A Linear Matrix Inequality Approach to H_∞ Control

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Abstract

The continuous- and discrete-time H_∞ control problems are solved via elementary manipulations on linear matrix inequalities (LMI). Two interesting new features emerge through this approach: solvability conditions valid for both regular and singular problems, and an LMI-based parametrization of all H_∞-suboptimal controllers, including reduced-order controllers.

The solvability conditions involve Riccati inequalities rather than the usual indefinite Riccati equations. Alternatively, these conditions can be expressed as a system of three LMI's. Efficient convex optimization techniques are available to solve this system. Moreover, its solutions parametrize the set of H_∞ controllers and bear important connections with the controller order and the closed-loop Lyapunov functions.

Thanks to such connections, the LMI-based characterization of H_∞ controllers opens new perspectives for the refinement of H_∞ design. Applications to cancellation-free design and controller order reduction are discussed and illustrated by examples.

Keywords: State-space H_∞ control, linear matrix inequalities, Riccati inequalities, loop shaping.

1 Introduction

The state-space results of [5] (DGKF hereafter) are widely accepted as an efficient and numerically sound way of computing H_∞ controllers. Indeed, solving two algebraic Riccati equations (**ARE**) is all it takes to test for existence of adequate controllers. In addition,

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explicit state-space formulas are given for some particular solution called the "central controller." Finally, all suitable controllers are parametrized via a linear fractional transformation built around the central controller and involving a free dynamical parameter $Q(s)$ [5].

Yet, some inherent restrictions of DGKF's solution tend to limit the scope and performance of current state-space design techniques. First, the results of [5] are only applicable to "regular" plants, i.e., plants in which the transfer functions from controls to controlled outputs and from disturbance to measured outputs have no invariant zeros at infinity nor on the imaginary axis. Various extensions have been proposed for singular problems [20, 18, 22, 3]. While these extensions can satisfactorily handle plants with infinite zeros, the case of $j\omega$-axis zeros remains problematic when reasonable closed-loop damping is required. In particular, possible discontinuities of the optimal H_∞ gain near such singularities are, in our opinion, inadequately addressed [10].

Secondly, DGKF's approach overemphasizes the "central" solution among all possible choices of H_∞ controller. Indeed, it is unclear how to exploit the Q-parametrization for design purposes, mainly because there is no obvious connection between the free parameter $Q(s)$ and the controller or closed-loop properties. As a result, the diversity in H_∞ controllers is hardly exploited and applications rely almost exclusively on the central controller in spite of certain undesirable properties of this controller. For instance, its tendency in loop shaping problems to cancel the stable poles of the plant [19]. Such cancellations may lead to unacceptable designs in the presence of lightly damped modes. Another drawback of the central controller is its potentially high order (typically equal to the order of the augmented plant). Reduced-order H_∞ design is therefore desirable and the Q-parametrization seems inadequate for this purpose.

The approach developed in this paper is an alternative to the state-space characterization of [5] and to the Q-parametrization. Using a linear matrix inequality (**LMI**) formulation, it extends the concept of convex state-space parametrization of H_∞ controllers introduced in [9]. Here the usual H_∞ Riccati equations are replaced by Riccati inequalities and the solution set of these inequalities is used to parametrize all suboptimal H_∞ controllers, including reduced-order ones. Specifically, controllers are generated from the set of pairs (X, Y) of symmetric matrices satisfying a system of matrix inequalities. With the assumptions of [5], this system reads:

$$\begin{cases} A^T X + XA + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X + C_1^T C_1 \; < \; 0 \\ AY + YA^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2)Y + B_1 B_1^T \; < \; 0 \\ X > 0, \qquad Y > 0, \qquad \rho(XY) \le \gamma^2. \end{cases}$$

The LMI-based characterization of achievable γ's and suboptimal H_∞ controllers proceeds from two simple observations: (1) via the Bounded Real Lemma, H_∞-like constraints can be converted into algebraic Riccati inequalities (**ARI**) and, in fact, into a linear matrix inequality; (2) the controller parameters enter this LMI affinely; hence they can be eliminated to obtain solvability conditions depending only on the plant data and two matrix variables R and S.

Interestingly, important design parameters such as the controller order or the damping of the closed-loop modes have a clear interpretation in terms of X, Y (see Section 9). Unlike the Q-parametrization, this formulation is therefore propitious to the design of better H_∞ controllers. In addition, both regular and singular H_∞ problems are encompassed within the same formalism, and the resulting solvability conditions are valid for both types of problems. Note that DGKF's results readily follow from our characterization when the plant is regular. Yet, our main purpose here is not to reproduce the well-documented results of [5]. Rather, we hope to demonstrate that the LMI-based approach to H_∞ control could prove a valuable alternative and/or complement to existing H_∞ synthesis techniques. Some evidence of this claim is provided in Section 9.

For completeness, note that the importance of the Bounded Real Lemma in H_∞ theory was already recognized in [16, 17]. Yet, only ARE versions of this lemma were used in [16] which focused on the derivation of DGKF's results in the regular case. As for the work of [17], it stopped short of bringing out the concept of convex parametrization of H_∞ controllers. We also point out analogies between our derivation technique and the manipulations in [15] (see Lemma 3.1 below and its applications). Finally, Iwasaki & Skelton have independently extended on [9] and obtained similar results for the continuous-time case [12].

The paper is organized as follows. Section 2 gives a precise statement of the suboptimal H_∞ control problem and recalls its state-space formulation. Attention is restricted to infinite-horizon problems for linear time-invariant plants. Section 3 contains an instrumental lemma which permits the elimination of the controller parameters from the Bounded Real Lemma LMI. Solvability conditions for the general suboptimal H_∞ problem are derived in Sections 4 and 5 in the continuous- and discrete-time contexts, respectively. In Section 6, these conditions are turned into LMI's which define a convex set of free matrix variables (R, S). The issue of constructing adequate controllers from these free variables is addressed in Section 7. Section 8 compares our characterization to the classical ARE-based results of [11]. Finally, applications to the refinement of current H_∞ design techniques are discussed in Section 9.

The following notation will be used throughout the paper: $\sigma_{max}(M)$ for the largest singular value of a matrix M, $\operatorname{Ker} M$ and $\operatorname{Im} M$ for the null space and range of the linear operator associated with M, and M^+ for the Moore-Penrose pseudoinverse of M.

2 The Suboptimal H_∞ Problem

Consider a proper continuous- or discrete-time linear time-invariant plant P mapping exogenous inputs w and control inputs u to controlled outputs q and measured outputs y (Figure 2.1). That is,

$$\begin{pmatrix} Q(\sigma) \\ Y(\sigma) \end{pmatrix} = P(\sigma) \begin{pmatrix} W(\sigma) \\ U(\sigma) \end{pmatrix}$$

where σ stands for the Laplace variable s in the continuous-time context and for the Z-transform variable z in the discrete-time context. Given some dynamic output feedback law $u = K(\sigma)y$ and with the partitioning

$$P(\sigma) = \begin{pmatrix} P_{11}(\sigma) & P_{12}(\sigma) \\ P_{21}(\sigma) & P_{22}(\sigma) \end{pmatrix} \qquad (\sigma = s, z), \tag{2.1}$$

the closed-loop transfer function from disturbance w to controlled output q is:

$$\mathcal{F}(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}. \tag{2.2}$$

The suboptimal H_∞ control problem of parameter γ consists of finding a controller $K(\sigma)$ such that:

- the closed-loop system is internally stable,
- the H_∞ norm of $\mathcal{F}(P,K)$ (the maximum gain from w to q) is strictly less than γ.

Solutions of this problem (if any) will be called γ-**suboptimal** controllers.

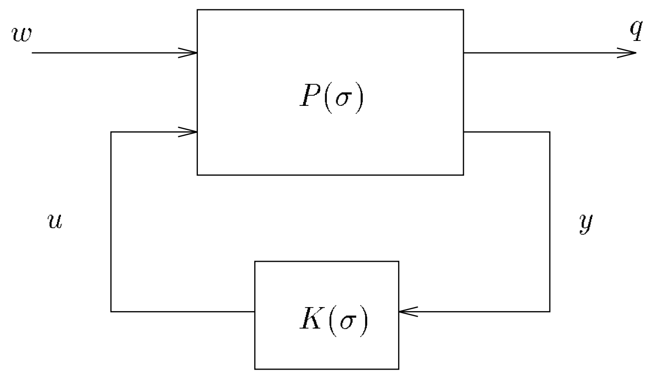

Figure 2.1: H_∞ problem.

As usual in state-space approaches to H_∞ control, we introduce some minimal realization of the plant P:

$$P(\sigma) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (\sigma I - A)^{-1} (B_1,\ B_2) \qquad (\sigma = s, z) \tag{2.3}$$

where the partitioning is conformable to (2.1). This realization corresponds to the state-space equations:

$$\begin{aligned} \dot{x} &= A\,x \ + B_1\,w + B_2\,u \\ q &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

The problem dimensions are summarized by:

$$A \in \mathbb{R}^{n \times n}; \qquad D_{11} \in \mathbb{R}^{p_1 \times m_1}; \qquad D_{22} \in \mathbb{R}^{p_2 \times m_2}.$$

Throughout the paper, the only assumptions on the plant parameters are:

(A1) (A, B_2, C_2) is stabilizable and detectable,

(A2) $D_{22} = 0$.

The first assumption is necessary and sufficient to allow for plant stabilization by dynamic output feedback. As for **(A2)**, it incurs no loss of generality while considerably simplifying calculations [11]. None of the customary "regularity" assumptions on the rank of D_{12} and D_{21} and on $j\omega$-axis invariant zeros of $P_{12}(s)$ and $P_{21}(s)$ [5] are needed here.

Assuming **(A2)** and given any proper real-rational controller $K(\sigma)$ of realization

$$K(\sigma) = D_K + C_K(\sigma I - A_K)^{-1} B_K; \qquad A_K \in \mathbb{R}^{k \times k} \qquad (\sigma = s, z)\,, \tag{2.4}$$

a (not necessarily minimal) realization of the closed-loop transfer function from w to z is obtained as:

$$\mathcal{F}(G, K)\,(\sigma) = D_{cl} + C_{cl}(\sigma I - A_{cl})^{-1} B_{cl} \tag{2.5}$$

where

$$\begin{aligned} A_{c\ell} &= \begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{pmatrix}; \qquad B_{c\ell} = \begin{pmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{pmatrix}; \\ C_{c\ell} &= (C_1 + D_{12} D_K C_2\ ,\ D_{12} C_K); \qquad D_{c\ell} = D_{11} + D_{12} D_K D_{21}. \end{aligned} \tag{2.6}$$

Gathering all controller parameters into the single variable

$$\Theta := \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \ \in \mathbb{R}^{(n+k) \times (n+k)} \tag{2.7}$$

and introducing the shorthands:

$$\begin{aligned} A_0 &= \begin{pmatrix} A & 0 \\ 0 & 0_k \end{pmatrix}; \qquad B_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}; \qquad C_0 = (C_1\ ,\ 0); \\ \mathcal{B} &= \begin{pmatrix} 0 & B_2 \\ I_k & 0 \end{pmatrix}; \quad \mathcal{C} = \begin{pmatrix} 0 & I_k \\ C_2 & 0 \end{pmatrix}; \quad \mathcal{D}_{12} = (0\ ,\ D_{12}); \quad \mathcal{D}_{21} = \begin{pmatrix} 0 \\ D_{21} \end{pmatrix}, \end{aligned} \tag{2.8}$$

the closed-loop matrices $A_{c\ell}, B_{c\ell}, C_{c\ell}, D_{c\ell}$ can be written as:

$$A_{c\ell} = A_0 + \mathcal{B}\Theta\mathcal{C}; \quad B_{c\ell} = B_0 + \mathcal{B}\Theta\mathcal{D}_{21}; \quad C_{c\ell} = C_0 + \mathcal{D}_{12}\Theta\mathcal{C}; \quad D_{c\ell} = D_{11} + \mathcal{D}_{12}\Theta\mathcal{D}_{21}. \tag{2.9}$$

Note that (2.8) involves only plant data and that $A_{c\ell}, B_{c\ell}, C_{c\ell}, D_{c\ell}$ depend affinely on the controller data Θ. This fact is instrumental to the solution derived in Section 4.

3 Useful Results

The following lemma plays a central role in our approach.

Lemma 3.1 *Given a symmetric matrix $\Psi \in \mathbb{R}^{m\times m}$ and two matrices P, Q of column dimension m, consider the problem of finding some matrix Θ of compatible dimensions such that*

$$\Psi + P^T\Theta^T Q + Q^T\Theta P < 0. \tag{3.1}$$

Denote by W_P, W_Q any matrices whose columns form bases of the null spaces of P and Q, respectively. Then (3.1) is solvable for Θ if and only if

$$\begin{cases} W_P^T\,\Psi\,W_P & < & 0 \\ W_Q^T\,\Psi\,W_Q & < & 0. \end{cases} \tag{3.2}$$

Proof: Necessity of (3.2) is clear; for instance, $P\,W_P = 0$ implies $W_P^T\Psi W_P < 0$ when pre- and post-multiplying (3.1) by W_P^T and W_P, respectively. For details on the sufficiency part, see Appendix A. An analogous result is also found in [15].

■

The proofs of our main theorems will also make extensive use of the following standard result on Schur complements and negative definite 2×2 block matrices.

Lemma 3.2 *The block matrix $\begin{pmatrix} P & M \\ M^T & Q \end{pmatrix}$ is negative definite if and only if*

$$\begin{cases} Q & < & 0 \\ P - MQ^{-1}M^T & < & 0 \end{cases} \tag{3.3}$$

In the sequel, $P - MQ^{-1}M^T$ will be referred to as the Schur complement of Q.

■

4 Solvability of Continuous-Time Problems

We first recall the Bounded Real Lemma for continuous-time systems. This lemma helps turning the H_∞ suboptimal constraints into an LMI.

Lemma 4.1 *Consider a continuous-time transfer function $T(s)$ of (not necessarily minimal) realization $T(s) = D + C(sI - A)^{-1}B$. The following statements are equivalent:*

(i) $\|D + C(sI - A)^{-1}B\|_\infty < \gamma$ *and A is stable in the continuous-time sense* ($\mathrm{Re}\,(\lambda_i(A)) < 0$);

(ii) there exists a symmetric **positive definite** *solution X to the LMI:*

$$\begin{pmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0. \tag{4.1}$$

Proof: See, e.g., [17], p. 82.

■

Note that the LMI (4.1) is equivalent to

$$\begin{cases} \sigma_{max}(D) < \gamma \\ A^T X + XA + \gamma^{-1} C^T C + \gamma\,(XB + \gamma^{-1} C^T D)(\gamma^2 I - D^T D)^{-1}(B^T X + \gamma^{-1} D^T C) < 0 \end{cases}$$

where we recognize the more familiar algebraic Riccati inequality associated with the Bounded Real Lemma.

Combining the Bounded Real Lemma 4.1 with Lemma 3.1, we obtain the following necessary and sufficient conditions for the existence of γ-suboptimal controllers of order k.

Theorem 4.2 *Consider a proper plant $P(s)$ of minimal realization (2.3) and assume (A1)-(A2). With the notation (2.8), define*

$$\mathcal{P} := (\mathcal{B}^T,\ 0_{(k+m_2)\times m_1},\ \mathcal{D}_{12}^T); \qquad \mathcal{Q} := (\mathcal{C},\ \mathcal{D}_{21},\ 0_{(k+p_2)\times p_1}) \tag{4.2}$$

and let $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ be two matrices whose columns span the null spaces of $\mathcal{P}$ and $\mathcal{Q}$, respectively.

Then the set of γ-suboptimal controllers of order k is non empty if and only if there exists some $(n+k)\times(n+k)$ **positive definite** *matrix $X_{c\ell}$ such that:*

$$W_{\mathcal{P}}^T \Phi_{X_{c\ell}} W_{\mathcal{P}} < 0; \qquad W_{\mathcal{Q}}^T \Psi_{X_{c\ell}} W_{\mathcal{Q}} < 0 \tag{4.3}$$

where

$$\begin{aligned} \Phi_{X_{c\ell}} &:= \begin{pmatrix} A_0\, X_{c\ell}^{-1} + X_{c\ell}^{-1} A_0^T & B_0 & X_{c\ell}^{-1} C_0^T \\ B_0^T & -\gamma I & D_{11}^T \\ C_0 X_{c\ell}^{-1} & D_{11} & -\gamma I \end{pmatrix} \\ \Psi_{X_{c\ell}} &:= \begin{pmatrix} A_0^T X_{c\ell} + X_{c\ell} A_0 & X_{c\ell} B_0 & C_0^T \\ B_0^T X_{c\ell} & -\gamma I & D_{11}^T \\ C_0 & D_{11} & -\gamma I \end{pmatrix}. \end{aligned} \tag{4.4}$$

Proof: From the Bounded Real Lemma, $K(s) = D_K + C_K(sI - A_K)^{-1} B_K$ is a kth-order γ-suboptimal controller if and only if the LMI

$$\begin{pmatrix} A_{c\ell}^T X_{c\ell} + X_{c\ell} A_{c\ell} & X_{c\ell} B_{c\ell} & C_{c\ell}^T \\ B_{c\ell}^T X_{c\ell} & -\gamma I & D_{c\ell}^T \\ C_{c\ell} & D_{c\ell} & -\gamma I \end{pmatrix} < 0 \tag{4.5}$$

holds for some $X_{c\ell} > 0$ in $\mathbb{R}^{(n+k)\times(n+k)}$. Using the expressions (2.9) of $A_{c\ell}, B_{c\ell}, C_{c\ell}, D_{c\ell}$, this LMI can also be written as:

$$\Psi_{X_{c\ell}} + \mathcal{Q}^T\, \Theta^T\, \mathcal{P}_{X_{c\ell}} + \mathcal{P}_{X_{c\ell}}^T\, \Theta\, \mathcal{Q} < 0 \tag{4.6}$$

where $\Theta = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$, $\Psi_{X_{c\ell}}$ and $\mathcal{Q}$ are defined above, and

$$\mathcal{P}_{X_{c\ell}} := (\mathcal{B}^T X_{c\ell},\ 0,\ \mathcal{D}_{12}^T). \tag{4.7}$$

Hence the set of γ-suboptimal controllers of order k is nonempty if and only if (4.6) holds for some $\Theta \in \mathbb{R}^{(k+m_2)\times(k+p_2)}$ and $X_{c\ell} > 0$.

We can now invoke Lemma 3.1 to eliminate Θ and obtain solvability conditions depending only on $X_{c\ell}$ and the plant parameters. Specifically, let $W_{\mathcal{P}_{X_{c\ell}}}$ and $W_{\mathcal{Q}}$ denote matrices whose columns form bases of $\operatorname{Ker}\mathcal{P}_{X_{c\ell}}$ and $\operatorname{Ker}\mathcal{Q}$, respectively. Then by Lemma 3.1, (4.6) holds for some Θ if and only if

$$W_{\mathcal{P}_{X_{c\ell}}}^T \Psi_{X_{c\ell}} W_{\mathcal{P}_{X_{c\ell}}} < 0; \qquad W_{\mathcal{Q}}^T \Psi_{X_{c\ell}} W_{\mathcal{Q}} < 0. \tag{4.8}$$

To complete the proof, it suffices to rewrite the first inequality in (4.8) as $W_{\mathcal{P}}^T \Phi_{X_{c\ell}} W_{\mathcal{P}} < 0$ where $W_{\mathcal{P}}$ denotes any basis of $\operatorname{Ker}\mathcal{P}$. To this end, observe that $\mathcal{P}_{X_{c\ell}} = \mathcal{P}\begin{pmatrix} X_{c\ell} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ with $\mathcal{P}$ as in (4.2). Hence $W_{\mathcal{P}_{X_{c\ell}}} := \begin{pmatrix} X_{c\ell}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} W_{\mathcal{P}}$ is a basis of $\operatorname{Ker}\mathcal{P}_{X_{c\ell}}$ whenever $W_{\mathcal{P}}$ is a basis of $\operatorname{Ker}\mathcal{P}$. Consequently, $W_{\mathcal{P}_{X_{c\ell}}}^T \Psi_{X_{c\ell}} W_{\mathcal{P}_{X_{c\ell}}} < 0$ is equivalent to

$$W_{\mathcal{P}}^T \left\{ \begin{pmatrix} X_{c\ell}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \Psi_{X_{c\ell}} \begin{pmatrix} X_{c\ell}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \right\} W_{\mathcal{P}} \;=\; W_{\mathcal{P}}^T \Phi_{X_{c\ell}} W_{\mathcal{P}} < 0.$$

■

The characterization of Theorem 4.2 is awkward because it involves both $X_{c\ell}$ and its inverse, and the role of each specific plant parameter is somewhat blurred. Fortunately, the conditions (4.3) can be further reduced to a pair of Riccati inequalities of lower dimensions which exactly parallel the usual H_∞ Riccati equations. This simpler characterization is obtained by partitioning $X_{c\ell}$ and $X_{c\ell}^{-1}$ conformably to $A_{c\ell}$, computing $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ explicitly, and carrying out the block matrix products. Formulas and calculations are simpler when introducing the following shorthands:

$$\begin{gathered} \hat{B}_2 := B_2 D_{12}^+; \qquad \hat{A} := A - \hat{B}_2 C_1; \qquad \hat{B}_1 := B_1 - \hat{B}_2 D_{11}; \\ \hat{C}_1 := (I - D_{12}D_{12}^+)C_1; \qquad \hat{D}_{11} := (I - D_{12}D_{12}^+)D_{11}, \end{gathered} \tag{4.9}$$

and

$$\begin{gathered} \tilde{C}_2 := D_{21}^+ C_2; \qquad \tilde{A} := A - B_1\tilde{C}_2; \qquad \tilde{C}_1 := C_1 - D_{11}\tilde{C}_2; \\ \tilde{B}_1 := B_1(I - D_{21}^+ D_{21}); \qquad \tilde{D}_{11} := D_{11}(I - D_{21}^+ D_{21}). \end{gathered} \tag{4.10}$$

Note that the next result is valid for both regular *and singular* H_∞ problems.

Theorem 4.3 (**γ-suboptimal controllers for continuous-time plants**)
Consider a proper continuous-time plant $P(s)$ of order n and minimal realization (2.3) and assume **(A1)-(A2)**. *Let W_{12} and W_{21} denote bases of the null spaces of $(I-D_{12}^{+}D_{12})B_2^T$ and $(I-D_{21}D_{21}^{+})C_2$, respectively. With the notation (4.9)-(4.10), the suboptimal H_∞ problem of parameter γ is solvable if and only if*

(i) $\gamma > \max\left(\sigma_{max}(\hat{D}_{11}), \sigma_{max}(\tilde{D}_{11})\right)$,

(ii) there exist pairs of symmetric matrices (R,S) in $\mathbb{R}^{n\times n}$ such that

$$W_{12}^T\left\{\hat{A}R+R\hat{A}^T-\gamma\hat{B}_2\hat{B}_2^T+\begin{pmatrix}\hat{C}_1R\\ \hat{B}_1^T\end{pmatrix}^T\begin{pmatrix}\gamma I & -\hat{D}_{11}\\ -\hat{D}_{11}^T & \gamma I\end{pmatrix}^{-1}\begin{pmatrix}\hat{C}_1R\\ \hat{B}_1^T\end{pmatrix}\right\}W_{12}<0 \tag{4.11}$$

$$W_{21}^T\left\{\tilde{A}^TS+S\tilde{A}-\gamma\tilde{C}_2^T\tilde{C}_2+\begin{pmatrix}\tilde{B}_1^TS\\ \tilde{C}_1\end{pmatrix}^T\begin{pmatrix}\gamma I & -\tilde{D}_{11}^T\\ -\tilde{D}_{11} & \gamma I\end{pmatrix}^{-1}\begin{pmatrix}\tilde{B}_1^TS\\ \tilde{C}_1\end{pmatrix}\right\}W_{21}<0 \tag{4.12}$$

$$R>0; \qquad S>0; \qquad \lambda_{\min}(RS)\ge 1. \tag{4.13}$$

Moreover, the set of γ-suboptimal controllers of order k is non empty if and only if (ii) holds for some R,S which further satisfy the rank constraint:

$$\mathrm{Rank}\,(I-RS)\le k. \tag{4.14}$$

Prior to a formal proof of this result, we give some insight into its interpretation and implications. When D_{12} and D_{21}^T have full column rank, the projectors $I-D_{12}^{+}D_{12}$ and $I-D_{21}D_{21}^{+}$ are identically zero and W_{12}, W_{21} can be taken as the identity matrix. The constraints (4.11)-(4.12) then reduce to a pair of algebraic Riccati inequalities. Solutions R,S to these ARI's are further constrained by the positivity and coupling conditions (4.13). With the notation $X:=\gamma R^{-1}$ and $Y:=\gamma S^{-1}$, and the simplifying assumptions of [5]:

$$D_{11}=0; \qquad D_{12}^T(D_{12}\,,C_1)=(I,0); \qquad D_{21}(D_{21}^T\,,B_1^T)=(I,0),$$

(4.11)-(4.13) read:

$$A^TX+XA+X(\gamma^{-2}B_1B_1^T-B_2B_2^T)X+C_1^TC_1 \;<\; 0 \tag{4.15}$$
$$AY+YA^T+Y(\gamma^{-2}C_1^TC_1-C_2^TC_2)Y+B_1B_1^T \;<\; 0 \tag{4.16}$$
$$X>0; \qquad Y>0; \qquad \rho(XY)\le\gamma^2 \tag{4.17}$$

We recognize the usual H_∞ Riccati expressions in the left-hand sides of (4.15)-(4.16), while the constraints (4.17) clearly parallel those arising in [5]. Further details on the connection with the classical characterization in terms of Riccati *equations* can be found in Section 8. Simply note that X_∞^{-1} and Y_∞^{-1}, when existing, are maximal points in the set of pairs (R,S) solving (4.11)-(4.13). That is,

$$0<R<\gamma X_\infty^{-1}; \qquad 0<S<\gamma Y_\infty^{-1}$$

for all R, S satisfying (4.11)-(4.13).

When D_{12} or D_{21} are rank-deficient (singular problem), the ARI constraints are relaxed by means of projections onto the ranges of $(I - D_{12}^+ D_{12}) B_2^T$ and $(I - D_{21} D_{21}^+) C_2$, respectively. Since DGFK's results are not applicable to such problems, the characterization of Theorem 4.3 coupled with convex optimization techniques offers a computationally appealing substitute (see Section 6). The same remark applies to plants with $j\omega$-axis zeros. Note that other computational alternatives have been proposed in [20, 3]. Yet the LMI approach is potentially more reliable in case of discontinuity of the optimal γ near the singular problem under consideration (see Section 9). In addition, Theorem 4.3 provides a complete parametrization of H_∞ controllers in the singular case (see Section 7), a feature not available in previous approaches.

Finally, another novelty in Theorem 4.3 is the rank condition (4.14) characterizing reduced-order H_∞ controllers. For $k \geq n$ (full order or higher), this condition is trivially satisfied and (4.11)-(4.13) are necessary and sufficient for the existence of γ-suboptimal controllers of order k. This confirms the well-known fact that when the suboptimal H_∞ problem is solvable, there exists an adequate controller of order equal to the plant order n. All γ-suboptimal controllers are not necessarily of order n however. In fact, there will exist reduced-order controllers ($k < n$) as soon as (4.11)-(4.13) have a solution (R, S) which further satisfies $\text{Rank}\,(I - RS) = k$. Note that $\lambda_{\min}(RS) = 1$ for such pairs, or equivalently $\rho(XY) = \gamma^2$ in terms of $X := \gamma R^{-1}$ and $Y := \gamma S^{-1}$. This is consistent with the order reduction experienced in optimal central controllers when $\rho(X_\infty Y_\infty) = \gamma^2$ at γ_{opt}.

Proof of Theorem 4.3: From Theorem 4.2, the set of γ-suboptimal controllers of order k is non empty if and only if (4.3) holds for some $X_{c\ell} > 0$ in $\mathbb{R}^{(n+k)\times(n+k)}$. To express (4.3) in terms of the plant parameters, partition $X_{c\ell}$ and $X_{c\ell}^{-1}$ as

$$X_{c\ell} := \begin{pmatrix} S & N \\ N^T & * \end{pmatrix}; \qquad X_{c\ell}^{-1} := \begin{pmatrix} R & M \\ M^T & * \end{pmatrix} \tag{4.18}$$

where $R, S \in \mathbb{R}^{n\times n}$ and $M, N \in \mathbb{R}^{n\times k}$. Consider the first constraint $W_{\mathcal{P}}^T\, \Phi_{X_{c\ell}}\, W_{\mathcal{P}} < 0$. With the partition (4.18), $\Phi_{X_{c\ell}}$ in (4.4) reads:

$$\Phi_{X_{c\ell}} = \begin{pmatrix} AR + RA^T & AM & \vdots & B_1 & RC_1^T \\ M^T A^T & 0 & \vdots & 0 & M^T C_1^T \\ \cdots & \cdots & \vdots & \cdots & \cdots \\ B_1^T & 0 & \vdots & -\gamma I & D_{11}^T \\ C_1 R & C_1 M & \vdots & D_{11} & -\gamma I \end{pmatrix}. \tag{4.19}$$

Meanwhile, from

$$\mathcal{P} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ B_2^T & 0 & \underbrace{0}_{m_1} & D_{12}^T \end{pmatrix}$$

it follows that bases of $\operatorname{Ker}\mathcal{P}$ are of the form $W_{\mathcal{P}} = \begin{pmatrix} W_1 & 0 \\ 0 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{pmatrix}$ where $\mathcal{N}_R := \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ is any basis of the null space of $\left(B_2^T \,,\, D_{12}^T\right)$. Observing that the second row of $W_{\mathcal{P}}$ is identically zero, the condition $W_{\mathcal{P}}^T \, \Phi_{X_{c\ell}} \, W_{\mathcal{P}} < 0$ reduces to:

$$\begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{pmatrix}^T \begin{pmatrix} AR + RA^T & B_1 & RC_1^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 R & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{pmatrix} < 0$$

or equivalently,

$$\begin{pmatrix} \mathcal{N}_R & 0 \\ 0 & I_{m_1} \end{pmatrix}^T \begin{pmatrix} AR + RA^T & RC_1^T & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{pmatrix} \begin{pmatrix} \mathcal{N}_R & 0 \\ 0 & I_{m_1} \end{pmatrix} < 0. \tag{4.20}$$

The next step consists of expressing $\mathcal{N}_R$ explicitly in terms of the plant data. To this end, introduce a basis W_{12} of the null space of $(I - D_{12}^{+} D_{12}) B_2^T$ and an orthonormal basis U_{12} of the null space of D_{12}^T. Elementary linear algebra shows that a basis of the null space of $\left(B_2^T \,,\, D_{12}^T\right)$ is then given by:

$$\mathcal{N}_R = \begin{pmatrix} W_{12} & 0 \\ -\hat{B}_2^T W_{12} & U_{12} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -\hat{B}_2^T & U_{12} \end{pmatrix} \begin{pmatrix} W_{12} & 0 \\ 0 & I \end{pmatrix}. \tag{4.21}$$

Replace $\mathcal{N}_R$ with this expression in (4.20), carry out the matrix products, and observe that $\hat{B}_2 U_{12} = 0$. This simplifies (4.20) to:

$$\begin{pmatrix} W_{12}^T \left(\hat{A}R + R\hat{A}^T - \gamma \hat{B}_2 \hat{B}_2^T\right) W_{12} & W_{12}^T \left(RC_1^T\right) U_{12} & W_{12}^T \hat{B}_1 \\ U_{12}^T \left(C_1 R\right) W_{12} & -\gamma \, U_{12}^T U_{12} & U_{12}^T D_{11} \\ \hat{B}_1^T \, W_{12} & D_{11}^T U_{12} & -\gamma I \end{pmatrix} < 0.$$

Observing that $U_{12} U_{12}^T = I - D_{12} D_{12}^{+}$, we can further eliminate U_{12} to obtain:

$$\begin{pmatrix} W_{12}^T \left(\hat{A}R + R\hat{A}^T - \gamma \hat{B}_2 \hat{B}_2^T\right) W_{12} & W_{12}^T \left(R\hat{C}_1^T\right) & W_{12}^T \hat{B}_1 \\ \left(\hat{C}_1 R\right) W_{12} & -\gamma I & \hat{D}_{11} \\ \hat{B}_1^T \, W_{12} & \hat{D}_{11}^T & -\gamma I \end{pmatrix} < 0$$

which is equivalent to (4.11) by a Schur complement argument.

Similarly, the condition $W_{\mathcal{Q}}^T \, \Psi_{X_{c\ell}} \, W_{\mathcal{Q}} < 0$ is equivalent to S satisfying (4.12). Hence $X_{c\ell}$ satisfies (4.3) if and only if R, S satisfy (4.11)-(4.12). Moreover, $X_{c\ell} \in \mathbb{R}^{(n+k)\times(n+k)}$ and $X_{c\ell} > 0$ is equivalent to R, S satisfying (4.13) and (4.14) (see, e.g., [15, 9]).

Summing up, if $X_{c\ell} > 0$ of dimension $n + k$ solves (4.3) then (4.11)-(4.14) hold for the symmetric matrices R, S given by (4.18). Conversely, if the system (4.11)-(4.14) admits a solution (R, S), then $X_{c\ell} > 0$ of dimension $n + k$ can be reconstructed from R, S to satisfy (4.18) [15]. From (4.11)-(4.12), this $X_{c\ell}$ further solves (4.3): the proof is complete.

■

5 Solvability of Discrete-Time Problems

The machinery developed in the previous section for continuous-time problems is easily transposed to the discrete-time context and leads to qualitatively similar results as shown next. We begin by recalling the Bounded Real Lemma for discrete-time systems.

Lemma 5.1 *Consider a discrete-time transfer function $T(z)$ of (not necessarily minimal) realization $T(z) = D + C(zI - A)^{-1}B$. The following statements are equivalent:*

(i) $\|D + C(sI - A)^{-1}B\|_\infty < 1$ *and A is stable in the discrete-time sense* ($|\lambda_i(A)| < 1$)*;*

(ii) $\inf_{T \text{ invertible}} \sigma_{max} \begin{pmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{pmatrix} < 1$*,*

(iii) there exists $X = X^T > 0$ such that $\begin{pmatrix} A^TXA - X & A^TXB & C^T \\ B^TXA & B^TXB - I & D^T \\ C & D & -I \end{pmatrix} < 0$*,*

(iv) there exists $X = X^T > 0$ such that $\begin{pmatrix} -X^{-1} & A & B & 0 \\ A^T & -X & 0 & C^T \\ B^T & 0 & -I & D^T \\ 0 & C & D & -I \end{pmatrix} < 0$ *.*

Proof: See, e.g., [6].

■

Applying this lemma to the realization (2.6) of the closed-loop system, the controller

$$K(z) = D_K + C_K(zI - A_K)^{-1}B_K \; ; \qquad A_K \in \mathbb{R}^{k \times k}$$

is γ-suboptimal if and only if the LMI $\begin{pmatrix} -X_{c\ell}^{-1} & A_{c\ell} & B_{c\ell} & 0 \\ A_{c\ell}^T & -X_{c\ell} & 0 & C_{c\ell}^T \\ B_{c\ell} & 0 & -\gamma I & D_{c\ell}^T \\ 0 & C_{c\ell} & D_{c\ell} & -\gamma I \end{pmatrix} < 0$ holds for some $X_{c\ell} > 0$ in $\mathbb{R}^{(n+k)\times(n+k)}$. With $\Theta := \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$ and the decompositions (2.9), this is equivalent to

$$\Psi_{X_{c\ell}} + \mathcal{Q}^T\Theta^T\mathcal{P} + \mathcal{P}^T\Theta\,\mathcal{Q} < 0 \tag{5.1}$$

where

$$\Psi_{X_{c\ell}} = \begin{pmatrix} -X_{c\ell}^{-1} & A_0 & B_0 & 0 \\ A_0^T & -X_{c\ell} & 0 & C_0^T \\ B_0^T & 0 & -\gamma I & D_{11}^T \\ 0 & C_0 & D_{11} & -\gamma I \end{pmatrix} \tag{5.2}$$

$$\mathcal{P} = (\mathcal{B}^T\,,\,0\,,\,0\,,\,\mathcal{D}_{12}^T) = \begin{pmatrix} 0 & I & \vdots & 0 & 0 & \vdots & 0 & 0 \\ B_2^T & 0 & \vdots & 0 & 0 & \vdots & 0 & D_{12}^T \end{pmatrix} \tag{5.3}$$

$$\mathcal{Q} = (0\,,\,\mathcal{C}\,,\,\mathcal{D}_{21}\,,\,0) = \begin{pmatrix} 0 & 0 & \vdots & 0 & I & \vdots & 0 & 0 \\ 0 & 0 & \vdots & C_2 & 0 & \vdots & D_{21} & 0 \end{pmatrix} \tag{5.4}$$

From Lemma 3.1, (5.1) is feasible in Θ if and only if

$$W_{\mathcal{P}}^T\, \Psi_{X_{c\ell}}\, W_{\mathcal{P}} < 0\,; \qquad W_{\mathcal{Q}}^T\, \Psi_{X_{c\ell}}\, W_{\mathcal{Q}} < 0 \tag{5.5}$$

where $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ denote bases of $\operatorname{Ker}\mathcal{P}$ and $\operatorname{Ker}\mathcal{Q}$.

As in the continuous-time case, focus on the first constraint and observe that, conformably to (5.3), $W_{\mathcal{P}}$ is of the form:

$$W_{\mathcal{P}} = \begin{pmatrix} W_1 & \vdots & 0 & 0 & 0 \\ 0 & \vdots & 0 & 0 & 0 \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ 0 & \vdots & I_n & 0 & 0 \\ 0 & \vdots & 0 & I_k & 0 \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ 0 & \vdots & 0 & 0 & I_{m_1} \\ W_2 & \vdots & 0 & 0 & 0 \end{pmatrix} \tag{5.6}$$

where $\mathcal{N}_R = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ is any basis of the null space of $\left(B_2^T\,,\ D_{12}^T\right)$. With this $W_{\mathcal{P}}$ and the partitioning (4.18) of $X_{c\ell}$ and $X_{c\ell}^{-1}$, the condition $W_{\mathcal{P}}^T \Psi_{X_{c\ell}} W_P < 0$ reduces to:

$$\begin{pmatrix} W_1^T & \vdots & 0 & \vdots & 0 & W_2^T \\ 0 & \vdots & I_{n+k} & \vdots & 0 & 0 \\ 0 & \vdots & 0 & \vdots & I_{m_1} & 0 \end{pmatrix} \begin{pmatrix} -R & \vdots & A \quad 0 & \vdots & B_1 & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \cdots \\ A^T & \vdots & \multirow{2}{*}{$-X_{c\ell}$} & \vdots & 0 & C_1^T \\ 0 & \vdots & & \vdots & 0 & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \cdots \\ B_1^T & \vdots & 0 \quad 0 & \vdots & -\gamma I & D_{11}^T \\ 0 & \vdots & C_1 \quad 0 & \vdots & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} W_1 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & I_{n+k} & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & I_{m_1} \\ W_2 & 0 & 0 \end{pmatrix} < 0.$$

Carrying out block multiplications and forming the Schur complement of the block $-X_{c\ell}$, this is equivalent to $X_{c\ell} > 0$ and

$$\begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{pmatrix}^T \left\{ \begin{pmatrix} -R & B_1 & 0 \\ B_1^T & -\gamma I & D_{11}^T \\ 0 & D_{11} & -\gamma I \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \\ C_1 & 0 \end{pmatrix} X_{c\ell}^{-1} \begin{pmatrix} A^T & 0 & C_1^T \\ 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{pmatrix}^T \begin{pmatrix} ARA^T - R & B_1 & ARC_1^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1RA^T & D_{11} & -\gamma I + C_1RC_1^T \end{pmatrix} \begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{pmatrix} < 0 \tag{5.7}$$

Using the explicit expression (4.21) for $\mathcal{N}_R = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ with identical definitions of W_{12} and U_{12}, (5.7) can be further simplified as in the proof of Theorem 4.3. This leads to the following discrete-time counterpart of Theorem 4.3.

Theorem 5.2 (γ-suboptimal controllers for discrete-time plants)
Consider a proper discrete-time plant $P(z)$ of order n and minimal realization (2.3) and assume **(A1)-(A2)**. *Let W_{12} and W_{21} denote bases of the null spaces of $(I - D_{12}^+ D_{12})B_2^T$ and $(I - D_{21}D_{21}^+)C_2$, respectively. With the notation (4.9)-(4.10), the suboptimal H_∞ problem of parameter γ is solvable if and only if*

(i) $\gamma > \max\left(\sigma_{max}(\hat{D}_{11}), \sigma_{max}(\tilde{D}_{11})\right)$,

(ii) there exist pairs of symmetric matrices (R, S) *in* $\mathbb{R}^{n\times n}$ *such that*

$$\hat{C}_1 R \hat{C}_1^T + \gamma^{-1}\hat{D}_{11}\hat{D}_{11}^T < \gamma I; \qquad \tilde{B}_1^T S \tilde{B}_1 + \gamma^{-1}\tilde{D}_{11}^T\tilde{D}_{11} < \gamma I \tag{5.8}$$

$$W_{12}^T\left\{\hat{A}R\hat{A}^T - R - \gamma\hat{B}_2\hat{B}_2^T + \begin{pmatrix}\hat{C}_1 R\hat{A}^T\\ \hat{B}_1^T\end{pmatrix}^T\begin{pmatrix}\gamma I - \hat{C}_1 R\hat{C}_1^T & -\hat{D}_{11}\\ -\hat{D}_{11}^T & \gamma I\end{pmatrix}^{-1}\begin{pmatrix}\hat{C}_1 R\hat{A}^T\\ \hat{B}_1^T\end{pmatrix}\right\}W_{12} < 0 \tag{5.9}$$

$$W_{21}^T\left\{\tilde{A}^T S\tilde{A} - S - \gamma\tilde{C}_2^T\tilde{C}_2 + \begin{pmatrix}\tilde{B}_1^T S\tilde{A}\\ \tilde{C}_1\end{pmatrix}^T\begin{pmatrix}\gamma I - \tilde{B}_1^T S\tilde{B}_1 & -\tilde{D}_{11}^T\\ -\tilde{D}_{11} & \gamma I\end{pmatrix}^{-1}\begin{pmatrix}\tilde{B}_1^T S\tilde{A}\\ \tilde{C}_1\end{pmatrix}\right\}W_{21} < 0 \tag{5.10}$$

$$R > 0; \qquad S > 0; \qquad \lambda_{\min}(RS) \geq 1. \tag{5.11}$$

Moreover, the set of γ*-suboptimal controllers of order* k *is non empty if and only if (ii) holds for some* R, S *which further satisfy the rank constraint:*

$$\text{Rank}\,(I - RS) \leq k. \tag{5.12}$$

■

Clearly the continuous- and discrete-time characterizations are similar in nature. Discrete-time Riccati expressions replace continuous-time ones, and the only qualitative difference lies in the additional constraints (5.8) on R and S. Even this difference becomes immaterial when (5.9)-(5.10) are written as LMI's in R and S (see Section 6).

6 LMI Formulation and Convexity Properties

The solvability conditions obtained in Theorems 4.3 and 5.2 involve Riccati inequalities instead of equations. At first this could seem a drawback since ARI's cannot be solved by the standard numerical techniques for Riccati equations [13]. However, these ARI's as well as the positivity and coupling constraints turn out to depend convexly on the unknown variables R, S. In fact, they can be rewritten as LMI's in R and S. Hence this characterization is not only numerically tractable, but also falls within the scope of efficient convex optimization algorithms such as [14, 2].

The LMI reformulation is a by-product of the proofs of Theorems 4.3 and 5.2. For continuous-time systems for instance, we have shown that $W_{\mathcal{P}}^T \Phi_{X_{c\ell}} W_{\mathcal{P}} < 0$ is equivalent to the LMI (4.20). In addition, the positivity and coupling constraints (4.13) are equivalent to $\begin{pmatrix} R & I \\ I & S\end{pmatrix} \geq 0$ [15]. Hence Theorem 4.3 has the following LMI reformulation.

The continuous-time γ-suboptimal H_∞ problem is solvable if and only if there exist symmetric matrices R, S satisfying the following LMI system:

$$\left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} AR + RA^T & RC_1^T & B_1 \\ C_1 R & -\gamma I & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma I \end{array}\right) \left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right) < 0 \quad (6.1)$$

$$\left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} A^T S + SA & SB_1 & C_1^T \\ B_1^T S & -\gamma I & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma I \end{array}\right) \left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right) < 0 \quad (6.2)$$

$$\left(\begin{array}{cc} R & I \\ I & S \end{array}\right) \geq 0 \quad (6.3)$$

where $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null spaces of (B_2^T, D_{12}^T) and (C_2, D_{21}), respectively. In addition, there exist γ-suboptimal controllers of order $k < n$ (reduced order) if and only if (6.1)-(6.3) hold for some R, S which further satisfy:

$$\text{Rank}\,(I - RS) \leq k. \quad (6.4)$$

For numerical stability, $\mathcal{N}_R$ and $\mathcal{N}_S$ should be chosen orthonormal. Such bases are easily computed via SVD's of $\left(\begin{array}{c} B_2 \\ D_{12} \end{array}\right)$ and (C_2, D_{21}). Note that the solution set of (6.1)-(6.3) is independent of the particular choice of bases $\mathcal{N}_R$ and $\mathcal{N}_S$. Also, the subscripts R, S do not mark any dependence on R or S and are simply meant to distinguish between the two LMI's.

The following counterpart is obtained for discrete-time systems based on (5.7).

The discrete-time γ-suboptimal H_∞ problem is solvable if and only if there exist symmetric matrices R, S satisfying the following LMI system:

$$\left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} ARA^T - R & ARC_1^T & B_1 \\ C_1 RA^T & -\gamma I + C_1 RC_1^T & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma I \end{array}\right) \left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right) < 0 \quad (6.5)$$

$$\left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} A^T SA - S & A^T SB_1 & C_1^T \\ B_1^T SA & -\gamma I + B_1^T SB_1 & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma I \end{array}\right) \left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right) < 0 \quad (6.6)$$

$$\left(\begin{array}{cc} R & I \\ I & S \end{array}\right) \geq 0 \quad (6.7)$$

where $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null spaces of (B_2^T, D_{12}^T) and (C_2, D_{21}), respectively. In addition, there exist γ-suboptimal controllers of order $k < n$ (reduced order) if and only if (6.5)-(6.7) hold for some R, S which further satisfy:

$$\text{Rank}\,(I - RS) \leq k.$$

Note that the inner block matrices in (6.1)-(6.2) and (6.5)-(6.6) depend only on the open-loop plant parameters A, B_1, C_1, D_{11}. Meanwhile, the control interconnection parameters B_2, C_2, D_{12}, D_{21} specify the projectors $\mathcal{N}_R$ and $\mathcal{N}_S$.

For fixed γ, the inequalities (6.1)-(6.3) or (6.5)-(6.7) are affine in R, S and define a convex set of pairs (R, S). Hence efficient interior-point methods from convex optimization [14, 2] can be used to test whether this set is nonempty and to generate particular members. In fact, the system (6.1)-(6.3) is jointly affine in R, S, γ. Hence the computation of the smallest feasible γ (usually called γ_{opt}) is also a convex program which can be solved with the same algorithms.

In contrast, the reduced-order problem is nonconvex due to the additional rank constraint (6.4). Finding appropriate optimization techniques for this problem constitutes a challenge for future research, but experimental results have been very encouraging so far.

7 Controller Reconstruction and Related Computational Issues

The theorems of Sections 4 and 5 are existence theorems which do not address the computation of the controller itself. This issue is now discussed in some detail. Suppose that we have computed some solution (R, S) of the system of LMI's (6.1)-(6.3) and that

$$Rank(I - RS) = k \leq n.$$

To construct an H_∞ controller from this data, recall that R, S are related by (4.18) to solutions $X_{c\ell}$ of the Bounded Real Lemma inequality. We therefore begin by computing a positive definite matrix $X_{c\ell} \in \mathbb{R}^{(n+k)\times(n+k)}$ compatible with (4.18). To this end, compute two *full-column-rank* matrices $M, N \in \mathbb{R}^{n\times k}$ such that

$$MN^T = I - RS. \tag{7.1}$$

An adequate $X_{c\ell}$ is then obtained as the unique solution of the linear equation:

$$\begin{pmatrix} S & I \\ N^T & 0 \end{pmatrix} = X_{c\ell} \begin{pmatrix} I & R \\ 0 & M^T \end{pmatrix}. \tag{7.2}$$

Note that (7.2) is always solvable when $S > 0$ and M has full column rank (see Lemma 7.5 in [15]).

From the proof of Theorem 4.3, (R, S) solves (4.11)-(4.13) if and only if $X_{c\ell}$ given by (4.18) is positive definite and satisfies (4.3). In turn, this guarantees the existence of a solution $\Theta = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$ to the Bounded Real Lemma inequality (4.6) in virtue of Theorem 4.2. And from the Bounded Real Lemma, $K(s) = D_K + C_K(sI - A_K)^{-1}B_K$ is then a γ-suboptimal controller.

It is now clear how to reconstruct adequate controllers from given R, S. First, compute $X_{c\ell} > 0$ by (7.1)-(7.2). Then, write for this $X_{c\ell}$ the Bounded Real Lemma inequality:

$$\begin{pmatrix} A_{c\ell}^T X_{c\ell} + X_{c\ell} A_{c\ell} & X_{c\ell} B_{c\ell} & C_{c\ell}^T \\ B_{c\ell}^T X_{c\ell} & -\gamma I & D_{c\ell}^T \\ C_{c\ell} & D_{c\ell} & -\gamma I \end{pmatrix} = \Psi_{X_{c\ell}} + \mathcal{Q}^T \Theta^T \mathcal{P}_{X_{c\ell}} + \mathcal{P}_{X_{c\ell}}^T \Theta \mathcal{Q} < 0 \qquad (7.3)$$

and solve this inequality for the controller parameters $\Theta = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$. The notation in (7.3) is that of Theorem 4.2. Since (7.3) is an LMI in Θ, it can be solved by the same optimization algorithms. Note that the minimal order of the controller is essentially determined by the size of $X_{c\ell}$ or equivalently, by the rank of $I - RS$.

This construction associates with each solution (R, S) of (6.1)-(6.3) the convex set of controller parameters determined by (7.3). Note that the particular choice of M, N is immaterial and simply amounts to a change of coordinates on the controller state. Conversely, with each controller realization (A_K, B_K, C_K, D_K), we can associate the convex set of matrices $X_{c\ell}$ solving the Bounded Real Lemma LMI (7.3). In turn, this determines via (4.18) a convex subset of the pairs (R, S) satisfying (6.1)-(6.3). Hence there is an exhaustive, though not one-to-one, correspondence between the set of γ-suboptimal controllers and the convex set of pairs (R, S) solving (6.1)-(6.3). Note that controllers of order $k > n$ can also be recovered by using "oversized" Θ in (7.3).

From a computational point of view, suboptimal controllers can therefore be derived from feasible pairs (R, S) by solving the LMI (7.3) for Θ. This approach offers the most flexibility for design purposes. Alternatively, more efficient "explicit" schemes can be used when no further constraint is placed on the controller or the closed-loop properties. The first of these schemes relies on an explicit description of the solution set of the LMI (7.3). Specifically, consider the proof of Lemma 3.1 and the condition (A.6) in particular. Defining

$$\Lambda_{22} := -\Psi_{22} + \Psi_{12}^T \Psi_{11}^{-1} \Psi_{12} > 0; \qquad \Lambda_{33} := -\Psi_{33} + \Psi_{13}^T \Psi_{11}^{-1} \Psi_{13} > 0, \qquad (7.4)$$

and using the usual Schur complement argument, (A.6) constrains Θ_{11} to:

$$-\Lambda_{22} + (\Theta_{11} + \Lambda_{32})^T \Lambda_{33}^{-1} (\Theta_{11} + \Lambda_{32}) < 0.$$

With the notation $\tilde{\Theta}_{11} := \Lambda_{33}^{-1/2} \Theta_{11} \Lambda_{22}^{-1/2}$ and $\tilde{\Lambda} := \Lambda_{33}^{-1/2} \Lambda_{32} \Lambda_{22}^{-1/2}$, this constraint also reads

$$(\tilde{\Theta}_{11} + \tilde{\Lambda})^T (\tilde{\Theta}_{11} + \tilde{\Lambda}) < I.$$

Hence $\tilde{\Theta}_{11}$ must be of the form $-\tilde{\Lambda} + U$ where U is any matrix of compatible dimensions satisfying $\sigma_{max}(U) < 1$.

Consequently, all solutions of (7.3) are obtained by selecting the Θ_{ij}'s in (A.2) as follows:

- $\Theta_{11} = -\Lambda_{32} + \Lambda_{33}^{1/2} U \Lambda_{22}^{1/2}$ with U arbitrary subject to $\sigma_{max}(U) < 1$;
- Θ_{12} and Θ_{21} are arbitrary;

- $\Theta_{22} = \Sigma + \Upsilon$ where Υ is an arbitrary skew-symmetric matrix and $\Sigma = \Sigma^T$ satisfies:

$$\Sigma < \frac{1}{2}\left\{ -\Psi_{44} + \begin{pmatrix} \Psi_{14} \\ \Psi_{24} + \Theta_{21}^T \\ \Psi_{34} + \Theta_{12} \end{pmatrix}^T \Pi^{-1} \begin{pmatrix} \Psi_{14} \\ \Psi_{24} + \Theta_{21}^T \\ \Psi_{34} + \Theta_{12} \end{pmatrix} \right\} \tag{7.5}$$

  with Π as in (A.5).

The implementation of this scheme requires only standard linear algebra. Note however that the preliminary congruence transformation may be ill-conditioned in some cases.

The second explicit scheme consists of implementing the formulas in [9]. These formulas parallel the central controller formulas of [11] except that R, S replace X_∞, Y_∞. This approach involves only straightforward linear algebra and is numerically well-conditioned.

8 Comparison with Classical Results

The classical H_∞ state-space formulas of [11, 5] are only applicable to plants which satisfy the regularity assumptions:

(A3) D_{12} has full column rank and D_{21} has full row rank,

(A4) $P_{12}(s)$ and $P_{21}(s)$ have no invariant zero on the imaginary axis.

Under these assumptions, [11, 5] provide a characterization of feasible γ's in terms of the stabilizing solutions of two H_∞ Riccati equations which exactly match our Riccati inequalities.

This section sheds some light on the connection between the ARE-based and LMI-based characterizations, as well as on the special role played by the stabilizing solutions of the H_∞ Riccati equations. For simplicity, we assume $D_{11} = 0$. The following monotonicity result is instrumental to the argument.

Lemma 8.1 *Consider a plant $P(s)$ satisfying (A1)-(A4) and suppose the ARI*

$$\hat{A}^T X + X\hat{A} + X(\gamma^{-2}\hat{B}_1\hat{B}_1^T - \hat{B}_2\hat{B}_2^T)X + \hat{C}_1^T\hat{C}_1 < 0 \tag{8.1}$$

has a solution $X_0 = X_0^T \in \mathbb{R}^{n\times n}$. Then:

(i) The Hamiltonian matrix

$$H_\gamma = \begin{pmatrix} \hat{A} & \gamma^{-2}\hat{B}_1\hat{B}_1^T - \hat{B}_2\hat{B}_2^T \\ -\hat{C}_1^T\hat{C}_1 & -\hat{A}^T \end{pmatrix} \tag{8.2}$$

has no eigenvalue on the imaginary axis.

(ii) If moreover $X_0 > 0$, the ARE

$$\hat{A}^T X + X\hat{A} + X(\gamma^{-2}\hat{B}_1\hat{B}_1^T - \hat{B}_2\hat{B}_2^T)X + \hat{C}_1^T\hat{C}_1 = 0 \tag{8.3}$$

has a stabilizing solution X_∞ satisfying

$$0 \le X_\infty < X_0. \tag{8.4}$$

Proof: See Appendix B.

■

This lemma shows that whenever the set of *positive definite* solutions of the ARI (8.1) is nonempty, the corresponding ARE does have a nonnegative stabilizing solution which is *minimal* in this set. This result together with Theorem 4.3 explains the special role played by the stabilizing solutions X_∞ and Y_∞ of the ARE (8.3) and its dual:

$$\tilde{A}Y + Y\tilde{A}^T + Y(\gamma^{-2}\tilde{C}_1^T\tilde{C}_1 - \tilde{C}_2^T\tilde{C}_2)Y + \tilde{B}_1\tilde{B}_1^T = 0. \tag{8.5}$$

Specifically, solvability of the γ-suboptimal H_∞ problem implies the existence of symmetric matrices R_0 and S_0 satisfying (4.11)-(4.13). Observing that $W_{12} = I$ and $W_{21} = I$ under Assumption **(A3)**, it follows that $X_0 := \gamma R_0^{-1}$ and $Y_0 := \gamma S_0^{-1}$ solve the ARI counterparts of (8.3) and (8.5) and further satisfy:

$$X_0 > 0; \qquad Y_0 > 0; \qquad \rho(X_0Y_0) < \gamma^2. \tag{8.6}$$

Note that $\rho(X_0Y_0) \le \gamma^2$ has been strengthened to a strict inequality, which is always possible by perturbing the ARI solutions. Invoking Lemma 8.1, the ARE's (8.3) and (8.5) must then have stabilizing solutions X_∞ and Y_∞ such that:

$$0 \le X_\infty < X_0\,; \qquad 0 \le Y_\infty < Y_0. \tag{8.7}$$

Observing that (8.6)-(8.7) imply $\rho(X_\infty Y_\infty) < \gamma^2$, we obtain exactly the necessary conditions of [5].

Conversely, suppose the ARE's (8.3) and (8.5) have stabilizing solutions satisfying

$$X_\infty \ge 0; \qquad Y_\infty \ge 0; \qquad \rho(X_\infty Y_\infty) < \gamma^2.$$

By standard results on the continuity of ARE stabilizing solutions under perturbation [4], the ARE

$$\hat{A}^TX + X\hat{A} + X(\gamma^{-2}\hat{B}_1\hat{B}_1^T - \hat{B}_2\hat{B}_2^T)X + \hat{C}_1^T\hat{C}_1 + \epsilon I = 0 \tag{8.8}$$

retains a stabilizing solution X_ϵ for $\epsilon > 0$ small enough. Remarking that solutions of (8.8) cannot be singular, we must have $X_\epsilon > 0$ by continuity. Hence X_ϵ is a solution of the ARI (8.1) satisfying $X_\infty < X_\epsilon$. Solutions of the Y-ARI are constructed similarly and by continuity, $\rho(X_\infty Y_\infty) < \gamma^2$ implies $\rho(X_\epsilon Y_\epsilon) < \gamma^2$ for ϵ small enough. Hence a solution of the LMI system (4.11)-(4.13) is given by $(\gamma X_\epsilon^{-1}, \gamma Y_\epsilon^{-1})$ and it follows from Theorem 4.3 that the γ-suboptimal H_∞ problem is solvable.

For regular H_∞ problems, the solvability tests of [5] are computationally more efficient since solving ARE's is of lesser complexity than solving LMI's. For singular problems however, the convex LMI characterization offers a numerically sound alternative since DGKF's algorithm breaks down and discontinuities render solvability assessment delicate [10]. Thanks to convexity, our solvability test is also competitive with that of [20]. Note that the solution of [20] can be seen as an extension of [5] to the singular case since it also emphasizes extremal points and stabilizing solutions of ARE's of smaller size.

9 Applications

This section is meant to illustrate the potential of LMI-based H_∞ synthesis and to point out directions of future research. Most claims will receive no other justification than demonstrative examples and qualitative insight. Some of the issues raised below have already been worked out (Subsection 9.4), yet most of them remain under investigation. Throughout this section, $\mathcal{A}_\gamma$ denotes the set of pairs (R, S) satisfying (6.1)-(6.3). All computations were performed with MATLAB 4.0. Classical H_∞ synthesis was performed with the *μ-Analysis and Synthesis Toolbox*. For LMI-based synthesis, we used a MATLAB package called *LMI-Lab*. This package is currently developed at INRIA by the first author and A. Nemirovsky. The LMI solvers in this package are based on the projective method of Nesterov & Nemirovsky [14].

9.1 H_∞ problems with $j\omega$-axis zeros

The first application is concerned with plants where P_{12} or P_{21} have invariant zeros on the imaginary axis. The augmented plants encountered in loop shaping applications often fall in this category. The case of $j\omega$-axis zeros has been investigated in, e.g., [18, 3]. A difficulty with such plants is the possible discontinuity of γ_{opt} when perturbing the plant data [10]. Not only this may lead to wrong estimates of γ_{opt}, but also to H_∞ controllers yielding extremely poor closed-loop stability. Specifically, singular problems with a discontinuity of γ_{opt} share the following features:

(a) the upper continuity value γ_{opt}^+ corresponds to the optimal γ for the singular plant.

(b) γ_{opt} may drop sharply when the singular plant is regularized by slight perturbation of the plant data.

(c) This extra performance is acquired at the expense of closed-loop damping: all controllers yielding $\gamma < \gamma_{opt}^+$ will place some closed-loop poles very close to the imaginary axis.

(d) The convex set $\mathcal{A}_\gamma$ of pairs (R, S) moves to infinity when γ is decreased below the upper continuity value γ_{opt}^+.

From the closed-loop stability viewpoint, γ_{opt}^+ is therefore more meaningful than γ_{opt} for nearly singular problems. More detail and justifications can be found in [10].

This discontinuity phenomenon is easily monitored when using the LMI-based characterization of suboptimal γ's. First, singular plants can be handled directly and no regularization by perturbation is needed to compute γ_{opt}. Secondly, the computed γ_{opt} for nearly singular plants can be kept close to the upper continuity value γ_{opt}^+. This is achieved by preventing the norms R or S from becoming large, a strategy which is rooted in property (d) above.

The following simple example illustrates the discontinuity phenomenon and the advantages of the LMI-based characterization.

Example 9.1 Consider the plant data

$$A = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad C_1 = \begin{pmatrix} 1 & 0 \\ 0.5 & -1 \end{pmatrix}; \quad C_2 = (0, 1);$$

$$D_{11} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad D_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad D_{21} = 1; \quad D_{22} = 0.$$

This plant is singular since $P_{21}(s)$ has a zero at $s = 0$. We perturbed A to $A + \epsilon I$ and computed γ_{opt} for various values of ϵ (using the function `hinfsyn` in μ-*Tools*). The results appear in Table 9.1 and clearly show a discontinuity at $\epsilon = 0$.

ϵ	γ_{opt} with $A - \epsilon I$	γ_{opt} with $A + \epsilon I$
1.0×10^{-4}	0.90	2.00
1.0×10^{-6}	0.90	2.00
1.0×10^{-8}	0.90	0.90

Table 9.1: Discontinuity of γ_{opt} at $\epsilon = 0$.

Note that the values obtained for $\epsilon = 1.0 \times 10^{-8}$ should not be seen as a contradiction to our claim, but rather attributed to numerical ill condition. Indeed, the diagnosis at each γ-iteration depends on the correct assessment of the sign of the smallest eigenvalue of $X_{c\ell}$ (cf. Example 3.2 and Section 4 in [10]). Since this eigenvalue is of order ϵ, correct diagnosis becomes hopeless when ϵ is too small. This shows that wrong estimates of γ_{opt} may be obtained even when A is perturbed to $A + \epsilon I$ with $\epsilon > 0$.

The same data was fed to the LMI solver with $\epsilon = 0$ (recall that singular problems can be handled directly). The optimization program returned $\gamma_{opt} = 2$ and this value was achieved for a pair (R, S) with $\max(\lambda_{\max}(R), \lambda_{\max}(S)) = 3.81$. The same value was obtained by perturbing A to $A + \epsilon I$ for small $\epsilon > 0$. For $\epsilon = -10^{-4}$ by contrast, $\gamma_{opt} = 0.9$ was achieved with $\lambda_{\max}(S) = 3.0 \times 10^3$. For $\epsilon = -10^{-8}$ finally, $\gamma_{opt} = 0.9$ was achieved with $\lambda_{\max}(S) = 1.0 \times 10^6$.

This confirms that the lower continuity value 0.9 can only be attained for S's of large norm. Interestingly, the LMI solver displayed the following behavior for $\epsilon < 0$; it quickly reached the upper continuity value $\gamma_{opt}^+ = 2$, then slowly decreased to the optimum $\gamma_{opt} = 0.9$ as it painstakingly reached toward feasible S's of large norm.

9.2 Preventing pole/zero cancellations between the plant and the controller

The central controller has the undesirable property of cancelling all stable invariant zeros of $P_{12}(s)$ and $P_{21}(s)$ [19]. These exact cancellations are frequently encountered in mixed-sensitivity problems and become unacceptable when the plant contains some lightly damped modes. Some remedies have been proposed in the loop shaping context and generally consist of modifying the criterion to penalize cancellations [23, 19]. Outside of this context however, available state-space techniques offer, to our knowledge, no systematic remedy.

To illustrate the additional flexibility offered by LMI-based H_∞ design, we now describe a more systematic remedy to the cancellation problem. We first introduce the concept of suboptimal H_∞ controller with a prescribed degree of closed-loop damping.

Definition 9.2 *Let $\alpha > 0$ be given. A stabilizing controller $K(s) = D_K + C_K(sI - A_K)^{-1}B_K$ is called* **α-stable** *if all closed-loop modes lie in the half plane* $\mathrm{Re}(s) \le -\alpha$*. In other words, α-stability requires that all eigenvalues of $A_{c\ell}$ satisfy*

$$\mathrm{Re}(\lambda) \;\le\; -\alpha.$$

It is easily seen that a sufficient condition for $K(s)$ to be γ-suboptimal and α-stable is the existence of $X_{c\ell} > 0$ satisfying:

$$\begin{pmatrix} A_{c\ell}^T X_{c\ell} + X_{c\ell} A_{c\ell} & X_{c\ell} B_{c\ell} & C_{c\ell}^T \\ B_{c\ell}^T X_{c\ell} & -\gamma I & D_{c\ell}^T \\ C_{c\ell} & D_{c\ell} & -\gamma I \end{pmatrix} \;<\; 0 \tag{9.1}$$

$$A_{c\ell}^T X_{c\ell} + X_{c\ell} A_{c\ell} + 2\alpha\, X_{c\ell} \;<\; 0 \tag{9.2}$$

Note that the second condition is equivalent to $A_{c\ell} + \alpha I$ stable by the Lyapunov Theorem. The conditions (9.1)-(9.2) are only sufficient since the same Lyapunov function $X_{c\ell}$ is used to prove both the bounded realness and the α-stability.

Using the same technique as in Sections 4 and 5, we can derive necessary conditions in terms of R, S for (9.1)-(9.2) to hold for some controller $K(s)$. These conditions consist of the three LMI's (6.1)-(6.3) plus two constraints issued from (9.2):

$$W_{B_2^T}^T \left\{ AR + RA^T + 2\alpha R \right\} W_{B_2^T} < 0; \qquad W_{C_2}^T \left\{ A^T S + SA + 2\alpha S \right\} W_{C_2} < 0 \tag{9.3}$$

where $W_{B_2^T}, W_{C_2}$ denote arbitrary bases of the null spaces of B_2^T and C_2, respectively. The resulting LMI system is affine in R, S, γ, α, and we can either maximize α for a given γ, or minimize γ for some prescribed α. Since these LMI's are only necessary conditions, there is no guarantee that the computed pair (R, S) will make (9.1)-(9.2) solvable for $\Theta = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$. Yet, numerical experiments show that fairly high damping can be achieved with this scheme.

Back to our original problem, poorly damped cancellations can be prevented by seeking α-stable controllers with α large enough. An illustration of this design strategy is given in the next example.

Example 9.3 Consider the second-order flexible system $G(s) = \dfrac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$ with $\zeta = 0.0001$ and $\omega_0 = 100$ rd/s. The problem is to design a controller $K(s)$ which meets the following specifications:

- K must have an integral behavior in the low frequencies, and the gain of GK should be at least 20 dB at 0.1 rd/s,
- the roll-off must be at least -20 dB/dec after 100 rd/s and the gain GK must fall below -20 dB at 1000 rd/s.

These specifications are captured by the following loop shaping problem: find an internally stabilizing controller $K(s)$ such that

$$\left\| \begin{pmatrix} w_1 S \\ w_2 K S \\ w_3 T \end{pmatrix} \right\|_\infty < 1 \tag{9.4}$$

with $S = (1 + GK)^{-1}$, $T = GKS$, and the weighting functions:

$$w_1(s) = \frac{1}{s}; \qquad w_2(s) = 0.01; \qquad w_3(s) = \frac{50s}{s + 5000}.$$

Up to perturbing $w_1(s)$ to $\frac{1}{s+0.01}$, the usual H_∞ synthesis can be performed and yields $\gamma_{opt} \approx 0.11$. For $\gamma = 1$, the central controller computed by `hinfsyn` is

$$K_c(s) = 100.8 \, \frac{(s + 5000)\,(s^2 + 0.02s + 1.0e4)}{(s + 0.01)\,(s + 1.42)\,(s^2 + 1.12e4\, s + 5.0e7)}.$$

Clearly, the term $s^2 + 0.02s + 1.0e4$ cancels the denominator of $G(s)$. As a result, the lightly damped modes $-0.01 \pm j\, 100$ appear among the closed-loop modes. Due to these exact cancellations, stability is very poorly robust to parametric variations of ω_0 or ζ.

We now turn to the LMI-based design. Of course, the natural cancellation of the weighting function poles cannot and should not be prevented. Recalling that Y_∞ is responsible for all cancellations in this problem [19], $Y = \gamma S^{-1}$ should therefore be kept singular along the directions associated with the weighting function modes. Specifically, if the augmented state matrix is diagonalized to $A_{aug} = \begin{pmatrix} A_G & 0 \\ 0 & A_W \end{pmatrix}$ where A_G is the state matrix of $G(s)$, Y should be sought of the form:

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Defining $S_1 := \gamma Y_1^{-1}$, it can be shown that S_1 solves an LMI of smaller size and that (6.1)-(6.3) can be replaced by an analogous system of LMI's involving only R and S_1.

Up to this technical adjustment, the α-stability problem can be solved as suggested above. Note that the integrator in $w_1(s)$ needs not be perturbed here. In this example, we set $\alpha = 1$ and minimized γ subject to (6.1)-(6.3) and (9.3). The smallest achievable value was found to be 0.98 . Next we solved the system (9.1)-(9.2) to compute an adequate controller. The LMI solver was successful and returned the following controller:

$$K_{new}(s) = 1.2e5 \, \frac{(s + 1500)\,(s^2 - 91s + 6130)}{s\,(s + 38.2)\,(s^2 + 1.25e5\, s + 4.33e9)}.$$

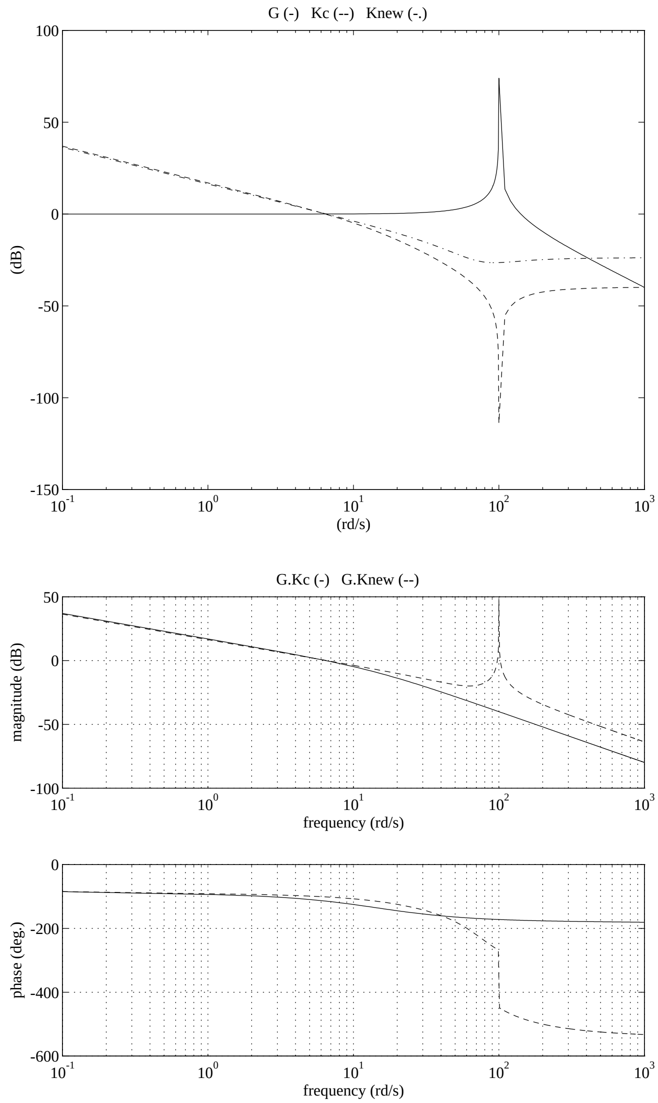

Figure 9.1: Comparison of K_c and K_{new}.

The gain response of K and the Bode plot of GK are compared in Figure 9.1 for the two controllers K_c and K_{new}. Note that K_{new} is cancellation-free and renders GK passive. This provides very strong robustness against variations of ζ and tolerates up to ± 40 % variation of ω_0. Inspection of the closed-loop modes confirms the α-stability since they all have real parts less than -2.0. Clearly, K_{new} has much nicer properties for control purposes.

When γ is decreased from 1 to $\gamma_{opt} = 0.11$, K_{new} gradually evolves toward a cancelling controller similar to K_c. More precisely, the smallest achievable γ as a function of α evolves as follows:

α	smallest γ
≥ 1	0.98
0.1	0.88
0.03	0.25
0.01	0.11

This indicates that $\gamma_{opt} = 0.11$ can only be achieved via cancellations of the lightly damped modes. A compromise must therefore be found between the H_∞ performance γ_{opt} and the closed-loop damping α.

Finally, note that the order of K_{new} can be halved by removing the poles and zeros of large magnitude. This yields the second-order controller $\tilde{K}_{new}(s) = 0.041 \dfrac{s^2 - 91s + 6130}{s(s + 38.2)}$.

9.3 Reduced-Order Design

The LMI-based parametrization introduced above is also useful for reduced-order H_∞ synthesis. Indeed, γ-suboptimal controllers of order $k < n$ have a simple characterization in this framework: they correspond to pairs (R, S) of $\mathcal{A}_\gamma$ for which $Rank(I - RS) = k$. Such pairs lie on the boundary of $\mathcal{A}_\gamma$ and saturate the constraint $\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0$ in $n - k$ directions. Thus, the reduced-order synthesis problem consists of maximizing the rank deficiency of $I - RS$ without leaving $\mathcal{A}_\gamma$.

For feasible γ's, this rank minimization problem can be expressed as an optimization problem under LMI constraints. Specifically, the synthesis of controllers of order $k < n$ amounts to minimizing for $(R, S) \in \mathcal{A}_\gamma$ the objective function:

$$\Psi(R, S) = \sum_{i=1}^{n-k} \lambda_i \begin{pmatrix} R & I \\ I & S \end{pmatrix} \tag{9.5}$$

where $\lambda_1(.) \leq \cdots \leq \lambda_{n-k}(.)$ denote the $n - k$ smallest eigenvalues of $\begin{pmatrix} R & I \\ I & S \end{pmatrix}$. There will exist suboptimal controllers of order k if and only if the global minimum of $\Psi(R, S)$ is 0. The objective function Ψ is not convex but in fact concave. Hence this problem is much harder than those discussed in the first two subsections. In particular, global convergence is not guaranteed.

The following experiment was performed to test this order reduction scheme. We generated plants admitting a stabilizing controller of order $k < n$, computed the closed-loop H_∞ norm γ_0 provided by this controller, and considered the problem of minimizing (9.5) under the LMI constraints (6.1)-(6.3) and for $\gamma \leq \gamma_0$. An initial feasible (R, S) was computed with *LMI-Lab*, and the function `constr` from MATLAB's *Optimization Toolbox* was used to minimize (9.5). Even though `constr` was not designed for non-differentiable problems, results were very encouraging. In most cases, the $n - k$ smallest eigenvalues were zeroed not only for $\gamma = \gamma_0$, but also for much smaller γ's, sometimes all the way to the optimum γ_{opt}. In other words, order reductions at least equal to what was known to be feasible were obtained, and sometimes for γ's much smaller than γ_0. These early results suggest that LMI-based reduced-order synthesis is indeed numerically tractable.

Codes for non-differentiable optimization are currently tried on this problem. It would also be worth comparing our LMI-based characterization of all reduced-order H_∞ controllers to the results obtained in [1] or in [21] for the special case of singular H_∞ control.

9.4 Reliable Computation of Optimal Central Controllers

When the H_∞ optimal gain γ_{opt} is characterized by

$$\rho(X_\infty Y_\infty) = \gamma_{opt}^2,$$

the computation of the central controller K_c is ill-conditioned near and at the optimum. This is due to the cancellation(s) at infinity which occur in the pole/zero structure of K_c. Such cancellations induce a feedthrough term in K_c as well as some order reduction at γ_{opt}.

These numerical difficulties can be eliminated altogether by allowing for a feedthrough term in suboptimal K_c's as well. This is easily done by extending the notion of central controller on the basis of the parametrization derived above. By appropriate choice of the feedthrough matrix, we can obtain finite instead of infinite pole/zero cancellations in the controller at γ_{opt}. Only numerically stable computations are involved in the process. More details on this approach can be found in [7].

10 Conclusions

We have presented a LMI-based solution to the general continuous- and discrete-time (sub-optimal) H_∞ problems. Our solvability conditions parallel the usual ones except that Riccati equations are replaced by Riccati inequalities. This inequality formulation provides a parametrization of all H_∞-suboptimal controllers where the free parameters are pairs (R, S) of positive definite matrices which solve the Riccati inequalities and satisfy some coupling constraint. Both the computation of adequate (R, S) and the controller reconstruction reduce to solving LMI's, hence to convex optimization programs. Because of the connection between (R, S), the controller order, and the closed-loop properties, this approach holds promise for the improvement of current H_∞ design techniques.

Appendix A

Proof of Lemma 3.1: Let U_{PQ} be a basis of $\operatorname{Ker} P \cap \operatorname{Ker} Q$ and introduce matrices U_P, U_Q such that $W_P := [U_{PQ}, U_P]$ and $W_Q := [U_{PQ}, U_Q]$ are bases of $\operatorname{Ker} P$ and $\operatorname{Ker} Q$, respectively. Let p and q denote the dimension of $\operatorname{Ker} P$ and $\operatorname{Ker} Q$, respectively. Observing that $[U_{PQ}, U_P, U_Q]$ is then a basis of $\operatorname{Ker} P \oplus \operatorname{Ker} Q$, complete it into a basis $T = [U_{PQ}, U_P, U_Q, V]$ of $\mathbb{R}^m$. The matrix T is nonsingular and therefore (3.1) is equivalent to:

$$T^T \Psi\, T + (PT)^T \Theta^T\, (QT) + (QT)^T \Theta\, (PT) < 0. \qquad \text{(A.1)}$$

Block-partition PT, QT and $T^T\Psi T$ conformably to the partition $[U_{PQ}, U_P, U_Q, V]$ of T. By construction, we have

$$PT = (0, 0, P_1, P_2); \qquad QT = (0, Q_1, 0, Q_2).$$

Note that $[P_1, P_2] \in \mathbb{R}^{(k+p_2)\times p}$ with $p \le k + p_2$ and $[Q_1, Q_2] \in \mathbb{R}^{(k+m_2)\times q}$ with $q \le k + m_2$, and that both matrices have full column rank. With the notation

$$\begin{pmatrix} P_1^T \\ P_2^T \end{pmatrix} \Theta \, (Q_1, Q_2) = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \in \mathbb{R}^{p\times q} \tag{A.2}$$

and the partition

$$T^T \Psi \, T = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{13}^T & \Psi_{23}^T & \Psi_{33} & \Psi_{34} \\ \Psi_{14}^T & \Psi_{24}^T & \Psi_{34}^T & \Psi_{44} \end{pmatrix},$$

(A.1) reads:

$$\begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} + \Theta_{11}^T & \Psi_{24} + \Theta_{21}^T \\ \Psi_{13}^T & \Psi_{23}^T + \Theta_{11} & \Psi_{33} & \Psi_{34} + \Theta_{12} \\ \Psi_{14}^T & \Psi_{24}^T + \Theta_{21} & \Psi_{34}^T + \Theta_{12}^T & \Psi_{44} + \Theta_{22} + \Theta_{22}^T \end{pmatrix} < 0. \tag{A.3}$$

Here the Θ_{ij}'s are arbitrary since Θ is arbitrary and $[P_1, P_2]$ and $[Q_1, Q_2]$ have full column rank. Specifically, given any Θ_{ij}'s the matrix $\begin{pmatrix} P_1^T \\ P_2^T \end{pmatrix}^+ \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} (Q_1, Q_2)^+$ solves (A.2). Hence our problem reduces to finding conditions on the Ψ_{ij}'s which ensure feasibility of (A.3) for some Θ_{ij}'s.

By a Schur complement argument, (A.3) is equivalent to

$$\Pi = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} + \Theta_{11}^T \\ \Psi_{13}^T & \Psi_{23}^T + \Theta_{11} & \Psi_{33} \end{pmatrix} < 0 \tag{A.4}$$

$$\Psi_{44} + \Theta_{22} + \Theta_{22}^T - \begin{pmatrix} \Psi_{14} \\ \Psi_{24} + \Theta_{21}^T \\ \Psi_{34} + \Theta_{12} \end{pmatrix}^T \Pi^{-1} \begin{pmatrix} \Psi_{14} \\ \Psi_{24} + \Theta_{21}^T \\ \Psi_{34} + \Theta_{12} \end{pmatrix} < 0. \tag{A.5}$$

Given Θ_{11}, Θ_{12}, and Θ_{21}, we can always find Θ_{22} such that (A.5) is satisfied. Hence (3.1) is feasible if and only if (A.4) is feasible for some Θ_{11}.

Now, (A.4) is equivalent to

$$\begin{pmatrix} I & 0 & 0 \\ -\Psi_{12}^T\Psi_{11}^{-1} & I & 0 \\ -\Psi_{13}^T\Psi_{11}^{-1} & 0 & I \end{pmatrix} \Pi \begin{pmatrix} I & -\Psi_{11}^{-1}\Psi_{12} & -\Psi_{11}^{-1}\Psi_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} < 0.$$

That is,

$$\begin{pmatrix} \Psi_{11} & 0 & 0 \\ 0 & \Psi_{22} - \Psi_{12}^T\Psi_{11}^{-1}\Psi_{12} & \Theta_{11}^T + \Lambda_{32}^T \\ 0 & \Theta_{11} + \Lambda_{32} & \Psi_{33} - \Psi_{13}^T\Psi_{11}^{-1}\Psi_{13} \end{pmatrix} < 0 \tag{A.6}$$

where

$$\Lambda_{32} := \Psi_{23}^T - \Psi_{13}^T \Psi_{11}^{-1} \Psi_{12}. \tag{A.7}$$

Since Θ_{11} is arbitrary, this is feasible if and only if

$$\begin{cases} \Psi_{11} & < 0 \\ \Psi_{22} - \Psi_{12}^T \Psi_{11}^{-1} \Psi_{12} & < 0 \\ \Psi_{33} - \Psi_{13}^T \Psi_{11}^{-1} \Psi_{13} & < 0 \end{cases}$$

or equivalently, if and only if

$$\begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^T & \Psi_{22} \end{pmatrix} < 0 \; ; \qquad \begin{pmatrix} \Psi_{11} & \Psi_{13} \\ \Psi_{13}^T & \Psi_{33} \end{pmatrix} < 0.$$

This last condition is exactly (3.2) upon recalling the definition of W_P, W_Q, and the Ψ_{ij}'s. ■

Appendix B

Proof of Lemma 8.1:
(i): For simplicity, assume $D_{11} = 0$ and drop the hats on $A, B_1, B_2, \ldots$ The ARI (8.1) then reads

$$A^T X + XA + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X + C_1^T C_1 < 0 \tag{B.1}$$

and

$$H_\gamma = \begin{pmatrix} A & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{pmatrix}.$$

The proof is by contradiction. Suppose $H_\gamma \begin{pmatrix} u \\ v \end{pmatrix} = j\omega \begin{pmatrix} u \\ v \end{pmatrix}$ with $(u, v) \neq (0, 0)$. That is,

$$Au + Fv = j\omega\, u \tag{B.2}$$

$$-C_1^T C_1 u - A^T v = j\omega\, v \tag{B.3}$$

where $F := \gamma^{-2} B_1 B_1^T - B_2 B_2^T$. First, it is easily seen that $v = 0$ would violate **(A4)** since then $u \neq 0$ would satisfy $Au = j\omega u$ and $C_1 u = 0$. Hence $v \neq 0$.

Observing that solutions of (B.1) cannot be singular and defining $R_0 := X_0^{-1}$, (B.1) is equivalent to

$$\mathcal{R} := AR_0 + R_0 A^T + R_0 C_1^T C_1 R_0 + F < 0. \tag{B.4}$$

From (B.2), we get $v^H F v = j\omega\, v^H u - v^H A u$ and from (B.3): $A^T v = -C_1^T C_1 u - j\omega\, v$. Consequently,

$$\begin{aligned} v^H \mathcal{R} v &= (v^H A) R_0 v + v^H R_0 (A^T v) + v^H R_0\, C_1^T C_1\, R_0 v + v^H F v \\ &= \left\{ -u^H C_1^T C_1 + j\omega\, v^H \right\} R_0 v + v^H R_0 \left\{ -C_1^T C_1 u - j\omega\, v \right\} + \\ &\qquad v^H R_0\, C_1^T C_1\, R_0 v + \left\{ j\omega\, v^H u - (v^H A) u \right\} \\ &= -u^H C_1^T C_1 R_0 v - v^H R_0 C_1^T C_1 u + v^H R_0 C_1^T C_1 R_0 v + j\omega\, v^H u + \left\{ u^H C_1^T C_1 - j\omega\, v^H \right\} u \\ &= (R_0 v - u)^H\, C_1^T C_1\, (R_0 v - u) \;\geq 0 \end{aligned}$$

which contradicts $\mathcal{R} < 0$ since $v \neq 0$.

(ii): Since H_γ has no $j\omega$-axis eigenvalue, its stable invariant subspace $\begin{pmatrix} P \\ Q \end{pmatrix}$ is of dimension n. Assume that (C_1, A) has no stable unobservable mode. By standard results on H_∞ Riccati equations (see, e.g., [8]), Q is then invertible and $R_\infty := PQ^{-1}$ is an antistabilizing solution of:

$$AR + RA^T + RC_1^T C_1 R + F = 0. \tag{B.5}$$

Subtracting (B.5) from (B.4), we obtain

$$A(R_0 - R_\infty) + (R_0 - R_\infty)A^T + R_0 C_1^T C_1 R_0 - R_\infty C_1^T C_1 R_\infty < 0$$

or equivalently,

$$(A + R_\infty C_1^T C_1)(R_0 - R_\infty) + (R_0 - R_\infty)(A + R_\infty C_1^T C_1)^T + (R_0 - R_\infty)C_1^T C_1 (R_0 - R_\infty) < 0.$$

From $A + R_\infty C_1^T C_1$ antistable and Lyapunov's theorem, we conclude that

$$R_0 < R_\infty \tag{B.6}$$

which together with our assumption $X_0 > 0$ ensures that $R_\infty > 0$. Consequently, P is invertible and $X_\infty := QP^{-1} = R_\infty^{-1}$ is a stabilizing solution of the ARE (8.3). In addition, (B.6) and $R_0 > 0$ also give $0 < R_\infty^{-1} < R_0^{-1}$, that is,

$$0 < X_\infty < X_0. \tag{B.7}$$

The proof is complete upon removing the assumption on (C_1, A). Continuity under small perturbations of the data can be used to this end. Specifically, we can always perturb (C_1, A) to $(C_1^{(\epsilon)}, A^{(\epsilon)})$ in such a way that the "no stable unobservable mode" assumption holds for $\epsilon > 0$ small enough and that X_0 remains a solution of the perturbed ARI. Now, the stable invariant subspace $\begin{pmatrix} P_\epsilon \\ Q_\epsilon \end{pmatrix}$ of $H_\gamma^{(\epsilon)}$ depends continuously on ϵ and from the discussion above we have

$$0 < X_\infty^{(\epsilon)} = Q_\epsilon P_\epsilon^{-1} < X_0$$

for $\epsilon > 0$. Consequently, as $\epsilon \to 0$, $X_\infty^{(\epsilon)}$ has a finite limit X_∞ which is clearly a stabilizing solution of (8.3) and satisfies

$$0 \leq X_\infty \leq X_0.$$

Finally, the second inequality is easily strengthened upon replacing X_0 by $(1 - \epsilon)X_0$ in the previous argument.

■

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