bar{x}^{k+1} + \sum_{j=1}^m \bar{y}_j^{1+1} A_{j0}(\bar{x}^{k+1})]$.

- *Step* 5. Bisect M_k upon its longest edge. Let $\mathcal{P}_{k+1}$ be the partition of M_k.

- *Step* 6. Set $\mathcal{S}_{k+1} = (\mathcal{R}_k \setminus \{M_k\}) \cup \mathcal{P}_{k+1}, k \leftarrow k+1$ and go back to Step 1.

An important question concerns the convergence properties of the proposed BB algorithm. The answer turns out to be positive under the following standard regularity assumption [21].

$$(\mathbf{R}) \qquad \begin{array}{l} (\forall x \in X) \quad (\exists Z_1 \geq 0) \quad \mathrm{Tr}(Z_1 Q_{00}) = 1, \\ \mathrm{Tr}(Z_1 A_{j0}(x)) > 0, \ j = 1, 2, ..., m. \end{array}$$

Note that $(\mathbf{R})$ simply means that that whenever $x \in X$ is fixed there is no duality gap between (1)-(4) and its dual [16]. Such assumption is present in every primal-dual algorithms of semi-definite programming. Alternatively, we can replace this assumption by the strict feasibility of (3)-(4) for every fixed $x \in X$.

In step 4 of the algorithm a new incumbent is easily generated as follows. Let x^k be any point of M_k, then an optimal solution (y^k, t^k) of the LMI program

$$\min\{t : \ A_{00} + \sum_{j=1}^{m} y_j A_{j0}(x^k) \leq tQ, y \geq 0\}$$

yields a feasible solution (x^k, y^k, t^k) of the primal problem.

4 Application to a robust control problem

In this section, we provide a simple illustration of the proposed method for solving BMI problems which can serve as our preliminary computational result. The best illustration for this is a a problem which can be reduced to the form (1)-(4) with n small compared with m. Thus, we consider an $\mathcal{H}_\infty$ control problem in which both the controller and some plant's coefficients or structure can be designed simultaneously.

In this example, the plant type is given beforehand, but parameter values are adjusted off-line to improve the performance of the overall system. Our aim is to attenuate the effect of disturbances on the mass in a mass-spring arrangement. The controlled system is shown in Figure 1, where x is the position of mass, u is the control force input, w is the disturbance force, m is the mass ($m = 4$), k and c are the spring and damping coefficients which are allowed to take values in a parameter box H,

$$p := (k, c) \in H = [4, 12] \times [0.5, 1.5].$$

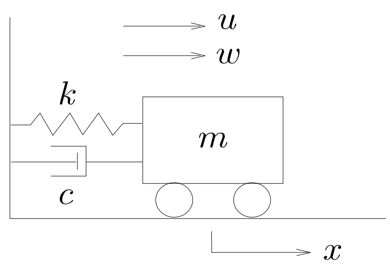

Figure 1: The mass-spring-damper system

The controlled output z and measurement output y are

$$z = [x, u]', \quad y = x.$$

Considering $[x, \dot{x}]'$ as the state vector, the generalized plant is given as

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & \frac{1}{m} & \frac{1}{m} \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right].$$

Now, the problem can be stated as to find an internally stabilizing proper controller $K(s)$ and a parameter $p \in H$ minimizing the H_∞ norm $||T_{zw}||_\infty$, where T_{zw} is the closed-loop transfer function from w to z. It is well known [8] that this problem has the BMI formulation

(P) minimize γ :

$$\begin{array}{l} \mathcal{N}_1' \begin{bmatrix} A(p)\widehat{R} + \widehat{R}A(p)' & \widehat{R}C_1' & B_1 \\ C_1\widehat{R} & -\gamma I & 0 \\ B_1' & 0 & -\gamma I \end{bmatrix} \mathcal{N}_1 < 0 \\ \mathcal{N}_2' \begin{bmatrix} A(p)'\widehat{S} + \widehat{S}A(p) & \widehat{S}B_1 & C_1' \\ B_1'\widehat{S} & -\gamma I & 0 \\ C_1 & 0 & -\gamma I \end{bmatrix} \mathcal{N}_2 < 0 \\ \begin{bmatrix} \widehat{R} & I \\ I & \widehat{S} \end{bmatrix} \geq 0 \\ p \in H, \end{array} \tag{18}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ denote any bases of the null spaces of $[B_2', D_{12}', 0]$ and $[C_2, D_{21}, 0]$, respectively.

It is important to note that the dimension of the complicating variable p in (18) is just 2 and compares favorably with the dimension 9 of the all decision variables $(\gamma, \hat{R}, \hat{S}, p)$ in (18). Thus, our BB algorithm with branching performed in a subset of space R^2 is really advantageous compared with BB algorithms operating in R^9. The optimal solution is

$$(k_{opt}, c_{opt}) = (11.969, 1.469),$$

$$R_{opt} = \begin{bmatrix} 0.1351 & -0.0248 \\ -0.0248 & 0.4043 \end{bmatrix},$$
$$S_{opt} = \begin{bmatrix} 9.2882e^{7} & 1.4606e^{-8} \\ 1.4606e^{-8} & 3.2739e^{-7} \end{bmatrix}$$

From Table 1, we can see how the duality gap is reduced along iterations. The benefit of the structure design is clear by looking at Table 2, where the performance of the plant with the optimal structure has been improved compared with the nominal plant. The step responses of given in Fig. 2 confirms that tracking performance of the optimal structure systems is much better than that of the nominal plant.

5 Concluding remarks

A new convergent BB algorithm of global optimization for solving BMI problems is proposed. First of all, the original problem is reduced to a convenient form such that its Lagrangian dual problem is just a convex LMI optimization problem which is used for computing lower bounds. Exploiting the most useful characterization of BMI that it becomes convex when a certain number of so called "complicating" are held fixed, the duality gap is reduced by an exhaustive partitioning on a reduced space of these variables which favorably limits the overall cost of the algorithm. The viability of the proposed algorithms is confirmed by solving a robust structure/control design and other computational examples.

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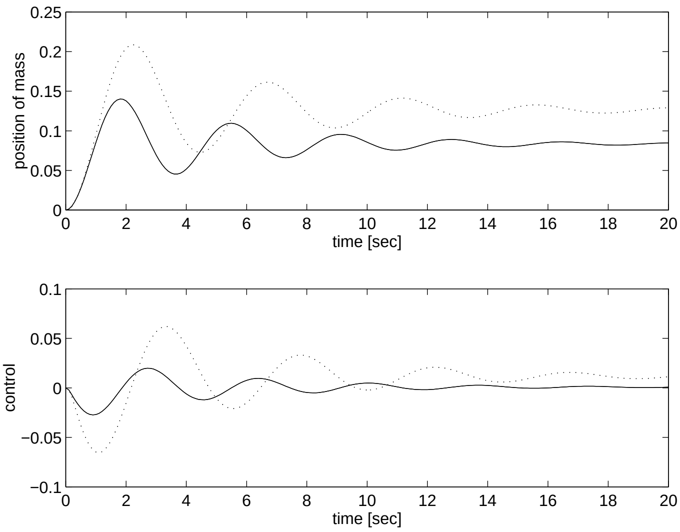

Figure 2: The tracking performance of the optimal structure plant (solid) and the nominal plant (dot) under the step disturbance

iteration	lower bound	current best value
1	0.240	0.579
2	0.293	0.536
3	0.321	0.518
4	0.323	0.518
5	0.339	0.509
6	0.336	0.428
7	0.344	0.424
8	0.347	0.424
9	0.347	0.392
10	0.351	0.391
11	0.347	0.391
12	0.353	0.391
13	0.353	0.391
14	0.353	0.376
15	0.354	0.376
16	0.356	0.375
17	0.357	0.375
18	0.357	0.369
19	0.357	0.369
20	0.359	0.368

Table 1: Performance of the algorithm. Total cputime is 598 sec.

parameter (k, c)	performance level γ_{opt}
(11.969,1.469) (optimal)	0.3681
(8,1) (nominal)	0.5791

Table 2: comparison with nominal case