Parameterized Linear Matrix Inequality Techniques in Fuzzy Control System Design

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***Abstract*—This paper proposes different parameterized linear matrix inequality (PLMI) characterizations for fuzzy control systems. These PLMI characterizations are, in turn, relaxed into pure LMI programs, which provides tractable and effective techniques for the design of suboptimal fuzzy control systems. The advantages of the proposed methods over earlier ones are then discussed and illustrated through numerical examples and simulations.**

***Index Terms*—Fuzzy systems, parameterized linear matrix inequality (PLMI).**

I. INTRODUCTION

THE well-known Tagaki–Sugeno (T–S) fuzzy model [13] is a convenient and flexible tool for handling complex nonlinear systems [11], where its consequent parts are linear systems connected by IF–THEN rules. Suppose that x is the state vector with dimension n_x, u is the control input with dimension n_u, w, and z are the disturbance and controlled output of the system with the same dimension n_{wz}, and L denotes the number of IF–THEN rules, where each ith plant rule has the form

$$\begin{aligned}&\text{IF} \quad z_1(t) \text{ is } N_{i1} \text{ and } \cdots\ z_p(t) \text{ is } N_{ip}\\ &\text{THEN} \quad \begin{bmatrix}\dot{x}\\ z\end{bmatrix} = \begin{bmatrix}A_i & B_{1i} & B_{2i}\\ C_i & D_{11i} & D_{12i}\end{bmatrix}\begin{bmatrix}x\\ w\\ u\end{bmatrix}.\end{aligned} \tag{1}$$

Here, z_i are premise variables assumed independent of the control u and N_{ij} are fuzzy sets. Denoting by $M_{ij}(z_i(t))$ the grade of membership of $z_i(t)$ in N_{ij} and normalizing the weight $w_i(t) = \prod_{j=1}^{p} N_{ij}(z_i(t))$ of each ith IF–THEN rule by

$$\alpha_i(t) = \frac{w_i(t)}{\sum_{j=1}^{L} w_j(t)} \geq 0, \qquad i = 1, 2, \ldots, L \tag{2}$$

$$\begin{aligned}\Rightarrow \quad \alpha(t) &= (\alpha_1(t), \ldots, \alpha_L(t)) \in \Gamma\\ &:= \left\{\alpha \in R^L \colon \sum_{i=1}^{L} \alpha_i = 1,\ \alpha_i \geq 0\right\}\end{aligned} \tag{3}$$

the state-space representation of the T–S model is

$$\begin{bmatrix}\dot{x}\\ z\end{bmatrix} = \begin{bmatrix}A(\alpha) & B_1(\alpha) & B_2(\alpha)\\ C(\alpha) & D_{11}(\alpha) & D_{12}(\alpha)\end{bmatrix}\begin{bmatrix}x\\ w\\ u\end{bmatrix} \tag{4}$$

where

$$\begin{bmatrix}A(\alpha) & B_1(\alpha) & B_2(\alpha)\\ C(\alpha) & D_{11}(\alpha) & D_{12}(\alpha)\end{bmatrix} = \sum_{i=1}^{L} \alpha_i(t)\begin{bmatrix}A_i & B_{1i} & B_{2i}\\ C_i & D_{11i} & D_{12i}\end{bmatrix}. \tag{5}$$

The simple and natural feedback control for T–S model is the so-called parallel distributed compensation (PDC), whose each ith plant rule is inferred similarly to (1) as

$$\begin{aligned}&\text{IF} \quad z_1(t) \text{ is } N_{i1} \text{ and } \cdots\ z_p(t) \text{ is } N_{ip}\\ &\text{THEN} \quad u = K_i x.\end{aligned} \tag{6}$$

The outcome is the state-feedback control law:

$$u(t) = K(\alpha(t))x \tag{7}$$

where

$$K(\alpha(t)) = \sum_{j=1}^{L} \alpha_j(t) K_j. \tag{8}$$

Since $\alpha(t)$ in (4) is available online, system (4) also belongs to the more general class of gain-scheduling control systems intensively studied in control theory in the past decade (see, e.g., [12], [1], and [2]). Gain-scheduling is a widely used method for the control of nonlinear plants or a family of linear models. Only recently, however, this technique has received a systematic treatment within the framework and tools based on LMIs [12], [1], [2]. LMI characterizations of the gain-scheduling control problem renders the design task both practical and appealing since LMIs can be globally and efficiently solved by interior-point methods in semidefinite programming. The representation relation (3) and (4) is often called a polytopic system [7] a class of parameter-dependent systems, which lends itself easily to practical computations. At first glance, it could appear that the additionally restricted structure (8) for PDC fuzzy control incurs conservatism in the synthesis problem in comparison with the most general structure (7) often considered in gain-scheduling control [14], [15]. In fact, the main contribution of [14] and [15] is to adapt the approach of [7] (for polytopic systems) to design PDC controller (8). However, by a main result presented in this paper, the existence of a general gain-scheduling freely structured controller (7) is equivalent to the existence of one with PDC structure (8). In other words, the PDC structure (7) very naturally arises in gain-scheduling control. Moreover,

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our presented results based on a general theory of gain-scheduling control [1], [2], [12] have the following essential advantages over those of [14] and [15]:

1) The resulting optimization formulations are much simpler with much fewer variables involved. Therefore, they are much more efficient computationally. In fact, our computational experiments show that the cpu time for solving problems of [14] and [15] are 2–4 times larger than that needed for solving our problems.
2) The controller performance are much better. In fact, our computational experience indicates that our controllers improve the performance by a significant order of magnitude of 10–15 as compared to those of [14] and [15].

It is important to note that the aforementioned advantages are also achieved by our new relaxation results for solving arising parameterized linear matrix inequalities (PLMIs). To see how PLMIs naturally arise in gain-scheduling control including fuzzy logic control, let us consider the stabilization problem where we seek K_j stabilizing (1), i.e., such that system

$$\dot{x}(t) = (A(\alpha) + B_2(\alpha)K(\alpha))x(t) \tag{9}$$

resulting from (1) and (8) by setting $w = 0$ is asymptotically stable.

By virtue of the Lyapunov theorem, (9) is asymptotically stable if there exists a quadratic Lyapunov function $V(x) = x^T X x$ such that

$$\begin{aligned}
&\frac{dV(x(t))}{dt} < 0 && (10)\\
\Leftrightarrow\ &(A(\alpha) + B_2(\alpha)K(\alpha))^T X + X(A(\alpha) + B_2(\alpha)K(\alpha))\\
&\quad < 0 \qquad \forall\, \alpha \in \Gamma\\
\Leftrightarrow\ &P(A(\alpha) + B_2(\alpha)K(\alpha))^T + (A(\alpha) + B_2(\alpha)K(\alpha))P\\
&\quad < 0, \qquad \forall\, \alpha \in \Gamma, \quad P = X^{-1}\\
\Leftrightarrow\ &PA(\alpha)^T + R(\alpha)^T B_2(\alpha)^T + A(\alpha)P + B_2(\alpha)R(\alpha)\\
&\quad < 0, \qquad \forall\, \alpha \in \Gamma && (11)\\
&R(\alpha) = K(\alpha)P. && (12)
\end{aligned}$$

The linearization technique (12) is rather standard and well known in control theory, even before LMI invention (see, e.g., [6]). Particularly, it is the main tool of [7, Ch. 7] in state-feedback control for polytopic systems. Later, it has been adapted in [14] and [15] for designing PDC of the form (8). In fact, with the PDC (8), $R(\alpha)$ has the form

$$R(\alpha) = \sum_{j=1}^{L} \alpha_j R_j, \qquad R_j = K_j P,\ j = 1, 2, \ldots, L. \tag{13}$$

Therefore, (11) can be rewritten as

$$\sum_{i,\,j=1}^{L} \alpha_i \alpha_j \mathcal{M}_{ij}(Z) < 0, \qquad \forall\, \alpha \in \Gamma \tag{14}$$

where

$$\begin{aligned}
Z &= (P, R)\\
R &= (R_1, R_2, \ldots, R_L)\\
\mathcal{M}_{ij}(Z) &= PA_i^T + R_j^T B_{2i}^T + A_i P + B_{2i} R_j && (15)
\end{aligned}$$

i.e., $\mathcal{M}_{ij}$ is an affine matrix-valued function of the variable Z.

Note that (14) is an LMI problem depending on the parameter α, i.e., one has to check the LMIs in (14) holds for all $\alpha \in \Gamma$, hence, the named PLMI.

As we shall see, PLMIs like (14) also arise in other control problems such as the regulator problem, H_2, H_∞ control problems and so forth. PLMI problems of the form (14) belong to the class of robust semidefinite programs, which is a very hard optimization problem whose NP-hardness is well known [4]. Therefore, it is natural to derive some convex (LMI) relaxations for (14) (see, e.g., [5], [3], and [17]) to make it computationally tractable. For $\alpha_i \geq 0$ it is obvious that one such convex (LMI) relaxation for (14) is obtained as [15]

$$\mathcal{M}_{ii}(Z) < 0, \quad \mathcal{M}_{ij}(Z) + \mathcal{M}_{ji}(Z) \leq 0, \quad 1 \leq i < j \leq L. \tag{16}$$

Unfortunately, conditions (16) are practically very restrictive and some potential improvements have been discussed in [14]–[16]. Other convex relaxations techniques solving a general PLMI including (14) as a particular case have been proposed in [17] and [3].

A major target of this paper is to give some new convex relaxation results for PLMI (14), which include and generalize all previous results in [14], [15], [17], and [3] as a particular case and are less conservative, i.e., they offer much better solutions while are still computationally efficient.

Since the work of [9], it is known that in many cases the control variable R in (11) can be eliminated by using the Projection Lemma [9] or the Finsler's Lemma. Such an elimination procedure not only makes LMI formulations much more appealing for computation but plays a key role for obtaining LMI characterizations in dynamic output feedback problems. In this paper, such elimination technique is adapted to obtain simpler PLMI characterizations with the two aforementioned advantages compared with (14), (15) and other arising in regulator and $\mathcal{H}_\infty$ control problems.

The structure of the paper is as follows. The main results on LMI relaxation for PLMIs are given in Section II. Then, based on this, different PLMI characterizations for stabilization, regulator, $\mathcal{H}_\infty$ control problems together with their LMI relaxations are considered in Sections III–V. The comparison between these LMI relaxations are illustrated by numerical examples in Section VI.

The notation of the paper is fairly standard. M^T is the transpose of the matrix M. For symmetric matrices, $M - N < 0$ ($M - N > 0$, respectively) means $M - N$ is negative definite (positive definite, respectively). In symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for terms that are induced by symmetry, e.g.,

$$K \begin{bmatrix} S & * \\ M & Q \end{bmatrix} * \equiv K \begin{bmatrix} S & M^T \\ M & Q \end{bmatrix} K^T.$$

Useful instrumental tools such as congruent transformation of matrices, Shur's complement and Finsler's lemma are given in the Appendix.

II. LMI Relaxations for PLMIs

The following intermediate result proves to be useful in the sequel.

Lemma 2.1: Given a 2×2-symmetric matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \qquad p_{ij} \in R$$

one has

$$x^T P x < 0 \qquad \forall x_i \geq 0, \quad x = (x_1, x_2) \neq 0 \tag{17}$$

if and only if there is $q_{12} \in R$ such that

$$\begin{bmatrix} p_{11} & q_{12} \\ q_{12} & p_{22} \end{bmatrix} < 0, \qquad p_{12} \leq q_{12}. \tag{18}$$

A sufficient condition for (17) and (18) is

$$p_{ii} < 0, \qquad p_{ii} + p_{ij} < 0, \qquad i, j = 1, 2. \tag{19}$$

Proof: First, let us prove (17) $\Leftrightarrow$ (18). Since the implication (18) $\Rightarrow$ (17) is obvious, we need only prove the inverse implication. It is trivial that $p_{11} < 0$, $p_{22} < 0$. If $p_{12} \leq 0$ in (17), then (18) is obvious with $q_{12} = 0$. On the other hand, when $p_{12} > 0$, taking $x_2 = 1$, then (17) implies

$$\begin{aligned} & p_{11}x_1^2 + 2p_{12}x_1 + p_{22} < 0 \\ \Leftrightarrow \quad & \max_{x_1 \geq 0}[p_{11}x_1^2 + 2p_{12}x_1 + p_{22}] = \frac{p_{11}p_{22} - p_{12}^2}{p_{11}} < 0 \\ \Rightarrow \quad & p_{11}p_{22} - p_{12}^2 > 0 \end{aligned}$$

i.e., (18) for $q_{12} = p_{12}$.

Since the implication (19) $\Rightarrow$ (17) is obvious when $p_{12} < 0$, let us consider the case $p_{12} \geq 0$. Then for every $x_1 \geq 0$, $x_2 \geq 0$, $(x_1, x_2) \neq 0$

$$\begin{aligned} & p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 \\ & \quad \leq (p_{11} + p_{12})x_1^2 + (p_{22} + p_{12})x_2^2 < 0 \end{aligned}$$

hence, (17) follows. □

The main LMI relaxation result which plays a crucial role hereafter is the following.

Theorem 2.2: PLMI (14) is fulfilled provided one of the following conditions holds:

1)

$$\mathcal{M}_{ii}(Z) < 0, \qquad i = 1, 2, \ldots, L \tag{20}$$

$$\begin{aligned} & \frac{1}{L-1}\mathcal{M}_{ii}(Z) + \frac{1}{2}(\mathcal{M}_{ij}(Z) + \mathcal{M}_{ji}(Z)) < 0, \\ & \quad 1 \leq i \neq j \leq L. \end{aligned} \tag{21}$$

2) There are symmetric matrices $Q_{ij} = Q_{ji}$, $i \neq j$ such that

$$\begin{aligned} & \begin{bmatrix} \frac{1}{L-1}\mathcal{M}_{ii}(Z) & Q_{ij} \\ Q_{ij} & \frac{1}{L-1}\mathcal{M}_{jj}(Z) \end{bmatrix} < 0, \\ & \tfrac{1}{2}(\mathcal{M}_{ij}(Z) + \mathcal{M}_{ji}(Z)) \leq Q_{ij}, \quad 1 \leq i < j \leq L. \end{aligned} \tag{22}$$

3) There are symmetric matrices $Q_{ij} = Q_{ji}$, $i \neq j$ such that

$$\begin{aligned} & \mathcal{M}_{ii} + \sum_{j \neq i}^{L} Q_{ij} < 0, \quad Q_{ij} \geq 0, \quad Q_{ij} \geq \tfrac{1}{2}(\mathcal{M}_{ij}(Z) + \mathcal{M}_{ji}(Z)), \\ & \quad 1 \leq i \neq j \leq L. \end{aligned} \tag{23}$$

Proof: Note that (14) can be rewritten as

$$\begin{aligned} \sum_{1 \leq i < j \leq L} \Bigg[& \frac{1}{L-1}\alpha_i^2 \mathcal{M}_{ii}(Z) + \frac{1}{L-1}\alpha_j^2 \mathcal{M}_{jj}(Z) \\ & + \alpha_i\alpha_j(\mathcal{M}_{ij}(Z) + \mathcal{M}_{ji}(Z)) \Bigg] \leq 0 \end{aligned} \tag{24}$$

and thus a sufficient condition for (24) is

$$\begin{aligned} & x^T \left[\frac{1}{L-1}\alpha_i^2 \mathcal{M}_{ii}(Z) + \frac{1}{L-1}\alpha_j^2 \mathcal{M}_{jj}(Z) \right] x \\ & \quad + \alpha_i\alpha_j x^T(\mathcal{M}_{ij}(Z) + \mathcal{M}_{ji}(Z))x \leq 0, \qquad \forall x \neq 0. \end{aligned} \tag{25}$$

Hence, parts 1) and 2) follow by applying Lemma 2.1.

For part 3), first note that since $Q_{ij} \geq 0$, (23) gives

$$\begin{aligned} & \sum_{i=1}^{L} \alpha_i^2 \mathcal{M}_{ii}(Z) + \sum_{i \neq j}^{L} \alpha_i\alpha_j Q_{ij} \\ & \quad \leq \sum_{i=1}^{L} \alpha_i^2 \left(\mathcal{M}_{ii}(Z) + \sum_{i \neq j}^{L} Q_{ij} \right) < 0 \end{aligned}$$

while by condition $Q_{ij} \geq (\mathcal{M}_{ij}(Z) + \mathcal{M}_{ji}(Z))/2$ in (23)

$$\sum_{i,j=1}^{L} \alpha_i\alpha_j \mathcal{M}_{ij}(Z) \leq \sum_{i=1}^{L} \alpha_i^2 \mathcal{M}_{ii}(Z) + \sum_{i \neq j}^{L} \alpha_i\alpha_j Q_{ij}$$

so (14) follows. □

Remark: While (19) implies (18) in Lemma 2.1, (20) and (21) are no longer a particular case of (22). A sufficient condition for (14) in [14] and [15] is

$$\begin{aligned} \mathbf{M}_{ii}(Z) + (L-1)Q < 0, & \qquad i = 1, 2, \ldots, L; \; Q \geq 0 \\ \mathbf{M}_{ij}(Z) + \mathbf{M}_{ji}(Z) - 2Q \leq 0, & \qquad i < j \end{aligned} \tag{26}$$

and can be shown a particular case of (20) and (21). Therefore, the introduction of the additional variable Q in (26) in [14] and [15] is superfluous.

III. Stabilization Problem

Return back to the stabilization problem for the system (4), i.e., to find a feedback control (7) and (8) such that the closed loop system (9) is asymptotically stable. Applying Theorem 2.2 to (14) with $\mathcal{M}_{ij}$ defined by (15) gives the following result.

Theorem 3.1: System (1) is stabilized by the PDC (8) if either one of LMIs system (20), (21) or (22) or (23) is feasible with $\mathcal{M}_{ij}$ defined by (15). Feedback gains K_i deriving the controller (8) are obtained as solutions of (20) and (21), (15) or (22), (15) or (23), (15) according to (12) by

$$K_i = R_i P^{-1}, \qquad i = 1, 2, \ldots, L. \tag{27}$$

□

Now, we will show how the control variable R in the LMI formulation of Theorem 3.1 can be eliminated to obtain a much simpler formulation. By Finsler's lemma the existence of $R(\alpha)$ satisfying (11) is equivalent to the existence of $P > 0$ and $\tau \in R$ such that

$$\begin{aligned} &\sum_{i,j=1}^{L} \alpha_i \alpha_j [PA_i^T + A_i P - \tau B_i B_j^T] < 0 \\ \Leftrightarrow \quad &\sum_{i,j=1}^{L} \alpha_i \alpha_j [PA_i^T + A_i P - B_i B_j^T] < 0, \qquad P \leftarrow P/|\tau| \end{aligned} \tag{28}$$

which is PLMI (14) with

$$\mathcal{M}_{ij}(Z) = PA_i^T + A_i P - B_i B_j^T, \qquad Z = P. \tag{29}$$

Obviously, once such P exists, $R(\alpha)$ satisfying (11) is given as

$$R(\alpha) = -B_2(\alpha)^T = -\sum_{j=1}^{L} \alpha_j B_{2j} \tag{30}$$

and thus by (12), $K(\alpha)$ is defined by

$$K(\alpha) = R(\alpha)P^{-1} = -\sum_{j=1}^{L} \alpha_j B_{2j} P^{-1} \tag{31}$$

which already has PDC structure (8). To sum up, we state the following theorem.

Theorem 3.2: There is a generally structured stabilizing controller (7) if and only if there is one with PDC structure (8) defined by

$$K_j = -B_{2j}^T P^{-1}, \qquad j = 1, 2, \ldots, L \tag{32}$$

where P is a solution of PLMI (14), (29), whose feasibility is implied by that of LMI systems (20), (21), (29) or (22), (29) or (23), (29).

Remark: Compared with (14) and (15), we see that (28) and (29) has the obvious advantages: it requires only one variable P instead of $P, R_1, \ldots, R_L$ in (14) and (15) and with much simpler form which makes it much more computationally tractable.

IV. Regulator Problem

Setting $B_{1i} = 0$, $D_{11i} = 0$, $D_{12i} = 0$ in (1) and (4), the regulator problem is to minimize the performance index

$$\int_0^\infty [z(t)^T \mathcal{Q} z(t) + u(t)^T \mathcal{R} u(t)]\, dt \tag{33}$$

subject to the initial condition $x(0) = x_0$ with some given $x_0 \in R^{n_x}$ and weighting matrices $\mathcal{Q} > 0$ and $\mathcal{R} > 0$.

Suppose that the function $V(x) = x^T X x$ with $X > 0$ and control u satisfies the following Hamilton–Jacoby inequality

$$\frac{dV(x(t))}{dt} + [z(t)^T \mathcal{Q} z(t) + u(t)^T \mathcal{R} u(t)] < 0 \tag{34}$$

$$\Leftrightarrow \quad \min_u \left[\frac{\partial V(x)}{\partial x} (A(\alpha)x + B_2(\alpha)u) + x^T C(\alpha)^T \mathcal{Q} C(\alpha) x + u^T \mathcal{R} u \right] < 0 \tag{35}$$

$$\Leftrightarrow \quad \frac{\partial V(x)}{\partial x} A(\alpha)x + x^T C(\alpha)^T \mathcal{Q} C(\alpha) x - \frac{1}{4} \frac{\partial V(x)}{\partial x} B_2(\alpha) \mathcal{R}^{-1} B_2(\alpha)^T \frac{\partial V^T(x)}{\partial x} < 0 \tag{36}$$

then for every $T > 0$, one has

$$\begin{aligned} V(T) - V(0) + \int_0^T [z(t)^T \mathcal{Q} z(t) + u(t)^T \mathcal{R} u(t)]\, dt < 0 \\ \Rightarrow \quad \int_0^T [z(t)^T \mathcal{Q} z(t) + u(t)^T \mathcal{R} u(t)]\, dt \le V(0) \\ = x_0^T X x_0 \end{aligned}$$

which implies that $x_0^T X x_0$ is an upper bound of (33).

Note that the equivalence between (35) and (36) is provided by the control signal

$$\begin{aligned} u := & -\frac{1}{2} \mathcal{R}^{-1} B_2(\alpha)^T \frac{\partial V^T(x)}{\partial x} \\ & \in \arg\min_u \left[\frac{\partial V(x)}{\partial x} [A(\alpha)x + B(\alpha)u] + x^T C(\alpha)^T \mathcal{Q} C(\alpha) x + u^T \mathcal{R} u \right]. \end{aligned} \tag{37}$$

From (4) and (8), we deduce

$$\begin{aligned} & z(t)^T \mathcal{Q} z(t) + u(t)^T \mathcal{R} u(t) \\ & = x^T(t) \begin{bmatrix} C(\alpha)^T & K(\alpha)^T \end{bmatrix} \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & \mathcal{R} \end{bmatrix} \begin{bmatrix} C(\alpha) \\ K(\alpha) \end{bmatrix} x(t) \end{aligned} \tag{38}$$

$$\begin{aligned} & \frac{dV(x(t))}{dt} \\ & = x^T(t)[(A(\alpha) + B_2(\alpha)K(\alpha))^T X \\ & \quad + X(A(\alpha) + B_2(\alpha)K(\alpha))]x(t). \end{aligned} \tag{39}$$

Hence, using a Schur's complement and a congruent transformation, leads to (40), as shown at the bottom of the next page, which by virtue of the structure (13) of $R(\alpha)$ is PLMI (14) with

$$\begin{aligned} \mathcal{M}_{ij}(Z) = & \begin{bmatrix} PA_i + A_i^T P + B_{2i} R_j + R_j^T B_{2i}^T & PC_i^T & R_i^T \\ C_i P & -\mathcal{Q}^{-1} & 0 \\ R_i & 0 & -\mathcal{R}^{-1} \end{bmatrix}, \\ & Z = (P, R_1, R_2, \ldots, R_L). \end{aligned} \tag{41}$$

Applying Theorem 2.2 to (40) and (41) gives the following result:

Theorem 4.1: An upper bound of (33) with the class of controller with PDC structure (8) is provided by one of the following LMI optimization problem

$$\min_{P, R_i, \nu} \nu: \begin{bmatrix} P & x_0 \\ x_0^T & \nu \end{bmatrix} > 0, \qquad (20), (21), (41) \tag{42}$$

$$\min_{P, Q_{ij}, R_i, \nu} \nu: \begin{bmatrix} P & x_0 \\ x_0^T & \nu \end{bmatrix} > 0, \qquad (22), (41) \tag{43}$$

$$\min_{P, Q_{ij}, R_i, \nu} \nu: \begin{bmatrix} P & x_0 \\ x_0^T & \nu \end{bmatrix} > 0, \qquad (23), (41). \tag{44}$$

Suboptimal controllers K_j for realizing (8) are defined from solutions of problems (42)–(44), by (27).

The PLMI-based result for computing an upper bound of [14] can be shown to be more conservative than (40), i.e., the result of [14] is a sufficient condition for feasibility of (40). As mentioned, the relaxation result (26) used in [14] is also more conservative that ours. Therefore, it is not difficult to see the upper bound given by [14] is more conservative than that given by (42)–(44). This will also be confirmed by computational experiments in Section VI.

Now, again we will try to eliminate the control variable $R(\alpha)$ in (40) by using Finsler's lemma. As clarified in Section VI, such elimination is really helpful and the corresponding upper bound is improved. Rewriting (40) as

$$\begin{bmatrix} PA(\alpha)^T + A(\alpha)P & * & * \\ C(\alpha)P & -\mathcal{Q}^{-1} & * \\ 0 & 0 & -\mathcal{R}^{-1} \end{bmatrix} + \begin{bmatrix} B_2(\alpha) \\ 0 \\ I \end{bmatrix} \cdot R(\alpha)\,[I \quad 0 \quad 0] + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} R(\alpha)^T\,[B_2(\alpha)^T \quad 0 \quad I] < 0 \tag{45}$$

by Finsler's lemma, the existence of $R(\alpha)$ is equivalent the existence of $P > 0$ and $\tau > 0$ such that

$$\begin{aligned} &\Leftrightarrow \begin{bmatrix} PA(\alpha)^T + A(\alpha)P & PC(\alpha)^T & 0 \\ C(\alpha)P & -\mathcal{Q}^{-1} & 0 \\ 0 & 0 & -\mathcal{R}^{-1} \end{bmatrix} - \tau \begin{bmatrix} B_2(\alpha) \\ 0 \\ I \end{bmatrix} \cdot [B_2(\alpha)^T \quad 0 \quad I] < 0 \\ &\Leftrightarrow \begin{bmatrix} PA(\alpha)^T + A(\alpha)P - \tau B_2(\alpha)B_2(\alpha)^T & PC(\alpha)^T & -\tau B_2(\alpha) \\ C(\alpha)P & -\mathcal{Q}^{-1} & 0 \\ -\tau B_2(\alpha)^T & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0 \\ &\Leftrightarrow \begin{bmatrix} PA(\alpha)^T + A(\alpha)P - B_2(\alpha)B_2(\alpha)^T & PC(\alpha)^T & -B_2(\alpha) \\ C(\alpha)P & -\tau\mathcal{Q}^{-1} & 0 \\ -B_2(\alpha)^T & 0 & -\tau\mathcal{R}^{-1} \end{bmatrix} < 0, P \leftrightarrow P/\tau \end{aligned} \tag{46}$$

which is PLMI (14) with the definition

$$\mathcal{M}_{ij}(Z) = \begin{bmatrix} PA_i^T + PA_i - B_{2i}B_{2j}^T & PC_i^T & -B_{2i} \\ C_iP & -\tau\mathcal{Q}^{-1} & 0 \\ -B_{2i}^T & 0 & -\tau\mathcal{R}^{-1} \end{bmatrix}, \quad Z = (P, \tau). \tag{47}$$

Obviously, when PLMI (46) is feasible, the function $V(x) = x^T P^{-1} x$ satisfies Hamilton–Jacoby inequality (35) or (36) and therefore one of controllers is defined according to (37) by

$$u = -\mathcal{R}^{-1}B(\alpha)^T P^{-1}x = -\mathcal{R}^{-1}\sum_{j=1}^{L}\alpha_j B_{2j}^T P^{-1}x \tag{48}$$

which also has PDC structure (8). The result is summarized in the following theorem.

Theorem 4.2: Using quadratic Lyapunov function for assessing the performance (33), the existence of the generally structured suboptimal controller (7) is equivalent to the existence of that with PDC structure (8).

An upper bound of (33) with the controller (8) is proved by either one of the following LMI optimization problems:

$$\min_{P,\tau,\nu} \nu\colon \begin{bmatrix} P & x_0 \\ x_0^T & \nu \end{bmatrix} > 0, \qquad (20), (21), (47) \tag{49}$$

$$\min_{P,\tau,\nu} \nu\colon \begin{bmatrix} P & x_0 \\ x_0^T & \nu \end{bmatrix} > 0, \qquad (22), (47) \tag{50}$$

$$\min_{P,\tau,\nu} \nu\colon \begin{bmatrix} P & x_0 \\ x_0^T & \nu \end{bmatrix} > 0, \qquad (23), (47) \tag{51}$$

and accordingly, a suboptimal controller K_j for realizing PDC (4) is

$$K_j = -\mathcal{R}^{-1}B_{2j}^T P^{-1}, \qquad j = 1, 2, \ldots, L. \tag{52}$$

□

Again, note that problems (49)–(51) involve only variables P, τ, and are much simpler than (42)–(44). These advantages will be clarified by numerical examples in Section VI.

V. $\mathcal{H}_\infty$ Control

The optimal $\mathcal{H}_\infty$ control problem consists in finding controller (7) for (4) such that

$$\begin{aligned} &\gamma \to \min: \gamma > 0, \\ &\quad \int_0^T \|z(t)\|^2\,dt < \gamma^2 \int_0^T \|w(t)\|^2\,dt\ \forall w, \\ &\quad \forall T > 0,\ x(0) = 0 \\ \Leftrightarrow\ &\gamma \to \min: \gamma > 0, \\ &\quad \gamma^{-1}\int_0^T \|z(t)\|^2\,dt < \gamma \int_0^T \|w(t)\|^2\,dt\ \forall w, \\ &\quad \forall T > 0,\ x(0) = 0. \end{aligned} \tag{53}$$

$$\begin{aligned} (34) \quad &\Leftrightarrow \quad (A(\alpha) + B_2(\alpha)K(\alpha))^T X + X(A(\alpha) + B_2(\alpha)K(\alpha)) + [C(\alpha)^T \quad K(\alpha)^T] \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & \mathcal{R} \end{bmatrix} \begin{bmatrix} C(\alpha) \\ K(\alpha) \end{bmatrix} < 0 \\ &\Leftrightarrow \quad \begin{bmatrix} (A(\alpha) + B_2(\alpha)K(\alpha))^T X + X(A(\alpha) + B_2(\alpha)K(\alpha)) & * & * \\ C(\alpha) & -\mathcal{Q}^{-1} & * \\ K(\alpha) & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0 \\ &\Leftrightarrow \quad \begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} (A(\alpha) + B_2(\alpha)K(\alpha))^T X + X(A(\alpha) + B_2(\alpha)K(\alpha)) & * & * \\ C(\alpha) & -\mathcal{Q}^{-1} & * \\ K(\alpha) & 0 & -\mathcal{R}^{-1} \end{bmatrix} [*] < 0 \\ &\Leftrightarrow \quad \begin{bmatrix} PA(\alpha)^T + R(\alpha)^T B_2(\alpha)^T + A(\alpha)P + B_2(\alpha)R(\alpha) & * & * \\ C(\alpha)P & -\mathcal{Q}^{-1} & * \\ R(\alpha) & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0 \\ &\qquad P = X^{-1}, \qquad R(\alpha) = K(\alpha)P \end{aligned} \tag{40}$$

Suppose that there exist $V(x) = x^T X x,\ X > 0$ and $u = K(\alpha)$ satisfying the following Hamilton–Jacoby–Isaac inequality

$$\frac{dV(x(t))}{dt} + \gamma^{-1}\|z(t)\|^2 - \gamma\|w(t)\|^2 < 0, \qquad \forall (x, w) \tag{54}$$

$$\begin{aligned}\Leftrightarrow \quad & \max_w \bigg(\frac{\partial V(x)}{\partial x} [(A(\alpha) + B_2(\alpha)K(\alpha))x + B_1(\alpha)w] \\ & \quad + \gamma^{-1}\|(C(\alpha) + D_{11}(\alpha)K(\alpha))x + D_{12}(\alpha)w\|^2 \\ & \quad - \gamma\|w\|^2 \bigg) < 0 \end{aligned} \tag{55}$$

$$\begin{aligned}\Leftrightarrow \quad & \min_u \max_w \bigg(\frac{\partial V(x)}{\partial x} (A(\alpha)x + B_2(\alpha)u + B_1(\alpha)w) \\ & \quad + \gamma^{-1}\|C(\alpha)x + D_{11}(\alpha)u + D_{12}(\alpha)w\|^2 \\ & \quad - \gamma\|w\|^2 \bigg) < 0 \end{aligned} \tag{56}$$

then for every T, taking the definite integral from 0 to T of both sides of (54) gives

$$\begin{aligned}\gamma^{-1}\int_0^T \|z(t)\|^2\,dt - \gamma\int_0^T \|w(t)\|^2\,dt &< V(x(0)) - V(x(T)) \\ &\le V(x(0)) = 0\end{aligned}$$

i.e., constraint of (53).

By a least square technique, it is easy to show that one of the controllers $u = K(\alpha)x$ satisfying (55) is

$$\begin{aligned} u = &-(D_{12}(\alpha)^T D_{12}(\alpha))^{-1} \\ &\cdot \left[D_{12}(\alpha)^T C(\alpha)x + \frac{1}{2}\gamma B_2(\alpha)^T \frac{\partial V^T(x)}{\partial x} \right] \end{aligned} \tag{57}$$

provided that $D_{12}(\alpha)$ is full-column rank, which can be assumed from now, without loss of generality.

Like (38) and (39), we can easily derive (58) and (59), shown at the bottom of the page. So, again using Schur's complement and congruent transformation as manipulation tools, shown in (60) at the bottom of the next page, which by structure (13) of $R(\alpha)$ is PLMI (14) with

$$\begin{aligned} &\mathcal{M}_{ij}(Z) \\ &= \begin{bmatrix} PA_i^T + R_j^T B_{2i}^T + A_i P + B_{2i}R_j & B_{1i} & PC_i^T + R_j^T D_{12i} \\ B_{1i}^T & -\gamma I & D_{11i}^T \\ C_i P + D_{12i}R_j & D_{11i} & -\gamma I \end{bmatrix} \\ &\quad Z = (P, R_1, R_2, \ldots, R_L, \gamma),\ R_i = K_i P. \end{aligned} \tag{61}$$

The following result is a direct consequence of Theorem 2.2.

Theorem 5.1: An upper bound of (53) within class of PDC structure (8) is provided by either of the following LMI optimization problem

$$\min_{P, R_i, \gamma} \gamma\text{: (20), (21), (61)} \tag{62}$$

$$\min_{P, R_i, \gamma} \gamma\text{: (22), (61)} \tag{63}$$

$$\min_{P, R_i, \gamma} \gamma\text{: (23), (61).} \tag{64}$$

A suboptimal $\mathcal{H}_\infty$ control gains K_i for realizing PDC (8) are defined by solutions of (62), (63) via (27).

Again, we can eliminate the control variable $R(\alpha)$ from (60) as follows. Rewrite (60) as

$$\begin{aligned} &\begin{bmatrix} PA(\alpha)^T + A(\alpha)P & * & * \\ B_1(\alpha)^T & -\gamma I & * \\ C(\alpha)P & D_{11}(\alpha) & -\gamma I \end{bmatrix} \\ &\quad + \begin{bmatrix} B_2(\alpha) \\ 0 \\ D_{12}(\alpha) \end{bmatrix} R(\alpha) [I \quad 0 \quad 0] \\ &\quad + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} R(\alpha)^T [B_2(\alpha)^T \quad 0 \quad D_{12}(\alpha)T] < 0 \end{aligned} \tag{65}$$

then again by Finsler's lemma the existence of $R(\alpha)$ in (65) is equivalent to the existence of P, τ such as (66), shown at the bottom of the next page, which is (14) with

$$\begin{aligned} &\mathcal{M}_{ij}(Z) \\ &= \begin{bmatrix} PA_i^T + A_i P - \tau B_{2i}B_{2j}^T & * & * \\ C_{1i}P - \tau D_{12i}B_{2j}^T & -\gamma I - \tau D_{12i}D_{12j}^T & * \\ B_{1i}^T & D_{11i}^T & -\gamma I \end{bmatrix}, \\ &\quad Z = (P, \tau, \gamma). \end{aligned} \tag{67}$$

When (66) holds true, it is obvious that the function $V(x) = x^T P^{-1} x$ satisfies Hamilton–Jacoby–Isaacs inequality (55). Then, by (57), we see that when $D_{12}(\alpha)$ is independent of α (i.e., $D_{12i} = D_{12}\,\forall i = 1, 2, \ldots, L$) as often verified on all control design problem, control (57) is adapted to

$$u = -(D_{12}^T D_{12})^{-1} \sum_{j=1}^{L} \alpha_j [D_{12}^T C_j + \gamma B_{2i}^T P^{-1}]x \tag{68}$$

i.e., it has structure (68).

Theorem 5.2: Suppose that $D_{12i} = D_{12}\ \forall i = 1, 2, \ldots, L$ and also that D_{12} is a full-column rank matrix in (1), (4), and the

$$\frac{dV}{dt} = \begin{bmatrix} x \\ w^T \end{bmatrix}^T \begin{bmatrix} (A(\alpha) + B_2(\alpha)K(\alpha))^T X + X(A(\alpha) + B_2(\alpha)K(\alpha)) & XB_1(\alpha) \\ B_1(\alpha)^T X & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{58}$$

$$\gamma^{-1}\|z(t)\|^2 = \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} (C(\alpha) + D_{12}(\alpha)K(\alpha))^T \\ D_{11}(\alpha)^T \end{bmatrix} \gamma^{-1} I \times [C(\alpha) + D_{12}(\alpha)K(\alpha) \quad D_{11}(\alpha)] \begin{bmatrix} x \\ w \end{bmatrix}. \tag{59}$$

class of quadratic Lyapunov function is used for checking $\mathcal{H}_\infty$ performance. Then the existence of general suboptimal controller (7) for problem (53) is equivalent to the existence of that with PDC structure (8).

Moreover, an upper bound of (53) is provided by either one of the following LMI optimization problems

$$\min_{P,\tau,\gamma} \gamma\text{: (20), (21), (67)} \tag{69}$$
$$\min_{P,\tau,\gamma} \gamma\text{: (22), (67)} \tag{70}$$
$$\min_{P,\tau,\gamma} \gamma\text{: (23), (67).} \tag{71}$$

In these cases, a suboptimal controller K_j for realizing PDC (8) is

$$K_j = -(D_{12}^T D_{12})^{-1}(D_{12}^T C_j + \gamma B_{2i}^T P^{-1}). \tag{72}$$

VI. Numerical Examples

By [14], the T–S model of the eccentric rotational proof mass actuator (TORA) system [8] (see Fig. 1) is described by (4) with

$$L=4, \quad B_{1i}=0, \quad D_{11i}=0, \quad C_i = I$$
$$D_{12i}=0, \quad \alpha = 0.99, \quad \epsilon = 0.1, \quad a=4$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \epsilon\sin(\alpha\pi)/(\alpha\pi) & 0 \\ 0 & 0 & 0 & 1 \\ -\epsilon/(1-\epsilon^2) & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 2\epsilon/\pi & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon/(1-\epsilon^2) & 0 & -\epsilon^2/(1-\epsilon^2) & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon/(1-\epsilon^2) & 0 & -\epsilon^2(1-a^2)/(1-\epsilon^2) & 0 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/(1-\epsilon^2) \end{bmatrix}, \qquad B_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$B_{23} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/(1-\epsilon^2) \end{bmatrix}, \qquad B_{24} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/(1-\epsilon^2) \end{bmatrix}.$$

$$\begin{aligned}
(54) \quad &\Leftrightarrow \begin{bmatrix} (A(\alpha)+B_2(\alpha)K(\alpha))^T X + X(A(\alpha)+B_2(\alpha)K(\alpha)) & XB_1(\alpha) \\ B_1(\alpha)^T X & -\gamma I \end{bmatrix} + \begin{bmatrix} (C(\alpha)+D_{12}(\alpha)K(\alpha))^T \\ D_{11}(\alpha)^T \end{bmatrix} \\
&\quad \cdot \gamma^{-1} I \left[C(\alpha)+D_{12}(\alpha)K(\alpha) \quad D_{11}(\alpha) \right] < 0 \\
&\Leftrightarrow \begin{bmatrix} (A(\alpha)+B_2(\alpha)K(\alpha))^T X + X(A(\alpha)+B_2(\alpha)K(\alpha)) & * & * \\ B_1(\alpha)^T X & -\gamma I & * \\ C(\alpha)+D_{12}(\alpha)K(\alpha) & D_{11}(\alpha) & -\gamma I \end{bmatrix} < 0 \\
&\Leftrightarrow \begin{bmatrix} P(A(\alpha)+B_2(\alpha)K(\alpha))^T + (A(\alpha)+B_2(\alpha))P & * & * \\ B_1(\alpha)^T & -\gamma I & * \\ (C(\alpha)+D_{12}(\alpha)K(\alpha))P & D_{11}(\alpha) & -\gamma I \end{bmatrix} < 0, \qquad P = X^{-1} \\
&\Leftrightarrow \begin{bmatrix} PA(\alpha)^T + R(\alpha)^T B_2(\alpha)^T + A(\alpha)P + B_2(\alpha)R(\alpha) & * & * \\ B_1(\alpha)^T & -\gamma I & * \\ C(\alpha)P + D_{12}(\alpha)R(\alpha) & D_{11}(\alpha) & -\gamma I \end{bmatrix} < 0 \qquad R(\alpha) = K(\alpha)P
\end{aligned} \tag{60}$$

$$\begin{aligned}
&\begin{bmatrix} PA(\alpha)^T + A(\alpha)P & * & * \\ B_1(\alpha)^T & -\gamma I & * \\ C(\alpha)P & D_{11}(\alpha) & -\gamma I \end{bmatrix} - \tau \begin{bmatrix} B_2(\alpha) \\ 0 \\ D_{12}(\alpha) \end{bmatrix} \left[B_2(\alpha)^T \quad 0 \quad D_{12}(\alpha)^T \right] < 0 \\
\Leftrightarrow &\begin{bmatrix} PA(\alpha)^T + A(\alpha)P - \tau B_2(\alpha)B_2(\alpha)^T & * & * \\ C(\alpha)P - \tau D_{12}(\alpha)B_2(\alpha)^T & -\gamma I - \tau D_{12}(\alpha)D_{12}(\alpha)^T & * \\ B_1(\alpha)^T & D_{11}(\alpha)^T & -\gamma I \end{bmatrix} < 0
\end{aligned} \tag{66}$$

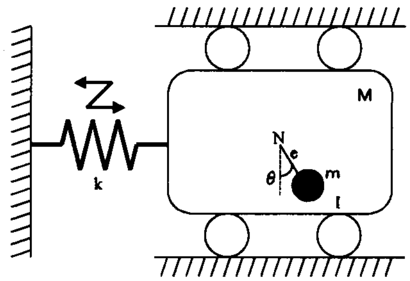

Fig. 1. TORA model.

TABLE I
LMI OPTIMIZATION COMPUTATIONAL RESULTS: ν_i(RESPECTIVELY ν_2, ν_3) IS AN UPPER BOUND GIVEN BY THE RESULT OF [14] [RESPECTIVELY (49), (51)]

$x(0)$	ν_1	ν_2	ν_3
(0,0,0.5,0)	10.4942	7.6535	4.2160
(0,0,1,0)	41.9766	23.9152	16.8634
(0,0,2,0)	167.9143	81.8160	67.4359
(0.5,0,0,0)	608.4980	69.3012	24.9822
(0.5,0,0.5,0)	619.1110	77.7380	34.6481
(0.5,0,1,0)	661.5580	98.8189	55.7873
(0.5,0,2,0)	823.5193	167.0413	121.9660
(1,0,0,0)	2433.6576	194.6976	99.9239
(1,0,0.5,0)	2434.6023	203.6533	110.6426
(1,0,1,0)	2476.1326	230.2337	138.5847
(1,0,2,0)	2646.8485	314.5188	223.1401
(2,0,0,0)	9735.4656	595.5697	399.6952
(2,0,0.5,0)	9715.6760	601.1897	407.5988
(2,0,1,0)	9738.6562	634.3817	442.5446
(2,0,2,0)	9902.4883	743.6955	554.5300

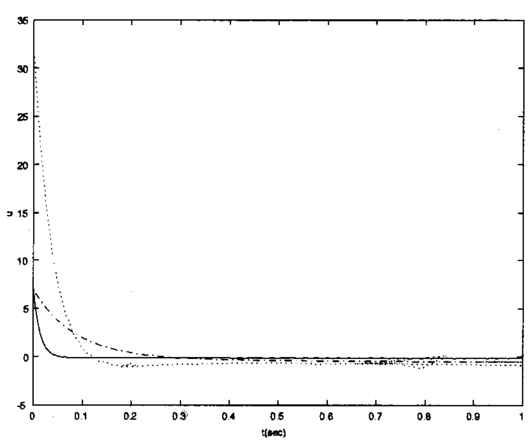

Fig. 2. Performance control simulation: [14] (dot line), (49) (dash-dot line), and (50) (solid line).

The state x of (4) in this case is (x_1, x_2, x_3, x_4), where $x_3 = \theta$ and $x_4 = \dot{\theta}$ is the angular position and velocity of the rotational proof mass, and $x_1 = \overline{x}_1 + \epsilon \sin x_3$ $x_2 = \overline{x}_2 + \epsilon x_4 \cos x_3$ with $\overline{x}_1 = q$, $\overline{x}_2 = \dot{q}$ the translational position and velocity of the cart.

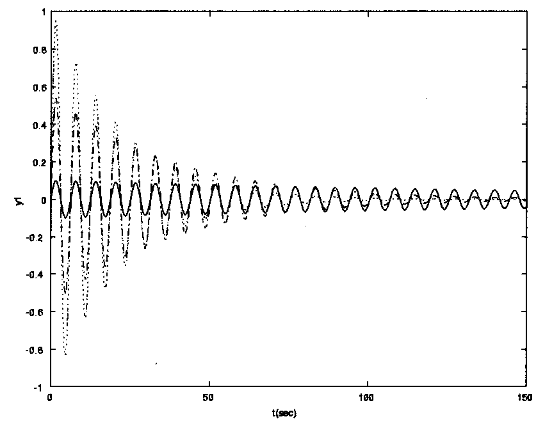

Fig. 3. Tracking performance of the angular position of rotational proof mass by control given by [14] (dot line), (49) (dash-dot line), (50) (solid line).

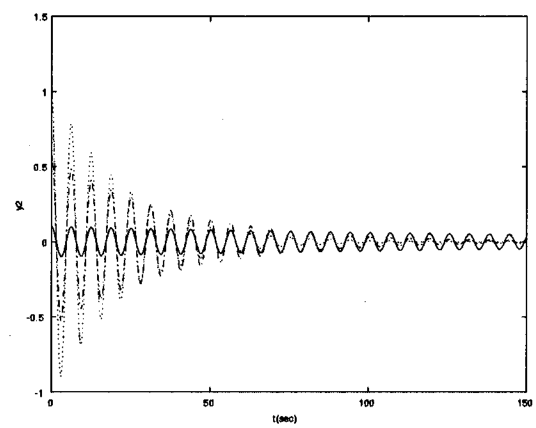

Fig. 4. Tracking performance of the angular velocity of rotational proof mass by control given by [14] (dot line), (49) (dash-dot line), (50) (solid line).

The problem is to regulate $(\overline{x}_1, \overline{x}_2, x_3, x_4)$ to the equilibrium (0, 0, 0, 0) so problem (33) is an appropriate formulation for this purpose.

The computational results using optimization formulations [14], (49), (51) with different initial condition $x(0)$ but $\mathcal{Q} = I$, $\mathcal{R} = 1$ are summarized in Table I. Computations are performed using LMI control tool box [10]. From Table I, we see the benefit of optimization formulations (49) and (50) with control variable $R(\alpha)$ eliminated: the control performance ν_2, ν_3 are improved dramatically compared with ν_1 based on optimization formulation (42) involving control variable $R(\alpha)$. Moreover, the cpu-time for computing solutions of (49) is 2–4 times less that needed for computing solution of their counterpart in [14]. From the MATLAB simulation results in Figs. 2–4 with initial condition (1, 0, 0, 0) as in [14], we see that indeed both tracking and controller's performance resulting from (49), (50) are better than that given in [14].

APPENDIX

- Congruent transformation of matrices: the matrix M is negative definite (positive definite, respectively) if and only if T^TMT is negative definite (positive definite, respectively) too for any nonsingular matrix T of appropriate dimension.

- Schur's complement:

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} < 0 \Leftrightarrow M_{22} < 0, \quad M_{11} - M_{12}M_{22}^{-1}M_{12}^T < 0$$

for any matrices M_{11}, M_{12}, M_{22} of appropriate dimensions.

- The Finsler's lemma: Given matrices P of dimension $n \times (n+m)$ and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}, \qquad M_{11} \in R^{n\times n},\ M_{22} \in R^{m\times m}$$

one has

$$\begin{aligned} &\exists X \in R^{n\times n}\colon M + \begin{bmatrix} I_n \\ O_{m\times n} \end{bmatrix} XP \\ &\qquad + P^TX^T \begin{bmatrix} I_n & O_{n\times m} \end{bmatrix} < 0 \\ \Leftrightarrow \quad & M_{22} < 0 \;\&\; \exists \tau > 0\colon M - \tau P^TP < 0. \end{aligned}$$

Here I_n ($O_{n\times m}$, respectively) is the identity matrix of dimension $n \times n$ (zero matrix of dimension $n \times m$, respectively).

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