Gain-Scheduled Filtering for Time-Varying Discrete Systems

Nguyen Thien Hoang, Hoang Duong Tuan, Pierre Apkarian, and Shigeyuki Hosoe

***Abstract*—This paper deals with the design of gain-scheduled filters, whose state-space realization depends on real-time parameters of plants. Similar to well-recognized advantages of gain-scheduled controllers in control theory, gain-scheduled filters are expected to provide enhanced performance in comparison with customary nonadjustable filters. Our construction technique is based on nonlinear fractional transformation (NFT) representations of systems that are a generalization of widely used linear fractional transformation (LFT) representations. Both generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ discrete-time filter design problems are investigated together with their extension to mixed designs. This study leads to new linear matrix inequality (LMI) formulations, which in turn provide an effective and reliable design tool. The proposed design technique is finally evaluated in the light of simulation examples.**

***Index Terms*—Linear fractional transformation (LFT), linear matrix inequality (LMI), nonlinear fractional transformation (NFT).**

I. Introduction

A COMMON tool to express the parameter dependence of a system is certainly the linear fractional transformation (LFT). Methods to transform many practical forms of parameter dependence into LFT representations are given in [24]. Besides, other forms of parameter dependence can be well approximated by LFT representations to be embedded into the context of linear parameter varying (LPV) control, as in [2], [14], and [15]. However, a critical issue with this transformation is the well-known "curse of dimensionality," i.e., the LFT systems have often too large dimensions in terms of parameters for practical and effective uses. One may also argue that the LFT is not the best system for representing systems with affine parameter dependence such as those of polytopic type. The so-called nonlinear fractional transformation (NFT) has been introduced in [18] for uncertain continuous-time systems to overcome these difficulties. An immediate advantage of the NFT is that it yields smaller representations that better lend themselves to numerical treatments.

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This paper investigates gain-scheduled filtering techniques for time-varying NFT system described as

$$\begin{bmatrix} x(k+1) \\ y(k) \\ z_\Delta(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\alpha(k)) & B_\Delta(\alpha(k)) & B(\alpha(k)) \\ C(\alpha(k)) & D_\Delta(\alpha(k)) & D(\alpha(k)) \\ C_\Delta(\alpha(k)) & D_{\Delta\Delta}(\alpha(k)) & D_z(\alpha(k)) \\ L(\alpha(k)) & D_{\Delta z}(\alpha(k)) & M(\alpha(k)) \end{bmatrix} \times \begin{bmatrix} x(k) \\ w_\Delta(k) \\ w(k) \end{bmatrix}$$
$$w_\Delta = \Delta(\alpha(k)) z_\Delta \tag{1}$$

where $A(\alpha(k)) \in \boldsymbol{R}^{n\times n}$, $B_\Delta(\alpha(k)) \in \boldsymbol{R}^{n\times m_\Delta}$, $B(\alpha)(k) \in \boldsymbol{R}^{n\times m}$, $D(\alpha(k)) \in \boldsymbol{R}^{p\times m}$, $C_\Delta(\alpha(k)) \in R^{m_\Delta\times n}$, $L(\alpha(k)) \in \boldsymbol{R}^{q\times n}$, and $x \in \boldsymbol{R}^n$ is the state, $y(k) \in \boldsymbol{R}^p$ is the measured output, $z(k) \in \boldsymbol{R}^q$ is the output to be estimated, and $w(k) \in R^m$ is the disturbance $w_\Delta(k) \in R^{m_\Delta}$. The extra variables $w_\Delta(k)$, $z_\Delta(k) \in R^{m_\Delta}$ are introduced to express the system nonlinear parameter dependence. The time-varying parameter $\alpha(k)$ is assumed to be gain scheduled, i.e., it is measured on line. Without loss of generality, it is allowed to vary in the unit simplex

$$\Gamma := \{(\alpha_1(k), \ldots, \alpha_s(k)) : \sum_{j=1}^{s} \alpha_j(k) = 1,\ \alpha_j(k) \geq 0\}.$$

The state-space data in (1) are assumed linear in $\alpha(k)$, i.e.,

$$\begin{bmatrix} A(\alpha(k)) & B_\Delta(\alpha(k)) & B(\alpha(k)) \\ C(\alpha(k)) & D_\Delta(\alpha(k)) & D(\alpha(k)) \\ C_\Delta(\alpha(k)) & D_{\Delta\Delta}(\alpha(k)) & D_z(\alpha(k)) \\ L(\alpha(k)) & D_{\Delta z}(\alpha(k)) & M(\alpha(k)) \\ 0 & \Delta(\alpha(k)) & 0 \end{bmatrix} = \sum_{j=1}^{s} \alpha_j(k) \begin{bmatrix} A_j & B_{\Delta j} & B_j \\ C_j & D_{\Delta j} & D_j \\ C_{\Delta j} & D_{\Delta\Delta j} & D_{zj} \\ L_j & D_{\Delta zj} & M_j \\ 0 & \Delta_j & 0 \end{bmatrix}. \tag{2}$$

The acronym NFT originates from the LFT. This can be viewed by the fact that if the slack variables $z_\Delta(k)$ and $w_\Delta(k)$ are removed from (1), then we can have the following equivalent representation of the parameter-dependent system:

$$\begin{bmatrix} x(k+1) \\ y(k) \\ z(k) \end{bmatrix} = \left(\begin{bmatrix} A(\alpha(k)) & B(\alpha(k)) \\ C(\alpha(k)) & D(\alpha(k)) \\ L(\alpha(k)) & M(\alpha(k)) \end{bmatrix} + \begin{bmatrix} B_\Delta(\alpha(k)) \\ D_\Delta(\alpha(k)) \\ D_{\Delta z}(\alpha(k)) \end{bmatrix} \Delta(\alpha(k)) \times (I - D_{\Delta\Delta}(\alpha(k))\Delta(\alpha(k)))^{-1} \times [\, C_\Delta(\alpha(k)) \quad D_z(\alpha(k)) \,] \right) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}. \tag{3}$$

One can see that (3) is highly nonlinear in the gain-scheduling parameter $\alpha(k)$ and includes well-known parameter-dependent classes as a particular case: The LFT representations correspond to the independence from $\alpha(k)$ of all system matrices except $\Delta(\alpha(k))$ in (1), whereas polytopic systems correspond to $\Delta(\alpha(k)) = 0$. As mentioned, most nonlinear parameter-dependent systems including the NFT representations (1), or its equivalence (3), can be alternatively expressed by the LFT, but as it will be seen through some simple examples, this often leads to impractical representations, whereas in stark contrast, the NFT-based ones are easily handled.

Correspondingly, the filters for estimation of the output $z(k)$ of systems (1) and (3) are also time-varying and share an NFT structure

$$\begin{bmatrix} x_F(k+1) \\ z_{\Delta F}(k) \\ z_F(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_F(\alpha(k)) & \mathbf{B}_{\Delta F}(\alpha(k)) & \mathbf{B}_F \\ \mathbf{C}_F(\alpha(k)) & \mathbf{D}_{\Delta F}(\alpha(k)) & \mathbf{D}_{yF} \\ \mathbf{L}_F(\alpha(k)) & \mathbf{D}_{zF}(\alpha(k)) & \mathbf{D}_F \end{bmatrix} \begin{bmatrix} x_F(k) \\ w_{\Delta F}(k) \\ y(k) \end{bmatrix}$$
$$w_{\Delta F}(k) = \mathbf{\Delta}_F(\alpha(k)) z_{\Delta F}(k) \tag{4}$$

where

$$\begin{bmatrix} \mathbf{A}_F(\alpha(k)) & \mathbf{B}_{\Delta F}(\alpha(k)) \\ \mathbf{C}_F(\alpha(k)) & \mathbf{D}_{\Delta F}(\alpha(k)) \\ \mathbf{L}_F(\alpha(k)) & \mathbf{D}_{zF}(\alpha(k)) \\ 0 & \mathbf{\Delta}_F(\alpha(k)) \end{bmatrix} = \sum_{j=1}^{s} \alpha_j(k) \begin{bmatrix} \mathbf{A}_{Fj} & \mathbf{B}_{\Delta Fj} \\ \mathbf{C}_{Fj} & \mathbf{D}_{\Delta Fj} \\ \mathbf{L}_{Fj} & \mathbf{D}_{zFj} \\ 0 & \mathbf{\Delta}_{Fj} \end{bmatrix} \tag{5}$$

and their sizes are the same as those of the matrices $A(\alpha(k))$, $B_\Delta(\alpha(k))$, $C_\Delta(\alpha(k))$, $D_{\Delta\Delta}(\alpha(k))$, $L(\alpha(k))$, $D_{\Delta z}(\alpha(k))$, and $\Delta(\alpha(k))$ in (1). The matrices on the right-hand side of (5) are computed offline using efficient LMI software. Then, the estimation $z_F(k)$ of the output $z(k)$ is easily updated online according to (4) and (5). Note that we can have just the parameter-independent matrices $\mathbf{B}_F,\ \mathbf{D}_{yF},\ \mathbf{D}_F$ because of the cross products of those matrices with the parameter-dependent matrices that constitute the measured output $\{y(k)\}$ of (1). This is necessary to obtain the convex formulations and will be clearly clarified from the context of Section III. The following mixed generalized $\mathcal{H}_2/\mathcal{H}_\infty$ error criterion is used to estimate $z(k)$

$$\max_{\alpha\in\Gamma}[\rho\|z - z_F\|_{pk}^2 + (1-\rho)\|z - z_F\|_2^2] \to \min \tag{6}$$

where $\|.\|_{pk}$ and $\|.\|_2$ denote the norms inducing the generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, respectively. As in [13], the generalized $\mathcal{H}_2$-norm is appropriate for handling time-domain peak errors, whereas $\mathcal{H}_\infty$ is most suitable for treating energy errors. Minimization of the peak-error and minimization of the energy-error are proved to be conflicting in [2], [8], and [13]. Therefore, a parameter $0 \le \rho \le 1$ is introduced in (6) to attain some balance between peak error and energy error constraints.

For linear time invariant (LTI) $\mathcal{H}_\infty$ filtering with different approaches, e.g., interpolation approaches, Riccati equation-based approaches and LMI-based approaches, one can refer to a variety of papers [3], [9], [11], [17], [22] and references therein. LTI $\mathcal{H}_2$ filtering has been intensively addressed in the literature; see, e.g., [5]–[7], [11], [12], and [23]. Forms of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control have been introduced in [2], [8], and [13], whereas one for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering has been used in [19], [20]. A comprehensive collection of performance criteria for control purposes is given by [10] and [13]. Some related topics such as filter order reduction and filtering for systems with stochastic uncertainties can be found in [16], [18], and [21]. Note that robust Kalman filtering exclusively detailed in [11] addresses somewhat narrower class of uncertainties via the use of Riccati equations, yielding LTI filters with the simple Luenberger observer structure. In the time-varying case, the differential Riccati equation-based approaches exhibit computational impracticality for real-time applications [10], wherein the LMI-based approaches can stay viable [2], [14]. Up to date, in the robust control literature, the LPV control for both analog and discrete uncertain systems has been considered by [2], [14], and [15]. However, the counterpart of the gain-scheduled filtering for robust control problems of NFT systems (1) remains open and very challenging.

The layout of the paper is as follows. Section II develops LMI-based norm characterizations of NFT systems, which are then used in Section III to derive new LMI-based formulations for the design of NFT filters. Validity and effectiveness of the proposed techniques are assessed via a number of numerical experiments in Section IV.

Notations in this paper are standard. Particularly, M^T is the transpose of the matrix M, whereas $M - N < 0$ ($M - N > 0$, resp.) means that $M - N$ is negative definite (positive definite, resp.) for symmetric matrices M and N. In symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for terms that are induced by symmetry, e.g.,

$$(*)\begin{bmatrix} S + (*) & * \\ M & Q \end{bmatrix} K^T \equiv K \begin{bmatrix} S + S^T & M^T \\ M & Q \end{bmatrix} K^T.$$

In addition, in long matrix inequalities involving matrix functions of the parameter $\alpha(k)$, we use, e.g.,

$$\begin{bmatrix} M_{11} & * \\ M_{12} & M_{22} \end{bmatrix}(\alpha(k)) \equiv \begin{bmatrix} M_{11}(\alpha(k)) & M_{12}^T(\alpha(k)) \\ M_{12}(\alpha(k)) & M_{22}(\alpha)(k) \end{bmatrix} \tag{7}$$

to save space. The bold capital letters such as $\mathbf{X}$, $\mathbf{K}$, $\mathbf{R}$, etc., are used to emphasize matrix variables.

II. Characterizations for Norm Constraints

In this section, we provide LMI-based analysis for different performance criteria of NFT filters. In other words, we are interested in generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the augmented system formed by (1) and (4) with the estimation error $\bar{z}(k) = z(k) - z_F(k)$ rewritten in the compact form

$$\begin{bmatrix} \bar{x}(k+1) \\ \bar{z}_\Delta(k) \\ \bar{z}(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_\Delta & \mathcal{B} \\ \mathcal{C}_\Delta & \mathcal{D}_{\Delta\Delta} & \mathcal{D}_z \\ \mathcal{L} & \mathcal{D}_{\Delta z} & \mathcal{M} \end{bmatrix}(\alpha(k)) \begin{bmatrix} \bar{x}(k) \\ \bar{w}_\Delta(k) \\ w(k) \end{bmatrix}$$
$$\bar{w}_\Delta(k) = \bar{\Delta}(\alpha(k))\bar{z}_\Delta(k) \tag{8}$$

where

$$\bar{x}(k) = \begin{bmatrix} x(k) \\ x_F(k) \end{bmatrix}, \bar{w}_\Delta(k) = \begin{bmatrix} w_\Delta(k) \\ w_{\Delta F}(k) \end{bmatrix}, \bar{z}_\Delta(k) = \begin{bmatrix} z_\Delta(k) \\ z_{\Delta F}(k) \end{bmatrix}$$
$$\bar{\Delta}(\alpha(k)) = \begin{bmatrix} \Delta(\alpha(k)) & 0 \\ 0 & \mathbf{\Delta}_F(\alpha(k)) \end{bmatrix} \tag{9}$$

and other matrices are defined accordingly [see (30) below]. With these definitions, the generalized $\mathcal{H}_2$-norm of the system (8) is defined and analyzed first.

A. Generalized $\mathcal{H}_2$-Norm Characterization

The $\mathcal{H}_2$-norm of an LTI system is strictly connected with its transfer function. For instance, the $\mathcal{H}_2$-norm of the transfer function $H(z) = C(zI - A)^{-1}B + D$ is defined as [8]

$$|H(.)|_2 = \left(\int_0^{2\pi} \text{Trace}[H^*(e^{\jmath\omega})H(e^{\jmath\omega})]d\omega\right)^{1/2}.$$

When $H(z)$ is of finite impulse response (FIR), i.e., $H(z) = \sum_{i=0}^{n} h_i z^{-i}$, this becomes the square norm $|H|_2 = (\sum_{i=0}^{n} h_i^2)^{1/2}$. Stochastically, the $\mathcal{H}_2$-norm is interpreted as the standard deviation of the output caused by the normalized white noise input. Deterministically, the $\mathcal{H}_2$-norm means the square root of the output energy under the normalized impulsive input. The generalized $\mathcal{H}_2$-norm for systems is defined as the peak of output values under normalized energy input. Clearly, these norm definitions are restricted to strictly proper continuous systems [13]. However, they are valid for both proper and strictly proper discrete-time systems. Indeed, the generalized $\mathcal{H}_2$-norm for system (8) is nothing but

$$\sup_{T, w\neq 0} \frac{\|\bar{z}(T)\|}{\left(\sum_{k=0}^{T} \|w(k)\|^2\right)^{1/2}}. \tag{10}$$

In other words, (8) is said to have the generalized $\mathcal{H}_2$-norm less than $\sqrt{\nu}$ if and only if the following relation holds for any input $w(k)$ and the corresponding output $\bar{z}(k)$:

$$\|\bar{z}(T)\|^2 \leq \nu \sum_{k=0}^{T} \|w(k)\|^2. \tag{11}$$

For continuous systems, it is obvious that $\sup_{\int_0^T \|w(t)\|^2 dt = 1} \|w(T)\|^2 = +\infty$, and therefore, the counterpart of (10) is not well-defined if there is a feed-through term in the output. That is why the generalized $\mathcal{H}_2$-norm is defined only for strictly proper continuous systems. Contrarily, it is obvious that $\sup_{\|w(0)\|^2 + \cdots + \|w(T)\|^2 = 1} \|w(T)\|^2 \leq 1$, and thus, the definition (10) is valid, whatever the class of discrete systems.

The stability, as well as the generalized $\mathcal{H}_2$-norm of (8), can be examined with the help of the Lyapunov function

$$V(\bar{x}(k)) = \bar{x}^T(k)X\bar{x}(k), \ \mathbf{X} > 0 \tag{12}$$

satisfying the two following inequalities:

$$V(\bar{x}(k+1)) - V(\bar{x}(k)) + \bar{z}_\Delta^T(k)\mathbf{R}_1(\alpha(k))\bar{z}_\Delta(k) + \bar{w}_\Delta^T(k)\mathbf{S}_1(\alpha(k))\bar{w}_\Delta(k) - \|w(k)\|^2 < 0 \tag{13}$$

$$\frac{1}{\nu}\|\bar{z}(k)\|^2 - V(\bar{x}(k)) + \bar{z}_\Delta^T(k)\mathbf{R}_2(\alpha(k))\bar{z}_\Delta(k) + \bar{w}_\Delta^T(k)\mathbf{S}_2(\alpha(k))\bar{w}_\Delta(k) - \|w(k)\|^2 < 0 \tag{14}$$

with matrices $R_i(\alpha(k)) > 0$ and $S_i(\alpha(k)) < 0$ belonging to the symmetric scaling class, which has been used in [2].

By additionally imposing

$$\bar{z}_\Delta^T(k)\mathbf{R}_i(\alpha(k))\bar{z}_\Delta(k) + \bar{w}_\Delta^T(k)\mathbf{S}_i(\alpha(k))\bar{w}_\Delta(k) \geq 0, \ i = 1, 2 \tag{15}$$

for all $\bar{w}_\Delta(k)$, $\bar{z}_\Delta(k)$ satisfying (8) (i.e., $\bar{w}_\Delta(k) = \bar{\Delta}(\alpha(k))\bar{z}_\Delta(k)$), (13) and (14) lead to

$$V(\bar{x}(T)) < \sum_{k=0}^{T-1} \|w(k)\|^2 \tag{16}$$

$$\|\bar{z}(T)\|^2 < \nu[V(\bar{x}(T)) + \|w(T)\|^2]. \tag{17}$$

Hence

$$\|\bar{z}(T)\|^2 < \nu \sum_{k=0}^{T} \|w(k)\|^2 \tag{18}$$

meaning that the generalized $\mathcal{H}_2$-norm of (8) is indeed less than $\sqrt{\nu}$.

Furthermore, in the zero input case ($w(k) = 0$), by (13) and (15)

$$V(\bar{x}(k+1)) - V(\bar{x}(k)) < 0 \tag{19}$$

which, according to Lyapunov theory, guarantees $\|\bar{x}(k)\| \to 0$ as $k \to \infty$ for any initial condition $\bar{x}(0)$, thus showing the asymptotic Lyapunov stability of (8).

To sum up, we state that (13) and (14), together with (15), guarantee that (8) is stable with the generalized $\mathcal{H}_2$-norm less than $\sqrt{\nu}$.

Meanwhile, all inequalities (13)–(15) can be readily rewritten as matrix inequalities as follows.

- As $\bar{x}(k+1)$ and $\bar{z}_\Delta(k)$ are linearly dependent on $(\bar{x}(k), \bar{w}_\Delta(k), w(k))$ by (8), the left-hand sides of (13) and (14) are easily rearranged as quadratic functions in $(\bar{x}(k), \bar{w}_\Delta(k), w(k))$. By using the Schur's complement, these quadratic functions are equivalent to the following inequalities:

$$\begin{bmatrix} -\mathbf{X} & * & * & * \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \mathbf{S}_1 & * \\ 0 & -I \end{bmatrix} & * & * \\ \mathcal{A} & [\mathcal{B}_\Delta \quad \mathcal{B}] & -\mathbf{X}^{-1} & * \\ \mathcal{C}_\Delta & [\mathcal{D}_{\Delta\Delta} \quad \mathcal{D}_z] & 0 & -\mathbf{R}_1^{-1} \end{bmatrix} (\alpha(k)) < 0 \tag{20}$$

$$\begin{bmatrix} -\mathbf{X} & * & * & * \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \mathbf{S}_2 & * \\ 0 & -I \end{bmatrix} & * & * \\ \mathcal{C}_\Delta & [\mathcal{D}_{\Delta\Delta} \quad \mathcal{D}_z] & -\mathbf{R}_2^{-1} & * \\ \mathcal{L} & [\mathcal{D}_{\Delta z} \quad \mathcal{M}] & 0 & -\nu I \end{bmatrix} (\alpha(k)) < 0. \tag{21}$$

- By substituting $\bar{w}_\Delta(k) = \bar{\Delta}(\alpha(k))\bar{z}_\Delta(k)$, the left-hand side of (15) becomes a quadratic function in $\bar{z}_\Delta(k)$, which is also equivalent to the following matrix inequality via the Schur's complement:

$$\begin{bmatrix} \mathbf{R}_i & \bar{\Delta}^T \\ \bar{\Delta} & -\mathbf{S}_i^{-1} \end{bmatrix} (\alpha(k)) > 0, \ i = 1, 2. \tag{22}$$

It is clear that inequalities (20)–(22) are not LMIs. Here, we use the linearization techniques, which have been introduced in [2] to render those inequalities linear at the expense of the insertion of some slack variables.

Theorem 1: One has (11), guaranteeing the generalized $\mathcal{H}_2$-norm of system (8) less than $\sqrt{\nu}$ if there are a symmetric matrix $\mathbf{X} > 0$, scalings $\mathbf{R}_i(\alpha(k))$, $\mathbf{S}_i(\alpha(k))$, and slack matrices $\mathbf{V}, \mathbf{H}_i, \mathbf{F}_i$ satisfying inequalities (23)–(25), shown at the bottom of the page.

Proof: The deduction from (23)–(25) to (20)–(22), respectively, can be established along the lines of [2]. □

It is worth noting the following three points concerning the conservatism of Theorem 1. First, we have to resort to the single Lyapunov function (12) since we do not make any assumption on the varying rate of the gain-scheduled parameters. If information on this rate is available, then like [1], our result can be easily modified to yield the corresponding LMI-based characterization with parameter-dependent Lyapunov functions, which in general are very efficient at handling the case of slowly varying parameters. Second, the symmetric scalings are used instead of the more general full-block scalings [15] to handle uncertainties. Based on our experience in robust control, the latter are actually not much better than the former. Moreover, the used symmetric scaling class will lead to convex formulations for the filtering problems in the next section, whereas the full-block one does not. Third, the slack matrices $\mathbf{V}$, $\mathbf{H}_i$, and $\mathbf{F}_i$ are still parameter independent. This is unavoidable in later attractive convex formulations for the filter design problems.

B. $\mathcal{H}_\infty$ -Norm Characterization

The $\mathcal{H}_\infty$ norm for system (8) is well understood as

$$\sup_{T,w\neq 0} \frac{\left(\sum_{i=0}^{T} \|\bar{z}(k)\|^2\right)^{1/2}}{\left(\sum_{i=0}^{T} \|w(k)\|^2\right)^{1/2}} = \sup_{\|w(0)\|^2+\cdots\cdots+\|w(T)\|^2=1} \left(\|\bar{z}(0)\|^2+\cdots\cdot+\|\bar{z}(T)\|^2\right)^{1/2} \tag{26}$$

i.e., (8) has the $\mathcal{H}_\infty$-norm less than $\sqrt{\gamma}$ if and only if

$$\sum_{k=0}^{T} \|\bar{z}(k)\|^2 < \gamma \sum_{k=0}^{T} \|w(k)\|^2 \quad \forall \bar{z}, w. \tag{27}$$

Paralleling the results in [2] leads to the following LMI characterization.

Theorem 2: One has (27), guaranteeing the $\mathcal{H}_\infty$-norm of system (8) less than $\sqrt{\gamma}$ if there exist $\mathbf{Y} > 0$ and $\mathbf{R}(\alpha(k))$, $\mathbf{S}(\alpha(k))$, $\mathbf{V}, \mathbf{H}, \mathbf{F}$ satisfying (28) and (29), shown at the bottom of the page. □

$$\begin{bmatrix} -\mathbf{X} & * & * & * \\ \begin{bmatrix}0\\0\end{bmatrix} & \begin{bmatrix}\mathbf{S}_1 & * \\ 0 & -I\end{bmatrix} & * & * \\ \mathbf{V}^T\mathcal{A} & \mathbf{V}^T[\mathcal{B}_\Delta \quad \mathcal{B}] & \mathbf{X}-(\mathbf{V}+\mathbf{V}^T) & * \\ \mathbf{F}_1\mathcal{C}_\Delta & \mathbf{F}_1[\mathcal{D}_{\Delta\Delta} \quad \mathcal{D}_z] & 0 & \mathbf{R}_1-(\mathbf{F}_1+\mathbf{F}_1^T) \end{bmatrix}(\alpha(k)) < 0 \quad \forall \alpha(k)\in\Gamma \tag{23}$$

$$\begin{bmatrix} -\mathbf{X} & * & * & * \\ \begin{bmatrix}0\\0\end{bmatrix} & \begin{bmatrix}\mathbf{S}_2 & * \\ 0 & -I\end{bmatrix} & * & * \\ \mathbf{F}_2\mathcal{C}_\Delta & \mathbf{F}_2[\mathcal{D}_{\Delta\Delta} \quad \mathcal{D}_z] & \mathbf{R}_2-(\mathbf{F}_2+\mathbf{F}_2^T) & * \\ \mathcal{L} & [\mathcal{D}_{\Delta z} \quad \mathcal{M}] & 0 & -\nu I \end{bmatrix}(\alpha(k)) < 0 \quad \forall \alpha(k)\in\Gamma \tag{24}$$

$$\begin{bmatrix} \mathbf{R}_i & \bar{\Delta}^T\mathbf{H}_i^T \\ \mathbf{H}_i\bar{\Delta} & \mathbf{S}_i+(\mathbf{H}_i+\mathbf{H}_i^T) \end{bmatrix}(\alpha(k)) \geq 0,\ \forall\alpha(k)\in\Gamma,\ i=1,2. \tag{25}$$

$$\begin{bmatrix} -\mathbf{Y} & * & * & * & * \\ \begin{bmatrix}0\\0\end{bmatrix} & \begin{bmatrix}\mathbf{S} & * \\ 0 & -I\end{bmatrix} & * & * & * \\ \mathbf{V}^T\mathcal{A} & \mathbf{V}^T[\mathcal{B}_\Delta \quad \mathcal{B}] & \mathbf{Y}-(\mathbf{V}+\mathbf{V}^T) & * & * \\ \mathbf{F}\mathcal{C}_\Delta & \mathbf{F}[\mathcal{D}_{\Delta\Delta} \quad \mathcal{D}_z] & 0 & \mathbf{R}-(\mathbf{F}+\mathbf{F}^T) & * \\ \mathcal{L} & [\mathcal{D}_{\Delta z} \quad \mathcal{M}] & 0 & 0 & -\gamma I \end{bmatrix}(\alpha(k)) < 0 \quad \forall \alpha(k)\in\Gamma \tag{28}$$

$$\begin{bmatrix} \mathbf{R} & \bar{\Delta}^T\mathbf{H}^T \\ \mathbf{H}\bar{\Delta} & \mathbf{S}+(\mathbf{H}+\mathbf{H}^T) \end{bmatrix}(\alpha(k)) \geq 0, \forall\alpha(k)\in\Gamma. \tag{29}$$

III. NFT Filter Designs

Returning to the design problem for NFT filters (4), an important preparation step is to write the matrices of system (8) in the following convenient forms:

$$
\begin{aligned}
&[\mathcal{A} \quad [\mathcal{B}_\Delta \quad \mathcal{B}]](\alpha(k)) \\
&\quad = \sum_{j=1}^{s} \alpha_j(k)([A_{aj} \quad B_{aj}] + \mathcal{I}_0 \mathbf{K}_{1j}[C_{aj} \quad D_{aj}]) \\
&[\mathcal{C}_\Delta \quad [\mathcal{D}_{\Delta\Delta} \quad \mathcal{D}_z]](\alpha(k)) \\
&\quad = \sum_{j=1}^{s} \alpha_j(k)([\mathcal{C}_{0j} \quad \mathcal{D}_{0j}] + \tilde{\mathcal{I}}_0 K_{2j}[C_{aj} \quad D_{aj}]) \\
&[\mathcal{L} \quad [\mathcal{D}_{\Delta z} \quad \mathcal{M}]](\alpha(k)) \\
&\quad = \sum_{j=1}^{s} \alpha_j(k)([L_{0j} \quad [D_{\Delta z0j} \quad M_j]] - \mathbf{K}_{3j}[C_{aj} \quad D_{aj}]
\end{aligned} \tag{30}
$$

with

$$
\begin{aligned}
&A_{aj} = \begin{bmatrix} A_j & 0_n \\ 0_n & 0_n \end{bmatrix}, \mathcal{I}_0 = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}, \tilde{\mathcal{I}}_0 = \begin{bmatrix} 0_{m_\Delta} \\ I_{m_\Delta} \end{bmatrix}, C_{aj} = \begin{bmatrix} 0_n & I_n \\ 0_{m_\Delta n} & 0_{m_\Delta n} \\ C_j & 0_{pn} \end{bmatrix} \\
&B_{aj} = \begin{bmatrix} B_{\Delta j} & 0_{nm_\Delta} & B_j \\ 0_{nm_\Delta} & 0_{nm_\Delta} & 0_{nm} \end{bmatrix}, D_{aj} = \begin{bmatrix} 0_{nm_\Delta} & 0_{nm_\Delta} & 0_{nm} \\ 0_{m_\Delta} & I_{m_\Delta} & 0_{m_\Delta m} \\ D_{\Delta j} & 0_{pm_\Delta} & D_j \end{bmatrix} \\
&\mathcal{C}_{0j} = \begin{bmatrix} C_{\Delta j} & 0_{m_\Delta n} \\ 0_{m_\Delta n} & 0_{m_\Delta n} \end{bmatrix}, \mathcal{D}_{0j} = \begin{bmatrix} D_{\Delta\Delta j} & 0_{m_\Delta} & D_{zj} \\ 0_{m_\Delta} & 0_{m_\Delta} & 0_{m_\Delta m} \end{bmatrix} \\
&L_{0j} = [L_j \quad 0_{qn}], D_{\Delta z0j} = [D_{\Delta zj} \quad 0_{qm_\Delta}]
\end{aligned} \tag{31}
$$

and the variables

$$
\begin{aligned}
\mathbf{K}_{1j} &= [\mathbf{A}_{Fj} \quad \mathbf{B}_{\Delta Fj} \quad \mathbf{B}_F] \\
\mathbf{K}_{2j} &= [\mathbf{C}_{Fj} \quad \mathbf{D}_{\Delta Fj} \quad \mathbf{D}_{yF}] \\
\mathbf{K}_{3j} &= [\mathbf{L}_{Fj} \quad \mathbf{D}_{zFj} \quad \mathbf{D}_F].
\end{aligned} \tag{32}
$$

The next subsection considers the case of the generalized $\mathcal{H}_2$ filter design.

A. NFT Generalized $\mathcal{H}_2$ Filter Design

All inequalities (23), (24), and (28), which characterize the existence of a filter (4), are not LMIs in the variables $\mathbf{X}$, $\mathbf{V}$, $\mathbf{F}_i$, $\mathbf{R}_i(\alpha(k))$, $\mathbf{S}_i(\alpha(k))$, $\mathbf{K}_{ij}$, and $\mathbf{\Delta}_{Fj}$. To translate them into LMIs, we choose the following linearly parameter-dependent class of scaling matrices $\mathbf{R}_i(\alpha(k))$, $\mathbf{S}_i(\alpha(k))$:

$$
\mathbf{R}_i(\alpha(k)) = \sum_{j=1}^{s} \alpha_j(k) \mathbf{R}_{ij}, \ \mathbf{S}_i(\alpha(k)) = \sum_{j=1}^{s} \alpha_j(k) \mathbf{S}_{ij}. \tag{33}
$$

It follows that (23)–(25) are rewritten as (34)–(36), shown at the bottom of the page, where

$$
\begin{aligned}
\mathbf{R}_{ij} &= \begin{bmatrix} \mathbf{R}_{11,ij} & \mathbf{R}_{12,ij} \\ \mathbf{R}_{12,ij}^T & \mathbf{R}_{22,ij} \end{bmatrix}, \ i = 1,2; \ j = 1,2,\ldots,s \\
\mathbf{H} &= \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix}.
\end{aligned} \tag{37}
$$

A careful analysis of inequalities (34)–(37) reveals the following bilinear terms involving the filter variables $\mathbf{K}_{1j}$, $\mathbf{K}_{2j}$, $\mathbf{\Delta}_{Fj}$, scaling variables $\mathbf{F}$, $\mathbf{H}$. and slack variable $\mathbf{V}$:

$$
\mathbf{V}^T \mathcal{I}_0 \mathbf{K}_{1j} = \begin{bmatrix} \mathbf{V}_{21}^T \\ \mathbf{V}_{22}^T \end{bmatrix} \mathbf{K}_{1j}, \ \mathbf{F}\tilde{\mathcal{I}}_0 \mathbf{K}_{2j} = \begin{bmatrix} \mathbf{F}_{12} \\ \mathbf{F}_{22} \end{bmatrix} \mathbf{K}_{2j}, \begin{bmatrix} \mathbf{H}_{21} \\ \mathbf{H}_{22} \end{bmatrix} \mathbf{\Delta}_{Fj}.
$$

We resort to the structure of matrices in (31) and (32) to linearize these terms according to the following steps.

- With the partitioning

$$
\mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix}, \mathbf{V}_{ij} \in R^{n \times n} \tag{38}
$$

defining

$$
\Pi_{\mathbf{V}} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \tag{39}
$$

as well as

$$
\Pi_{\mathbf{H}} = \begin{bmatrix} I & 0 \\ 0 & (\mathbf{H}_{12}\mathbf{H}_{22}^{-1})^T \end{bmatrix}, \ \Pi_{\mathbf{F}} = \begin{bmatrix} I & 0 \\ 0 & (\mathbf{F}_{12}\mathbf{F}_{22}^{-1})^T \end{bmatrix}.
$$

$$
\begin{bmatrix}
-\mathbf{X} & * & * & * \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \mathbf{S}_{1j} & * \\ 0 & -I \end{bmatrix} & * & * \\
\mathbf{V}^T(A_{aj} + \mathcal{I}_0 \mathbf{K}_{1j} C_{aj}) & \mathbf{V}^T(B_{aj} + \mathcal{I}_0 \mathbf{K}_{1j} D_{aj}) & \mathbf{X} - (\mathbf{V} + \mathbf{V}^T) & * \\
\mathbf{F}(\mathcal{C}_{0j} + \tilde{\mathcal{I}}_0 \mathbf{K}_{2j} C_{aj}) & \mathbf{F}(\mathcal{D}_{0j} + \tilde{\mathcal{I}}_0 \mathbf{K}_{2j} D_{aj}) & 0 & \mathbf{R}_{1j} - (\mathbf{F} + \mathbf{F}^T)
\end{bmatrix} < 0 \tag{34}
$$

$$
\begin{bmatrix}
-\mathbf{X} & * & * & * \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \mathbf{S}_{2j} & * \\ 0 & -I \end{bmatrix} & * & * \\
\mathbf{F}(\mathcal{C}_{0j} + \tilde{\mathcal{I}}_0 \mathbf{K}_{2j} C_{aj}) & \mathbf{F}(\mathcal{D}_{0j} + \tilde{\mathcal{I}}_0 \mathbf{K}_{2j} D_{aj}) & \mathbf{R}_{2j} - (\mathbf{F} + \mathbf{F}^T) & * \\
L_{0j} - \mathbf{K}_{3j} C_{aj} & [D_{\Delta z0j} \quad M_j] - \mathbf{K}_{3j} D_{aj} & 0 & -\nu I
\end{bmatrix} < 0 \tag{35}
$$

$$
\begin{bmatrix}
\mathbf{R}_{ij} & * \\
\mathbf{H} \begin{bmatrix} \Delta_j & 0 \\ 0 & \mathbf{\Delta}_{Fj} \end{bmatrix} & \mathbf{S}_{ij} + \mathbf{H} + \mathbf{H}^T
\end{bmatrix} \geq 0
$$

$$
i = 1,2; \ j = 1,2,\ldots,s \tag{36}
$$

- Define the new variables

$$\hat{\mathbf{V}} = \begin{bmatrix} \hat{\mathbf{V}}_1 & \hat{\mathbf{V}}_2 \\ \hat{\mathbf{V}}_3 & \hat{\mathbf{V}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21} \\ \mathbf{V}_{21}^T\mathbf{V}_{22}^{-T}\mathbf{V}_{21} & \mathbf{V}_{21}^T\mathbf{V}_{22}^{-T}\mathbf{V}_{21} \end{bmatrix} = \Pi_{\mathbf{V}}^T \mathbf{V} \Pi_{\mathbf{V}} \tag{40}$$

and

$$\begin{aligned} \hat{\mathbf{X}} &= \Pi_{\mathbf{V}}^T \mathbf{X} \Pi_{\mathbf{V}} \\ \hat{\mathbf{F}} &= \begin{bmatrix} \hat{\mathbf{F}}_1 & \hat{\mathbf{F}}_2 \\ \hat{\mathbf{F}}_3 & \hat{\mathbf{F}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12}\mathbf{F}_{22}^{-T}\mathbf{F}_{12}^T \\ \mathbf{F}_{12}\mathbf{F}_{22}^{-1}\mathbf{F}_{21} & \mathbf{F}_{12}\mathbf{F}_{22}^{-T}\mathbf{F}_{12}^T \end{bmatrix} = \Pi_{\mathbf{F}}^T \mathbf{F} \Pi_{\mathbf{F}} \end{aligned} \tag{41}$$

$$\begin{aligned} \hat{\mathbf{K}}_{1j} &= [\hat{\mathbf{A}}_{Fj} \quad \hat{\mathbf{B}}_{\Delta Fj} \quad \hat{\mathbf{B}}_F] \\ &= [\mathbf{V}_{21}^T \mathbf{A}_{Fj} \mathbf{V}_{22}^{-1} \mathbf{V}_{21} \quad \mathbf{V}_{21}^T \mathbf{B}_{\Delta Fj} \mathbf{H}_{22}^{-T} \mathbf{H}_{12}^T \quad \mathbf{V}_{21}^T \mathbf{B}_F] \\ \hat{\mathbf{K}}_{2j} &= [\,\hat{\mathbf{C}}_{Fj} \quad \hat{\mathbf{D}}_{\Delta Fj} \quad \hat{\mathbf{D}}_{yF}\,] \\ &= [\,\mathbf{F}_{12}\mathbf{C}_{Fj}\mathbf{V}_{22}^{-1}\mathbf{V}_{21} \quad \mathbf{F}_{12}\mathbf{D}_{\Delta Fj}\mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T \quad \mathbf{F}_{12}\mathbf{D}_{yF}\,] \\ \hat{\mathbf{K}}_{3j} &= [\,\hat{\mathbf{L}}_{Fj} \quad \hat{\mathbf{D}}_{zFj} \quad \hat{\mathbf{D}}_F\,] \\ &= [\,\mathbf{L}_{Fj}\mathbf{V}_{22}^{-1}\mathbf{V}_{21} \quad \mathbf{D}_{zFj}\mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T \quad \mathbf{D}_F\,] \end{aligned} \tag{42}$$

$$\begin{aligned} \hat{\mathbf{R}}_{ij} &= \Pi_{\mathbf{F}}^T \mathbf{R}_{ij} \Pi_{\mathbf{F}}, \hat{\mathbf{S}}_{ij} = \Pi_{\mathbf{H}}^T \mathbf{S}_{ij} \Pi_{\mathbf{H}} \\ \hat{\mathbf{H}} &= \begin{bmatrix} \hat{\mathbf{H}}_1 & \hat{\mathbf{H}}_2 \\ \hat{\mathbf{H}}_3 & \hat{\mathbf{H}}_2 \end{bmatrix} = \Pi_{\mathbf{H}}^T \mathbf{H} \Pi_{\mathbf{H}} \\ \hat{\boldsymbol{\Delta}}_{Fj} &= \mathbf{H}_{21}^T \boldsymbol{\Delta}_{Fj} \mathbf{F}_{22}^{-T} \mathbf{F}_{12}^T, \; j = 1, 2, \ldots, s. \end{aligned} \tag{43}$$

- Apply the congruence transformations

$$\begin{aligned} &\text{diag}\left[\Pi_{\mathbf{V}} \quad \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} \quad \Pi_{\mathbf{V}} \quad \Pi_{\mathbf{F}}\right] \\ &\text{diag}\left[\Pi_{\mathbf{V}} \quad \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} \quad \Pi_{\mathbf{F}} \quad I\right] \\ &\text{diag}\,[\Pi_{\mathbf{F}} \quad \Pi_{\mathbf{H}}] \end{aligned}$$

to (34)–(36), respectively.

- With the help of the structured matrices in (30) and (31), bring into play the following identities, whose lengthy algebraic verification is provided in the Appendix :

$$\begin{aligned} \Pi_{\mathbf{V}}^T \mathbf{V}^T A_{aj} \Pi_{\mathbf{V}} &= \hat{\mathbf{V}}^T A_{aj} \\ \Pi_{\mathbf{V}}^T \mathbf{V}^T \mathcal{I}_0 \mathbf{K}_{1j} C_{aj} \Pi_{\mathbf{V}} &= \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{1j} C_{aj} \\ \Pi_{\mathbf{V}}^T \mathbf{V}^T B_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} &= \hat{\mathbf{V}}^T B_{aj} \\ \Pi_{\mathbf{V}}^T \mathbf{V}^T \mathcal{I}_0 \mathbf{K}_{1j} D_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{1j} D_{aj} \\ \Pi_{\mathbf{F}}^T \mathbf{F} \mathcal{C}_{0j} \Pi_{\mathbf{V}} &= \hat{\mathbf{F}} \mathcal{C}_{0j} \\ \Pi_{\mathbf{F}}^T \mathbf{F} \tilde{\mathcal{I}}_0 \mathbf{K}_{2j} C_{aj} \Pi_{\mathbf{V}} &= \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{2j} C_{aj} \\ \Pi_{\mathbf{F}}^T \mathbf{F} \mathcal{D}_{0j} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} &= \hat{\mathbf{F}} \mathcal{D}_{0j} \\ \Pi_{\mathbf{F}}^T \mathbf{F} \tilde{\mathcal{I}}_0 \mathbf{K}_{2j} D_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{2j} D_{aj} \\ L_{0j} \Pi_{\mathbf{V}} &= L_{0j} \\ \mathbf{K}_{3j} C_{aj} \Pi_{\mathbf{V}} &= \hat{\mathbf{K}}_{3j} C_{aj} \\ D_{\Delta z0j} \Pi_{\mathbf{H}} &= D_{\Delta z0j} \\ \mathbf{K}_{3j} D_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} &= \hat{\mathbf{K}}_{3j} D_{aj}. \end{aligned} \tag{44}$$

As a result, the nonlinear matrix inequalities (34)–(36) are translated into the following LMIs with respect to the newly introduced variables $\hat{\mathbf{X}}$, $\hat{\mathbf{V}}$, $\hat{\mathbf{R}}_{1j}$, $\hat{\mathbf{R}}_{2j}$, $\hat{\mathbf{S}}_{1j}$, $\hat{\mathbf{S}}_{2j}$, $\hat{\mathbf{H}}$, $\hat{\mathbf{F}}$, $\hat{\boldsymbol{\Delta}}_F$, and $\hat{\mathbf{K}}_{1j}$, $\hat{\mathbf{K}}_{2j}$, $\hat{\mathbf{K}}_{3j}$, shown in (45)–(47) at the bottom of the page.

We recap these results in the next theorem.

Theorem 3: There is an NFT filter (4) that makes the estimation (11) fulfilled if LMI's (45)–(47) are feasible in $\hat{\mathbf{X}}$, $\hat{\mathbf{V}}$, $\hat{\mathbf{R}}_{ij}$, $\hat{\mathbf{S}}_{ij}$, $\hat{\mathbf{H}}$, $\hat{\mathbf{F}}$, $\hat{\boldsymbol{\Delta}}_F$, $\hat{\mathbf{K}}_{1j}$, $\hat{\mathbf{K}}_{2j}$, $\hat{\mathbf{K}}_{3j}$. The matrix data defining the

$$\begin{bmatrix} -\hat{\mathbf{X}} & * & * & * \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \hat{\mathbf{S}}_{1j} & * \\ 0 & -I \end{bmatrix} & * & * \\ \hat{\mathbf{V}}^T A_{aj} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{1j} C_{aj} & \hat{\mathbf{V}}^T B_{aj} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{1j} D_{aj} & \hat{\mathbf{X}} - (\hat{\mathbf{V}} + \hat{\mathbf{V}}^T) & * \\ \hat{\mathbf{F}} \mathcal{C}_{0j} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{2j} C_{aj} & \hat{\mathbf{F}} \mathcal{D}_{0j} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{2j} D_{aj} & 0 & \hat{\mathbf{R}}_{1j} - (\hat{\mathbf{F}} + \hat{\mathbf{F}}^T) \end{bmatrix} < 0 \tag{45}$$

$$\begin{bmatrix} -\hat{\mathbf{X}} & * & * & * \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \hat{\mathbf{S}}_{2j} & * \\ 0 & -I \end{bmatrix} & * & * \\ \hat{\mathbf{F}} \mathcal{C}_{0j} + \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{2j} C_{aj} & \hat{\mathbf{F}} \mathcal{D}_{0j} + \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{2j} D_{aj} & \hat{\mathbf{R}}_{2j} - (\hat{\mathbf{F}} + \hat{\mathbf{F}}^T) & * \\ L_{0j} - \hat{\mathbf{K}}_{3j} C_{aj} & [\,D_{\Delta z0j} \quad M_j\,] - \hat{\mathbf{K}}_{3j} D_{aj} & 0 & -\nu I \end{bmatrix} < 0 \tag{46}$$

$$\begin{bmatrix} \hat{\mathbf{R}}_{ij} & * \\ \begin{bmatrix} \hat{\mathbf{H}}_1 \Delta_j & \hat{\boldsymbol{\Delta}}_{Fj} \\ \hat{\mathbf{H}}_3 \Delta_j & \hat{\boldsymbol{\Delta}}_{Fj} \end{bmatrix} & \hat{\mathbf{S}}_{ij} + \hat{\mathbf{H}} + \hat{\mathbf{H}}^T \end{bmatrix} \geq 0, \; i = 1, 2 \qquad j = 1, 2, \ldots, s. \tag{47}$$

filter (4) can be derived from a solution to (45)–(47) through the formulas

$$\begin{aligned}[\mathbf{A}_{Fj} \quad \mathbf{B}_{\Delta Fj} \quad \mathbf{B}_F] &= [\hat{\mathbf{A}}_{Fj}\hat{\mathbf{V}}_3^{-T} \quad \hat{\mathbf{B}}_{\Delta Fj}\hat{\mathbf{H}}_2^{-1} \quad \hat{\mathbf{B}}_F]\\ [\mathbf{C}_{Fj} \quad \mathbf{D}_{\Delta Fj} \quad \mathbf{D}_{yF}] &= [\hat{\mathbf{C}}_{Fj}\hat{\mathbf{V}}_3^{-T} \quad \hat{\mathbf{D}}_{\Delta Fj}\hat{\mathbf{H}}_2^{-1} \quad \hat{\mathbf{D}}_{yF}]\\ [\mathbf{L}_{Fj} \quad \mathbf{D}_{zFj} \quad \mathbf{D}_F] &= [\hat{\mathbf{L}}_{Fj}\hat{\mathbf{V}}_3^{-T} \quad \hat{\mathbf{D}}_{zFj}\hat{\mathbf{H}}_2^{-1} \quad \hat{\mathbf{D}}_F]\\ \mathbf{\Delta}_{Fj} &= \hat{\mathbf{\Delta}}_{Fj}\hat{\mathbf{F}}_2^{-1}. \end{aligned} \tag{48}$$

Proof: For given matrices $\hat{\mathbf{V}}$, $\hat{\mathbf{F}}$, and $\hat{\mathbf{H}}$, matrices $\mathbf{V}$, $\mathbf{F}$, and $\mathbf{H}$ satisfying (40), (41), and (43) are

$$\mathbf{V} = \begin{bmatrix} \hat{\mathbf{V}}_1 & \hat{\mathbf{V}}_2\hat{\mathbf{V}}_3^{-T} \\ I & \hat{\mathbf{V}}_3^{-T} \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} \hat{\mathbf{F}}_1 & I \\ \hat{\mathbf{F}}_2^{-T}\hat{\mathbf{F}}_3 & \hat{\mathbf{F}}_2^{-T} \end{bmatrix}$$
$$\mathbf{H} = \begin{bmatrix} \hat{\mathbf{H}}_1 & I \\ \hat{\mathbf{H}}_2^{-T}\hat{\mathbf{H}}_3 & \hat{\mathbf{H}}_2^{-T} \end{bmatrix}. \tag{49}$$

Substituting these values of $\mathbf{V}$, $\mathbf{F}$, and $\mathbf{H}$ into (41)–(43) and then (48) can be easily obtained by restoring $\mathbf{A}_{Fj}$, $\mathbf{B}_{\Delta Fj}$, $\mathbf{B}_F$, $\mathbf{C}_{Fj}$, $\mathbf{D}_{\Delta Fj}$, $\mathbf{D}_{yF}$, $\mathbf{L}_{Fj}$, $\mathbf{D}_{zFj}$, $\mathbf{D}_F$, and $\mathbf{\Delta}_{Fj}$.

B. NFT $\mathcal{H}_\infty$ and Mixed Generalized $\mathcal{H}_2/\mathcal{H}_\infty$ Filter Designs

By similar arguments, an LMI characterization for the existence of $\mathcal{H}_\infty$ filters is as follows.

Theorem 4: There is an NFT filter (4) that makes the estimation (27) fulfilled if LMIs (50) and (51), shown at the bottom of the page, are feasible in $\hat{\mathbf{Y}}$, $\hat{\mathbf{V}}$, $\hat{\mathbf{R}}_j$, $\hat{\mathbf{S}}_j$, $\hat{\mathbf{H}}$, $\hat{\mathbf{F}}$, $\hat{\mathbf{\Delta}}_F$, $\hat{\mathbf{K}}_{1j}$, $\hat{\mathbf{K}}_{2j}$, and $\hat{\mathbf{K}}_{3j}$, as in (50) and (51). The matrix data defining the filter (4) can be derived from a solution to (50) and (51) according to (48).

Consequently, the mixed generalized $\mathcal{H}_2/\mathcal{H}_\infty$ filter design is merely the combination of Theorems 3 and 4.

Theorem 5: A suboptimal filter (4) for (6) can be obtained from the solution to the optimization problem in (52), shown at the bottom of the page, according to (48).

C. Particular Cases: LFT and LPV Filters

It is interesting to gain insight into the linearizing transforms (42) and (43) by considering two special cases.

If we impose $\mathbf{A}_{Fj} = \mathbf{A}_F$, $\mathbf{B}_{\Delta Fj} = \mathbf{B}_{\Delta F}$, $\mathbf{C}_{Fj} = \mathbf{C}_F$, $\mathbf{D}_{\Delta Fj} = \mathbf{D}_{\Delta F}$, $\mathbf{L}_{Fj} = \mathbf{L}_F$, and $\mathbf{D}_{zFj} = \mathbf{D}_{zF}$; $j = 1, 2, \ldots, s$ in (4), i.e., we obtain the LFT filter

$$\begin{bmatrix} x_F(k+1) \\ z_{\Delta F}(k) \\ z_F(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_F & \mathbf{B}_{\Delta F} & \mathbf{B}_F \\ \mathbf{C}_F & \mathbf{D}_{\Delta F} & \mathbf{D}_{yF} \\ \mathbf{L}_F & \mathbf{D}_{zF} & \mathbf{D}_F \end{bmatrix} \begin{bmatrix} x_F(k) \\ w_{\Delta F}(k) \\ y(k) \end{bmatrix}$$
$$w_{\Delta F}(k) = \sum_{j=1}^{s} \alpha_j(k)\mathbf{\Delta}_{Fj} z_{\Delta F}(k), \tag{53}$$

then the corresponding linearizing transform of the filter matrices is

$$\begin{aligned} \hat{\mathbf{K}}_{1j} &= \hat{\mathbf{K}}_1 \\ &= [\mathbf{V}_{21}^T\mathbf{A}_F\mathbf{V}_{22}^{-1}\mathbf{V}_{21} \quad \mathbf{V}_{21}^T\mathbf{B}_{\Delta F}\mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T \quad \mathbf{V}_{21}^T\mathbf{B}_F] \\ \hat{\mathbf{K}}_{2j} &= \hat{\mathbf{K}}_2 \\ &= [\mathbf{F}_{12}\mathbf{C}_F\mathbf{V}_{22}^{-1}\mathbf{V}_{21} \quad \mathbf{F}_{12}\mathbf{D}_{\Delta F}\mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T \quad \mathbf{F}_{12}\mathbf{D}_{yF}] \\ \hat{\mathbf{K}}_{3j} &= \hat{\mathbf{K}}_3 = [\mathbf{L}_F\mathbf{V}_{22}^{-1}\mathbf{V}_{21} \quad \mathbf{D}_{zF}\mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T \quad \mathbf{D}_F] \\ j &= 1, 2, \ldots, s \end{aligned} \tag{54}$$

and (45), (46), and (50) are obviously still LMIs in $\hat{\mathbf{K}}_i$, $i = 1, 2, 3$.

On the other hand, if $\Delta_j \equiv 0$, i.e., (1) becomes the usual linear parameter varying (LPV) system

$$\begin{bmatrix} x(k+1) \\ y(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\alpha(k)) & B(\alpha(k)) \\ C(\alpha(k)) & D(\alpha(k)) \\ L(\alpha(k)) & M(\alpha(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{55}$$

then accordingly, $\mathbf{\Delta}_{Fj} \equiv 0$. In other words, (4) is down to the LPV filter

$$\begin{bmatrix} x_F(k+1) \\ z_F(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_F(\alpha(k)) & \mathbf{B}_F \\ \mathbf{L}_F(\alpha(k)) & \mathbf{D}_F \end{bmatrix} \begin{bmatrix} x_F(k) \\ y(k) \end{bmatrix}. \tag{56}$$

In this case, the scaling variables $\mathbf{S}_{ij}$, $\mathbf{R}_{ij}$ (in Theorem 3), $\mathbf{S}_j$, $\mathbf{R}_j$ (in Theorem 4), and their related elements are no longer necessary. As a result, LMIs (47) and (51) in Theorem 3 and 4 disappear, whereas LMIs (45), (46), and (50) are reduced to

$$\begin{bmatrix} -\hat{\mathbf{Y}} & * & * & * & * \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \hat{\mathbf{S}}_j & * \\ 0 & -I \end{bmatrix} & * & * & * \\ \hat{\mathbf{V}}^T A_{aj} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{1j} C_{aj} & \hat{\mathbf{V}}^T B_{aj} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{1j} D_{aj} & \hat{\mathbf{Y}} - (\hat{\mathbf{V}} + \hat{\mathbf{V}}^T) & * & * \\ \hat{\mathbf{F}}\mathcal{C}_{0j} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{2j} C_{aj} & \hat{\mathbf{F}}\mathcal{D}_{0j} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{2j} D_{aj} & 0 & \hat{\mathbf{R}}_j - (\hat{\mathbf{F}} + \hat{\mathbf{F}}^T) & * \\ L_{0j} - \hat{\mathbf{K}}_{3j} C_{aj} & [D_{\Delta z0j} \quad M_j] - \hat{\mathbf{K}}_{3j} D_{aj} & 0 & 0 & -\gamma I \end{bmatrix} < 0 \tag{50}$$

$$\begin{bmatrix} \hat{\mathbf{R}}_j & * \\ \begin{bmatrix} \hat{\mathbf{H}}_1\Delta_j & \hat{\mathbf{\Delta}}_{Fj} \\ \hat{\mathbf{H}}_3\Delta_j & \hat{\mathbf{\Delta}}_{Fj} \end{bmatrix} & \hat{\mathbf{S}}_j + \hat{\mathbf{H}} + \hat{\mathbf{H}}^T \end{bmatrix} \geq 0$$
$$j = 1, 2, \ldots, s. \tag{51}$$

$$\min_{\mathbf{X}, \mathbf{Y}, \hat{\mathbf{K}}_{ij}, \hat{\mathbf{V}}, \hat{\mathbf{R}}_{ij}, \hat{\mathbf{S}}_{ij}, \hat{\mathbf{R}}_j, \hat{\mathbf{S}}_j, \hat{\mathbf{H}}, \hat{\mathbf{F}}, \hat{\mathbf{\Delta}}_F, \nu, \gamma} [\rho\nu + (1-\rho)\gamma] : \text{(45)-(47), (50), (51)} \tag{52}$$

(57)–(59), shown at the bottom of the page, with the redefinitions

$$A_{aj} = \begin{bmatrix} A_j & 0_n \\ 0_n & 0_n \end{bmatrix}, \; C_{aj} = \begin{bmatrix} 0_n & I_n \\ C_j & 0_{pn} \end{bmatrix}$$
$$B_{aj} = \begin{bmatrix} B_j \\ 0_{nm} \end{bmatrix}, D_{aj} = \begin{bmatrix} 0_{nm} \\ D_j \end{bmatrix} \tag{60}$$
$$\hat{\mathbf{K}}_{1j} = [\,\hat{\mathbf{A}}_{Fj} \quad \hat{\mathbf{B}}_F\,], \hat{\mathbf{K}}_{3j} = [\,\hat{\mathbf{L}}_{Fj} \quad \mathbf{D}_F\,] \tag{61}$$
$$[\,\mathbf{A}_{Fj} \quad \mathbf{B}_F] = [\,\hat{\mathbf{A}}_{Fj}\mathbf{V}_3^{-T} \quad \hat{\mathbf{B}}_F\,]$$
$$[\,\mathbf{L}_{Fj} \quad \mathbf{D}_F] = [\,\hat{\mathbf{L}}_{Fj}\hat{\mathbf{V}}_3^{-T} \quad \mathbf{D}_F\,]. \tag{62}$$

Note that LMI formulations for the (nonadjustable) LTI filter

$$\begin{bmatrix} x_F(k+1) \\ z_F(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_F & \mathbf{B}_F \\ \mathbf{L}_F & \mathbf{D}_F \end{bmatrix} \begin{bmatrix} x_F(k) \\ y(k) \end{bmatrix} \tag{63}$$

follow from LMIs (57)–(59) by the restriction

$$\hat{\mathbf{K}}_{1j} = \hat{\mathbf{K}}_1, \; \hat{\mathbf{K}}_{3j} = \hat{\mathbf{K}}_3, \; j = 1, 2, \ldots, s. \tag{64}$$

IV. NUMERICAL EXAMPLE

The power and flexibility of the proposed approach are demonstrated through the following example:

$$\begin{bmatrix} x(k+1) \\ y(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\alpha(k)) & B \\ C & D \\ L & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{65}$$

where

$$\begin{aligned} A(\alpha(k)) &= Q_0 + \alpha_1^3(k)Q_1 + \alpha_2^3(k)Q_2 + \alpha_1(k)\alpha_2^2(k)Q_3 \\ &\quad + \alpha_1(k)Q_4 + \alpha_2(k)Q_5 \end{aligned}$$
$$Q_0 = \begin{bmatrix} -0.3 & -0.6 \\ 0.4 & -0.1 \end{bmatrix}, Q_1 = \begin{bmatrix} -0.2 & 0.05 \\ 0.1 & 0.08 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.1 & 0.08 \\ 0.2 & 0.15 \end{bmatrix}$$
$$Q_3 = \begin{bmatrix} -0.3 & 0.1 \\ 0.15 & 0.05 \end{bmatrix}, Q_4 = \begin{bmatrix} 0.1 & 0.2 \\ 0.05 & 0.1 \end{bmatrix}, Q_5 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}$$
$$B = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, C = [-10 \quad 10], D = [0 \quad 1.5], L = [1 \quad 0]. \tag{66}$$

Two representations are used to handle the nonlinear uncertain parameters $\alpha_1(k)$ and $\alpha_2(k)$.

The NFT format

$$\begin{aligned} A(\alpha(k)) &= Q_0 + \alpha_1(k)Q_4 + \alpha_2(k)Q_5 \\ &\quad + [\,\alpha_1(k)I_2 \quad \alpha_2(k)I_2\,] \begin{bmatrix} Q_1 & Q_3 \\ 0_2 & Q_2 \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \alpha_1(k)I_2 & 0_2 \\ 0_2 & \alpha_2(k)I_2 \end{bmatrix} \begin{bmatrix} \alpha_1(k)I_2 \\ \alpha_2(k)I_2 \end{bmatrix} \end{aligned} \tag{67}$$

leads to NFT (1) with

$$A(\alpha(k)) = \alpha_1(k)(Q_0 + Q_4) + \alpha_2(k)(Q_0 + Q_5)$$
$$B_\Delta(\alpha(k)) = [\,\alpha_1(k)I_2 \quad \alpha_2(k)I_2\,] \begin{bmatrix} Q_1 & Q_3 \\ O & Q_2 \end{bmatrix}$$
$$\Delta(\alpha(k)) = \begin{bmatrix} \alpha_1(k)I_2 & 0_2 \\ 0_2 & \alpha_2(k)I_2 \end{bmatrix}, D_{\Delta z} = 0$$
$$C_\Delta(\alpha(k)) = \begin{bmatrix} \alpha_1(k)I_2 \\ \alpha_2(k)I_2 \end{bmatrix}, D_z = 0, D_\Delta = 0, D_{\Delta\Delta} = 0. \tag{68}$$

Alternatively, the LFT format

$$\begin{aligned} A(\alpha(k)) &= Q_0 + [I_2 \quad 0_2 \quad 0_2 \quad I_2 \quad 0_2 \quad 0_2]\,\Delta(\alpha(k)) \\ &\quad \times \left(I_{12} - \begin{bmatrix} 0_2 & Q_1 & 0_2 & 0_2 & Q_3 & 0_2 \\ 0_2 & 0_2 & I_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & Q_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & I_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \end{bmatrix} \Delta(\alpha(k)) \right)^{-1} \\ &\quad \times \begin{bmatrix} Q_4 \\ 0_2 \\ I_2 \\ Q_5 \\ 0_2 \\ I_2 \end{bmatrix} \end{aligned}$$
$$\Delta(\alpha(k)) = \begin{bmatrix} \alpha_1(k)I_6 & O_6 \\ O_6 & \alpha_2(k)I_6 \end{bmatrix} \tag{69}$$

$$\begin{bmatrix} -\hat{\mathbf{X}} & * & * \\ 0 & -I & * \\ \hat{\mathbf{V}}^T A_{aj} + \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{1j} C_{aj} & \hat{\mathbf{V}}^T B_{aj} + \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{1j} D_{aj} & \hat{\mathbf{X}} - (\hat{\mathbf{V}} + \hat{\mathbf{V}}^T) \end{bmatrix} < 0 \tag{57}$$

$$\begin{bmatrix} -\hat{\mathbf{X}} & * & * \\ 0 & -I & * \\ L_{0j} - \hat{\mathbf{K}}_{3j} C_{aj} & M_j - \hat{\mathbf{K}}_{3j} D_{aj} & -\nu I \end{bmatrix} < 0 \tag{58}$$

$$\begin{bmatrix} -\hat{\mathbf{Y}} & * & * & * \\ 0 & -I & * & * \\ \hat{\mathbf{V}}^T A_{aj} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{1j} C_{aj} & \hat{\mathbf{V}}^T B_{aj} + \begin{bmatrix} I \\ I \end{bmatrix} \mathbf{K}_{1j} D_{aj} & \hat{\mathbf{Y}} - (\hat{\mathbf{V}} + \hat{\mathbf{V}}^T) & * \\ L_{0j} - \hat{\mathbf{K}}_{3j} C_{aj} & M_j - \hat{\mathbf{K}}_{3j} D_{aj} & 0 & -\gamma I \end{bmatrix} < 0$$
$$j = 1, 2, \ldots, s \tag{59}$$

TABLE I
COMPUTATIONAL PERFORMANCES OF DIFFERENT FILTERS.

Filter	gen. $\mathcal{H}_2$ perf.	Comp. time for gen. $\mathcal{H}_2$	$\mathcal{H}_\infty$ perf.	Comp. time for $\mathcal{H}_\infty$
NFT (4)	0.1883	140 sec	0.2363	37.4 sec
LFT (53)	0.3597	108 sec	0.4326	29.4 sec
LPV (56)	0.4647	5.9 sec	0.5473	3.6 sec
LTI (63)	0.4947	4.2 sec	0.5861	2.9 sec

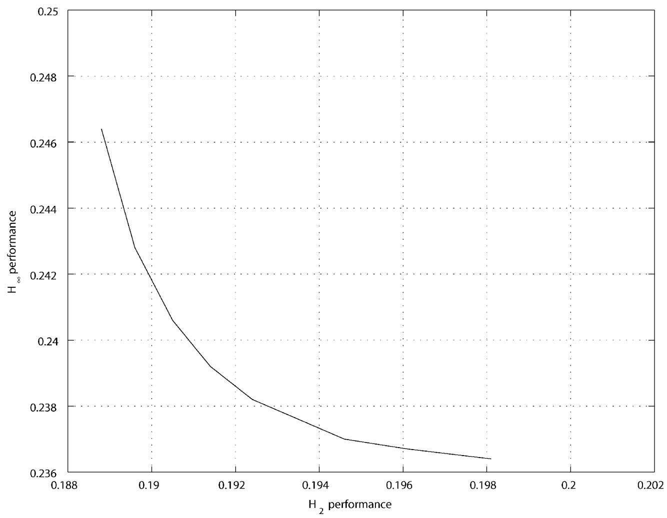

Fig. 1. Generalized $\mathcal{H}_2, \mathcal{H}_\infty$ performances of NFT-mixed filters.

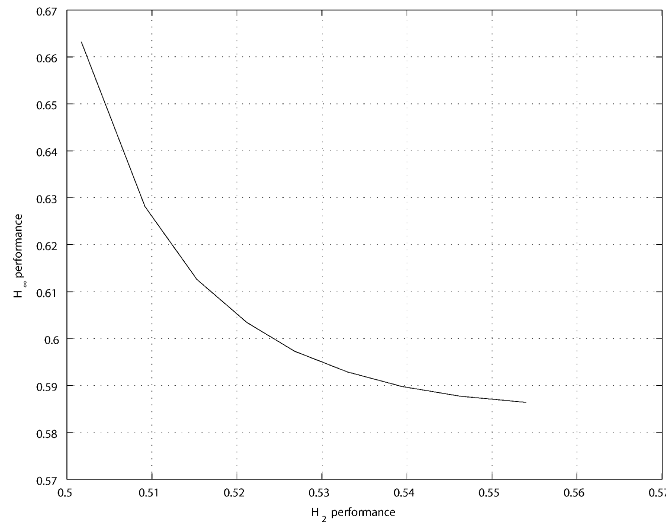

Fig. 2. Generalized $\mathcal{H}_2, \mathcal{H}_\infty$ performances of LTI-mixed filters.

leads to LFT (1) with

$$A = Q_0, B_\Delta = [\, I_2 \quad 0_2 \quad 0_2 \quad I_2 \quad 0_2 \quad 0_2 \,]$$
$$D_{\Delta\Delta} = \begin{bmatrix} 0_2 & Q_1 & 0_2 & 0_2 & Q_3 & 0_2 \\ 0_2 & 0_2 & I_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & Q_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & I_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \end{bmatrix}, \; C_\Delta = \begin{bmatrix} Q_4 \\ 0_2 \\ I_2 \\ Q_5 \\ 0_2 \\ I_2 \end{bmatrix}$$
$$D_z = 0, \; D_\Delta = 0, D_{\Delta z} = 0. \tag{70}$$

The Matlab LMI Control Toolbox [4] was used in all LMI-related computations. The dimension of 12 of the z_Δ's in the LFT representation (69) is very large in comparison with that equal to 4 in the NFT format (67). This deteriorates the computational efficiency and estimation performance so severely that our computer equipped by a 1.3-GHz AMD CPU was unable to solve the corresponding LMI formulations with the LFT model. In contrast, we easily solved the LMI formulations corresponding to the NFT model. Applying the result of Theorems 1 and 2, upper bounds on generalized $\mathcal{H}_2$, and $\mathcal{H}_\infty$ norms of NFT (1), (68), in the worst case, are found to be 2.2129 and 3.9961, respectively.

First, we consider the performance of the particularly designed LFT filters (53), LPV filters with structure (56), and LTI filters with structure (63). Table I displays different measures of performance. As expected, NFT filters result in substantial improvements of generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances over LFT, LPV, and LTI filters. Such improvement is reaffirmed through the comparison between Figs. 1 and 2, depicting the generalized $\mathcal{H}_2, \mathcal{H}_\infty$, and mixed performances of NFT and LTI mixed filters with different values of the tradeoff constant ρ.

TABLE II
MSE PERFORMANCES OF FILTERS

Filter	MSE performance
NFT-$\mathcal{H}_2$filter	0.0151
NFT-$\mathcal{H}_\infty$ filter	0.0143
NFT-mixed filter ($\rho = 0.9$)	0.0149
LTI-$\mathcal{H}_2$ filter	0.0287
LTI-$\mathcal{H}_\infty$ filter	0.0603
LTI-mixed filter ($\rho = 0.9$)	0.0312
Kalman filter	0.0624

Next, the actual performance of the NFT and LTI filters are evaluated via the mean square error (MSE) criterion $\mathcal{E}\{(z(k) - z_F(k))^2\}$. Process noise and measurement noise are mutually independent white Gaussian noises with the unity variance. The gain-scheduling parameter $\alpha_1(k)$ was randomly generated for every instant k, and then, $\alpha_2(k)$ was appropriately obtained. For each filter, the corresponding result was obtained over 1000 trials with 10 000 samples per every trial and listed in Table II. Noting that the variance of the signal $z(k)$, $\mathcal{E}\{z(k)^2\}$ of the time-varying plant is 4.5683, whereas that of the nominal (LTI) plant, i.e., $A(\alpha(k)) = Q_0$, is 3.8641, time-varying uncertainties essentially alter the statistics properties of the plant's response. In comparison with the variance of the signal of interest

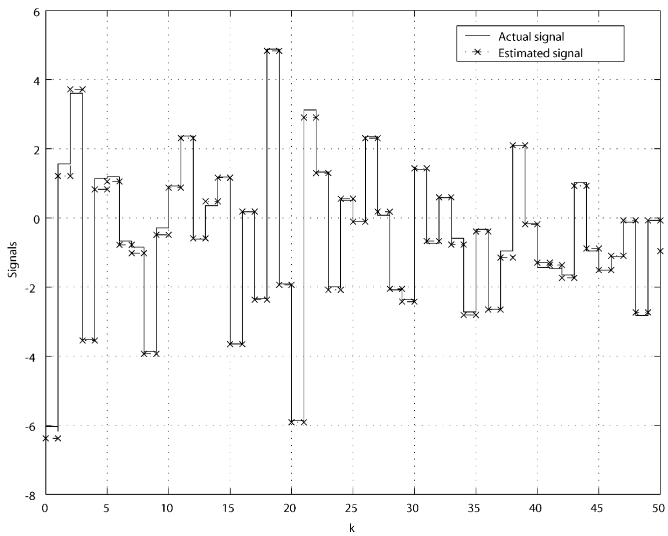

Fig. 3. Signal tracking of the NFT-mixed filter ($\rho = 0.9$).

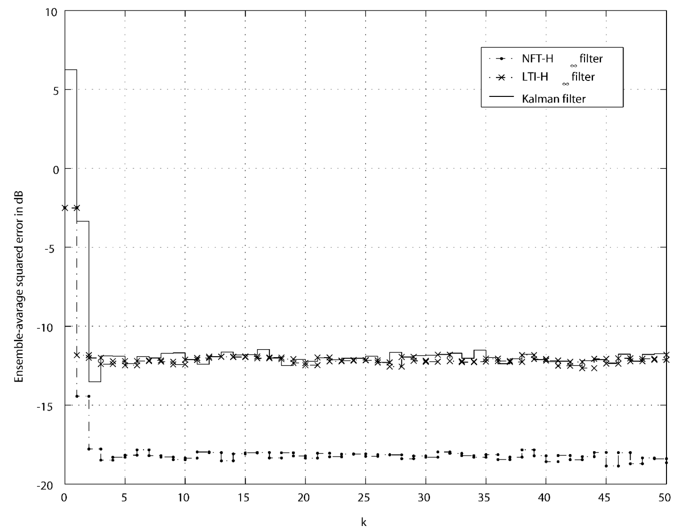

Fig. 5. Ensemble-average squares of errors $|z(k) - z_F(k)|^2$ by $\mathcal{H}_\infty$ filters and the Kalman filter.

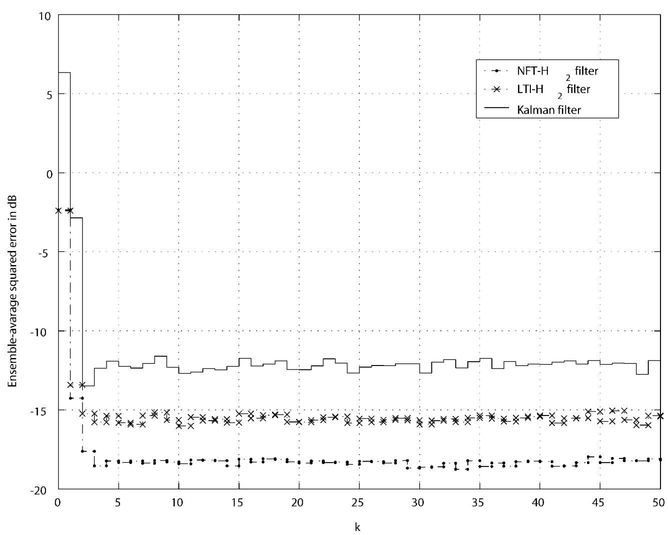

Fig. 4. Ensemble-average squares of errors $|z(k) - z_F(k)|^2\}$ by gen. $\mathcal{H}_2$ filters and the Kalman filter.

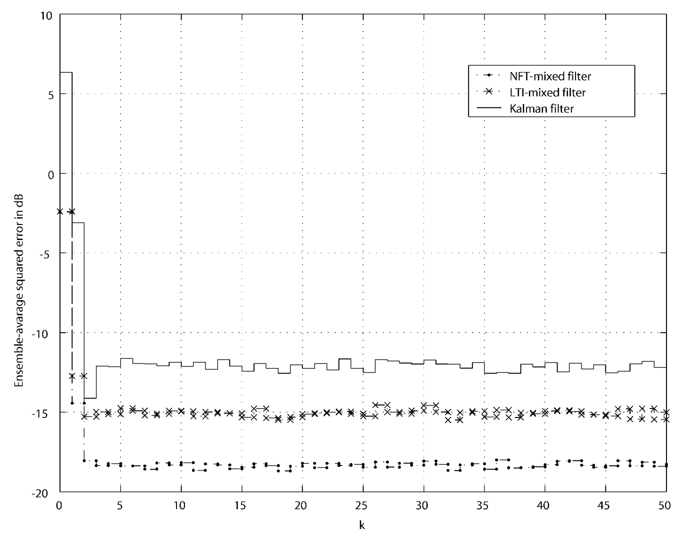

Fig. 6. Ensemble-average squares of errors $|z(k) - z_F(k)|^2$ by mixed filters ($\rho = 0.9$) and the Kalman filter.

$\mathcal{E}\{z(k)^2\} = 4.5683$, the results affirm that all the NFT filters achieve very good MSE performances. For the case of LTI filters, the LTI generalized $\mathcal{H}_2$ filter outperforms the Kalman filter designed for the nominal plant. Although the design noise conditions are met exactly, the MSE performance of the Kalman filter is slightly poorer than that of the LTI-$\mathcal{H}_\infty$ filter.

Furthermore, the tracking performances the NFT-mixed filter is shown in Fig. 3, confirming that it is a very good filter. For the sake of comparison and clarity between NFT filters, LTI filters, and the Kalman filter, the ensemble average squared error sequences over 1000 trials are depicted in Figs. 4–6. The figures all together demonstrate the superiority of NFT filters over the LTI and the Kalman filters.

V. Conclusions

In this paper, we have developed new techniques for the design of parameter-dependent filters. These filters explicitly depend on real-time-available system parameters and, thus, outperform customary nonadjustable filters. Our discussion has also investigated specific parameter structures attached to systems and filters. We have shown that the NFT structure is especially attractive, not only to encompass a wider set of parameter dependence but also for computational efficiency. Of most importance, our design techniques are based on LMI computations for which efficient and reliable software is now available. The validity of the proposed techniques has been confirmed through a number of simulations.

Finally, applications of similar techniques to equalization for fading communication channels are currently under study.

Appendix

We will verify the identities in (44). In the steps to follow, the left-hand sides of all the identities will be transformed to their equivalent forms. The verifications of the equivalence between these forms and the corresponding right-hand sides of identities are trivial; hence, they are omitted to save the space. Verifications of identities in (44) are done in order:

- Validity of $\Pi_{\mathbf{V}}^T \mathbf{V}^T A_{aj} \Pi_{\mathbf{V}} = \hat{\mathbf{V}}^T A_{aj}$:

$$\begin{aligned}\Pi_{\mathbf{V}}^T \mathbf{V}^T A_{aj} \Pi_{\mathbf{V}} &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}^T & \mathbf{V}_{21}^T \\ \mathbf{V}_{12}^T & \mathbf{V}_{22}^T \end{bmatrix} \begin{bmatrix} A_j & 0_n \\ 0_n & 0_n \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}^T A_j & 0_n \\ \mathbf{V}_{12}^T A_j & 0_n \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}_{11}^T A_j & 0_n \\ \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \mathbf{V}_{12}^T A_j & 0_n \end{bmatrix}\end{aligned}$$

- Validity of $\Pi_{\mathbf{V}}^T \mathbf{V}^T \mathcal{I}_0 \mathbf{K}_{1j} C_{aj} \Pi_{\mathbf{V}} = \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{1j} C_{aj}$:

$$\begin{aligned}\Pi_{\mathbf{V}}^T \mathbf{V}^T \mathcal{I}_0 \mathbf{K}_{1j} C_{aj} \Pi_{\mathbf{V}} &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}^T & \mathbf{V}_{21}^T \\ \mathbf{V}_{12}^T & \mathbf{V}_{22}^T \end{bmatrix} \begin{bmatrix} 0_n \\ I_n \end{bmatrix} [\mathbf{A}_{Fj} \quad \mathbf{B}_{\Delta Fj} \quad \mathbf{B}_F] \\ &\quad \times \begin{bmatrix} 0_n & I_n \\ 0_{m_\Delta n} & 0_{m_\Delta n} \\ C_j & 0_{pn} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{21}^T \\ \mathbf{V}_{22}^T \end{bmatrix} [\mathbf{B}_F C_j \quad \mathbf{A}_{Fj}] \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{21}^T \mathbf{B}_F C_j & \mathbf{V}_{21}^T \mathbf{A}_{Fj} \\ \mathbf{V}_{22}^T \mathbf{B}_F C_j & \mathbf{V}_{22}^T \mathbf{A}_{Fj} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}_{21}^T \mathbf{B}_F C_j & \mathbf{V}_{21}^T \mathbf{A}_{Fj} \mathbf{V}_{22}^{-1} \mathbf{V}_{21} \\ \mathbf{V}_{21}^T \mathbf{B}_F C_j & \mathbf{V}_{21}^T \mathbf{A}_{Fj} \mathbf{V}_{21}^{-1} \mathbf{V}_{21} \end{bmatrix}\end{aligned}$$

- Validity of $\Pi_{\mathbf{V}}^T \mathbf{V}^T B_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} = \hat{\mathbf{V}}^T B_{aj}$:

$$\begin{aligned}\Pi_{\mathbf{V}}^T \mathbf{V}^T B_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}^T & \mathbf{V}_{21}^T \\ \mathbf{V}_{12}^T & \mathbf{V}_{22}^T \end{bmatrix} \begin{bmatrix} B_{\Delta j} & 0_{nm_\Delta} & B_j \\ 0_{nm_\Delta} & 0_{nm_\Delta} & 0_{nm} \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}_{11}^T & \mathbf{V}_{21}^T \\ \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \mathbf{V}_{12}^T & \mathbf{V}_{21}^T \end{bmatrix} \begin{bmatrix} B_{\Delta j} & 0_{nm_\Delta} & B_j \\ 0_{nm_\Delta} & 0_{nm_\Delta} & 0_{nm} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}_{11}^T B_{\Delta j} & 0_{nm_\Delta} & \mathbf{V}_{11}^T B_j \\ \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \mathbf{V}_{12}^T B_{\Delta j} & 0_{nm_\Delta} & \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \mathbf{V}_{12}^T B_j \end{bmatrix}\end{aligned}$$

- Validity of $\Pi_{\mathbf{V}}^T \mathbf{V}^T \mathcal{I}_0 \mathbf{K}_{1j} D_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{1j} D_{aj}$:

$$\begin{aligned}\Pi_{\mathbf{V}}^T \mathbf{V}^T \mathcal{I}_0 \mathbf{K}_{1j} D_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}^T & \mathbf{V}_{21}^T \\ \mathbf{V}_{12}^T & \mathbf{V}_{22}^T \end{bmatrix} \begin{bmatrix} 0_n \\ I_n \end{bmatrix} [\mathbf{A}_{Fj} \quad \mathbf{B}_{\Delta Fj} \quad \mathbf{B}_F] \\ &\quad \times \begin{bmatrix} 0_{nm_\Delta} & 0_{nm_\Delta} & 0_{nm} \\ 0_{m_\Delta} & I_{m_\Delta} & 0_{m_\Delta m} \\ D_{\Delta j} & 0_{pm_\Delta} & D_j \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}_{11}^T & \mathbf{V}_{21}^T \\ \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \mathbf{V}_{12}^T & \mathbf{V}_{21}^T \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{A}_{Fj} & \mathbf{B}_{\Delta Fj} & \mathbf{B}_F \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0_{nm_\Delta} & 0_{nm_\Delta} & 0_{nm} \\ 0_{m_\Delta} & \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & 0_{m_\Delta m} \\ D_{\Delta j} & 0_{pm_\Delta} & D_j \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}_{11}^T & \mathbf{V}_{21}^T \\ \mathbf{V}_{21}^T \mathbf{V}_{22}^{-T} \mathbf{V}_{12}^T & \mathbf{V}_{21}^T \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{B}_F D_{\Delta j} & \mathbf{B}_{\Delta Fj} \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & \mathbf{B}_F D_j \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}_{21}^T \mathbf{B}_F D_{\Delta j} & \mathbf{V}_{21}^T \mathbf{B}_{\Delta Fj} \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & \mathbf{V}_{21}^T \mathbf{B}_F D_j \\ \mathbf{V}_{21}^T \mathbf{B}_F D_{\Delta j} & \mathbf{V}_{21}^T \mathbf{B}_{\Delta Fj} \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & \mathbf{V}_{21}^T \mathbf{B}_F D_j \end{bmatrix}\end{aligned}$$

- Validity of $\Pi_{\mathbf{F}}^T \mathbf{F} \mathcal{C}_{0j} \Pi_{\mathbf{V}} = \hat{\mathbf{F}} \mathcal{C}_{0j}$:

$$\begin{aligned}\Pi_{\mathbf{F}}^T \mathbf{F} \mathcal{C}_{0j} \Pi_{\mathbf{V}} &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{F}_{12}\mathbf{F}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} \begin{bmatrix} C_{\Delta j} & 0_{m_\Delta n} \\ 0_{m_\Delta n} & 0_{m_\Delta n} \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{F}_{12}\mathbf{F}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{11} C_{\Delta j} & 0 \\ \mathbf{F}_{21} C_{\Delta j} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{F}_{11} C_{\Delta j} & 0 \\ \mathbf{F}_{12}\mathbf{F}_{22}^{-1}\mathbf{F}_{21} C_{\Delta j} & 0 \end{bmatrix}\end{aligned}$$

- Validity of $\Pi_{\mathbf{F}}^T \mathbf{F} \tilde{\mathcal{I}}_0 \mathbf{K}_{2j} C_{aj} \Pi_{\mathbf{V}} = \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{2j} C_{aj}$:

$$\begin{aligned}\Pi_{\mathbf{F}}^T \mathbf{F} \tilde{\mathcal{I}}_0 \mathbf{K}_{2j} C_{aj} \Pi_{\mathbf{V}} &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{F}_{12}\mathbf{F}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} \begin{bmatrix} 0_{m_\Delta} \\ I_{m_\Delta} \end{bmatrix} [\mathbf{C}_{Fj} \quad \mathbf{D}_{\Delta Fj} \quad \mathbf{D}_{yF}] \\ &\quad \times \begin{bmatrix} 0_n & I_n \\ 0_{m_\Delta n} & 0_{m_\Delta n} \\ C_j & 0_{pn} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{F}_{12}\mathbf{F}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{12} \\ \mathbf{F}_{22} \end{bmatrix} [\mathbf{D}_{yF} C_j \quad \mathbf{C}_{Fj}] \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{F}_{12}\mathbf{F}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{12}\mathbf{D}_{yF} C_j & \mathbf{F}_{12}\mathbf{C}_{Fj} \\ \mathbf{F}_{22}\mathbf{D}_{yF} C_j & \mathbf{F}_{22}\mathbf{C}_{Fj} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{F}_{12}\mathbf{D}_{yF} C_j & \mathbf{F}_{12}\mathbf{C}_{Fj}\mathbf{V}_{22}^{-1}\mathbf{V}_{21} \\ \mathbf{F}_{12}\mathbf{D}_{yF} C_j & \mathbf{F}_{12}\mathbf{C}_{Fj}\mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix}\end{aligned}$$

- Validity of $\Pi_{\mathbf{F}}^T \mathbf{F} \mathcal{D}_{0j} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} = \hat{\mathbf{F}} \mathcal{D}_{0j}$:

$$\begin{aligned}\Pi_{\mathbf{F}}^T \mathbf{F} \mathcal{D}_{0j} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & \mathbf{F}_{12}\mathbf{F}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} \begin{bmatrix} D_{\Delta\Delta j} & 0_{m_\Delta} & D_{zj} \\ 0_{m_\Delta} & 0_{m_\Delta} & 0_{m_\Delta m} \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{12}\mathbf{F}_{22}^{-1}\mathbf{F}_{21} & \mathbf{F}_{12} \end{bmatrix} \begin{bmatrix} D_{\Delta\Delta j} & 0_{m_\Delta} & D_{zj} \\ 0_{m_\Delta} & 0_{m_\Delta} & 0_{m_\Delta m} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{F}_{11} D_{\Delta\Delta j} & 0_{m_\Delta} & \mathbf{F}_{11} D_{zj} \\ \mathbf{F}_{12}\mathbf{F}_{22}^{-1}\mathbf{F}_{21} D_{\Delta\Delta j} & 0_{m_\Delta} & \mathbf{F}_{12}\mathbf{F}_{22}^{-1}\mathbf{F}_{21} D_{zj} \end{bmatrix}\end{aligned}$$

- Validity of $\Pi_{\mathbf{F}}^T \mathbf{F}\tilde{\mathcal{I}}_0 \mathbf{K}_{2j} D_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix} \hat{\mathbf{K}}_{2j} D_{aj}$:

$$\begin{aligned}
&\Pi_{\mathbf{F}}^T \mathbf{F}\tilde{\mathcal{I}}_0 \mathbf{K}_{2j} D_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ 0 & \mathbf{F}_{12}\mathbf{F}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} \begin{bmatrix} 0_{m_\triangle} \\ I_{m_\triangle} \end{bmatrix} [\, \mathbf{C}_{Fj} \quad \mathbf{D}_{\Delta Fj} \quad \mathbf{D}_{yF} \,] \\
&\quad \times \begin{bmatrix} 0_{nm_\triangle} & 0_{nm_\triangle} & 0_{nm} \\ 0_{m_\triangle} & I_{m_\triangle} & 0_{m_\triangle m} \\ D_{\Delta j} & 0_{pm_\triangle} & D_j \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & 0 \\ 0 & 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{12}\mathbf{F}_{22}^{-1}\mathbf{F}_{21} & \mathbf{F}_{12} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{C}_{Fj} & \mathbf{D}_{\Delta Fj} & \mathbf{D}_{yF} \end{bmatrix} \\
&\quad \times \begin{bmatrix} 0_{nm_\triangle} & 0_{nm_\triangle} & 0_{nm} \\ 0_{m_\triangle} & \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & 0_{m_\triangle m} \\ D_{\Delta j} & 0_{pm_\triangle} & D_j \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{12}\mathbf{F}_{22}^{-1}\mathbf{F}_{21} & \mathbf{F}_{12} \end{bmatrix} \\
&\quad \times \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{D}_{yF} D_{\Delta j} & \mathbf{D}_{\Delta Fj}\mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & \mathbf{D}_{yF} D_j \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{F}_{12}\mathbf{D}_{yF} D_{\Delta j} & \mathbf{F}_{12}\mathbf{D}_{\Delta Fj}\mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & \mathbf{F}_{12}\mathbf{D}_{yF} D_j \\ \mathbf{F}_{12}\mathbf{D}_{yF} D_{\Delta j} & \mathbf{F}_{12}\mathbf{D}_{\Delta Fj}\mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & \mathbf{F}_{12}\mathbf{D}_{yF} D_j \end{bmatrix}
\end{aligned}$$

- Validity of $L_{0j}\Pi_{\mathbf{V}} = L_{0j}$:

$$L_{0j}\Pi_{\mathbf{V}} = [\, L_j \quad 0_{qn} \,] \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} = [\, L_j \quad 0_{qn} \,]$$

- Validity of $\mathbf{K}_{3j} C_{aj} \Pi_{\mathbf{V}} = \hat{\mathbf{K}}_{3j} C_{aj}$:

$$\begin{aligned}
&\mathbf{K}_{3j} C_{aj} \Pi_{\mathbf{V}} \\
&= [\, \mathbf{L}_{Fj} \quad \mathbf{D}_{zFj} \quad \mathbf{D}_F \,] \begin{bmatrix} 0_n & I_n \\ 0_{m_\triangle n} & 0_{m_\triangle n} \\ C_j & 0_{pn} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\
&= [\, \mathbf{D}_F C_j \quad \mathbf{L}_{Fj} \,] \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix} \\
&= [\, \mathbf{D}_F C_j \quad \mathbf{L}_{Fj}\mathbf{V}_{22}^{-1}\mathbf{V}_{21} \,]
\end{aligned}$$

- Validity of $D_{\Delta z0j}\Pi_{\mathbf{H}} = D_{\Delta z0j}$:

$$D_{\Delta z0j}\Pi_{\mathbf{H}} = [D_{\Delta zj} \quad 0_{qm_\triangle}] \begin{bmatrix} I & 0 \\ 0 & \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T \end{bmatrix} = [D_{\Delta zj} \quad 0_{qm_\triangle}]$$

- Validity of $\mathbf{K}_{3j} D_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} = \hat{\mathbf{K}}_{3j} D_{aj}$:

$$\begin{aligned}
&\mathbf{K}_{3j} D_{aj} \begin{bmatrix} \Pi_{\mathbf{H}} & 0 \\ 0 & I \end{bmatrix} \\
&= [\, \mathbf{L}_{Fj} \quad \mathbf{D}_{zFj} \quad \mathbf{D}_F \,] \begin{bmatrix} 0_{nm_\triangle} & 0_{nm_\triangle} & 0_{nm} \\ 0_{m_\triangle} & I_{m_\triangle} & 0_{m_\triangle m} \\ D_{\Delta j} & 0_{pm_\triangle} & D_j \end{bmatrix} \\
&\quad \times \begin{bmatrix} I & 0 & 0 \\ 0 & \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & 0 \\ 0 & 0 & I \end{bmatrix} \\
&= [\, \mathbf{L}_{Fj} \quad \mathbf{D}_{zFj} \quad \mathbf{D}_F \,] \begin{bmatrix} 0_{nm_\triangle} & 0_{nm_\triangle} & 0_{nm} \\ 0_{m_\triangle} & \mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T & 0_{m_\triangle m} \\ D_{\Delta j} & 0_{pm_\triangle} & D_j \end{bmatrix} \\
&= [\, \mathbf{D}_F D_{\Delta j} \quad \mathbf{D}_{zFj}\mathbf{H}_{22}^{-T}\mathbf{H}_{12}^T \quad \mathbf{D}_F D_j \,].
\end{aligned}$$

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