Robust Filtering for Discrete Nonlinear Fractional Transformation Systems

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***Abstract*—<AU: PLEASE REDUCE PAPER LENGTH TO 5 PAGES, THANKS?> In this brief, we consider robust filtering problems for uncertain discrete-time systems. The uncertain plants under consideration possess nonlinear fractional transformation (NFT) representations which are a generalization of the classical linear fractional transformation (LFT) representations. The proposed NFT is more practical than the LFT, and moreover, it leads to substantial performance gains as well as computational savings. For this class of systems, we derive linear-matrix inequality characterizations for $\mathcal{H}_2$, $\mathcal{H}_\infty$, and mixed filtering problems. Our approach is finally validated through a number of examples.**

***Index Terms*—Linear-matrix inequality (LMI), nonlinear fractional transformation (NFT), robust filtering.**

I. Introduction

In recent years, robust filtering has been intensively studied in the literature (see, e.g., [4], [7], [11]–[13] and references therein). This is mainly due to the emergence of linear-matrix inequalities (LMIs) as an efficient and practical tool to solve robust controller and filter design problems. The LMI setting is really fit to handle robust optimization since many realistic uncertainty constraints can be adequately and accurately expressed by LMIs in a straightforward manner. In contrast to the Riccati-equation-based approaches, which only work for the restricted family of filters with simple Luenberger observer structure (see, e.g., [7] and references therein), the LMI-based approaches extend to filters with general structure and can handle a much wider class of uncertain systems [4], [11], [12]. Very often, the uncertain systems are assumed linear in the uncertain parameters [4], [11], [12]. The more general situation where uncertain parameters enter the system data in a nonlinear way has been addressed in [8], [12]. The results of [8] provide matrix inequalities, which are still highly nonlinear in scaling variables, while those of [12] are in the form of exact LMIs. As shown in [12], it is crucial to express nonlinear parameter dependence of a system in a tractable form, which in turn leads to LMI characterizations. For this purpose, the nonlinear fractional transformation (NFT) introduced in [12] seems to be an eligible candidate.

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The aim of this brief is to extend results of [12] to the case of discrete-time systems. That is to solve robust filtering problems for the discrete-time uncertain linear systems in the NFT form

$$\begin{bmatrix} x(k+1) \\ y(k) \\ z_\Delta(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\alpha) & B_\Delta(\alpha) & B(\alpha) \\ C(\alpha) & D_\Delta(\alpha) & D(\alpha) \\ C_\Delta(\alpha) & D_{\Delta z}(\alpha) & D_z(\alpha) \\ L(\alpha) & D_{\Delta\Delta}(\alpha) & M(\alpha) \end{bmatrix} \begin{bmatrix} x(k) \\ w_\Delta(k) \\ w(k) \end{bmatrix}$$
$$w_\Delta(k) = \Delta(\alpha) z_\Delta(k) \tag{1}$$

where $A(\alpha) \in \boldsymbol{R}^{n\times n}$, $B_\Delta(\alpha) \in \boldsymbol{R}^{n\times m_\Delta}$, $B(\alpha) \in \boldsymbol{R}^{n\times m}$, $D(\alpha) \in \boldsymbol{R}^{p\times m}$, $C_\Delta(\alpha) \in R^{m_\Delta \times n}$, $L(\alpha) \in \boldsymbol{R}^{q\times n}$ and $x \in \boldsymbol{R}^n$ is the state, $y \in \boldsymbol{R}^p$ is the measured output, $z \in \boldsymbol{R}^q$ is the output to be estimated and $w \in R^m$ is the noise, the variables $w_\Delta \in R^{m_\Delta}$ and $z_\Delta \in R^{m_\Delta}$ are introduced to express the uncertain components of the system. With preliminary normalization if necessary, the uncertain parameter α is assumed to lie in the unit simplex Γ

$$\Gamma := \left\{ (\alpha_1, \ldots, \alpha_s) : \sum_{j=1}^{s} \alpha_j = 1,\ \alpha_j \geq 0 \right\}.$$

In sharp contrast with the linear fractional transformation (LFT) [14], all the state–space matrix data in (1) are allowed to depend linearly on the uncertain parameter α

$$\begin{bmatrix} A(\alpha) & B_\Delta(\alpha) & B(\alpha) \\ C(\alpha) & D_\Delta(\alpha) & D(\alpha) \\ C_\Delta(\alpha) & D_{\Delta z}(\alpha) & D_z(\alpha) \\ L(\alpha) & D_{\Delta\Delta}(\alpha) & M(\alpha) \\ 0 & \Delta(\alpha) & 0 \end{bmatrix} = \sum_{j=1}^{s} \alpha_j \begin{bmatrix} A_j & B_{\Delta j} & B_j \\ C_j & D_{\Delta j} & D_j \\ C_{\Delta j} & D_{\Delta z j} & D_{zj} \\ L_j & D_{\Delta\Delta j} & M_j \\ 0 & \Delta_j & 0 \end{bmatrix}. \tag{2}$$

The NFT is advantageuos to the LFT since the NFT yields representations with smaller dimensionality which in turn result in the better efficacy of numerical treatments. It will be seen later via a number of examples in Section IV that the NFT offers not only substantial performance gains but also significant computational reduction. For robust filtering problems of LFT systems and their treatments one can refer to [3], [10] or resort to the simplification of linear parameter-varying (LPV) control [1] and references therein.

It is worth stressing that the strictly proper filter structure

$$\begin{bmatrix} x_F(k+1) \\ z_F(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_F & \mathbf{B}_F \\ \mathbf{L}_F & 0 \end{bmatrix} \begin{bmatrix} x_F(k) \\ y(k) \end{bmatrix} \tag{3}$$

used in [4], [11], and [13], in essence, corresponds to the class of one-step-ahead predictors. Intrinsically, filtering problems are solved by using the proper structure

$$\begin{bmatrix} x_F(k+1) \\ z_F(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_F & \mathbf{B}_F \\ \mathbf{L}_F & \mathbf{D}_F \end{bmatrix} \begin{bmatrix} x_F(k) \\ y(k) \end{bmatrix} \quad (4)$$

with $\mathbf{A}_F \in \boldsymbol{R}^{n\times n}$ and $\mathbf{L}_F \in \boldsymbol{R}^{q\times n}$. Furthermore, the estimation criterion of filters is based on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ criterion

$$\max_{\alpha\in\Gamma}[\rho\nu + (1-\rho)\gamma] \rightarrow \min \quad (5)$$

where ν and γ, respectively, denote the squares of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the transfer function made out from (1) and (4) which maps the noise sequence $\{w(k)\}$ to the estimation error sequence $\{z(k) - z_F(k)\}$. The $\mathcal{H}_2$ norm constraint introduced in Section II is the error variance criterion and the $\mathcal{H}_\infty$ norm constraint is the error energy criterion. Therefore, (5) makes a compromise between these two constraints with tradeoff constant ρ $(0 < \rho < 1)$.

This paper develops an effective approach toward robust filtering problems. The contribution is twofold. For the class of NFT systems, first we give a new characterization of the $\mathcal{H}_2$ norm constraint then we derive new LMI formulations of $\mathcal{H}_2$, $\mathcal{H}_\infty$ and mixed filtering problems.

We organize the paper as follows. Section II outlines characterizations of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the above NFT systems. Section III presents LMI synthesis conditions for the robust filtering problems. Section IV provides validation for our techniques through numerical examples. Due to space limitations, the presentation is rather brief. The interested reader can refer to the full version of the paper [5] for more technical details.

Notations used in the paper are fairly standard. $\mathcal{E}(\ )$ denotes the expectation operation. M^T is the transpose of the matrix M. For symmetric matrices, $M < N$ or $M > N$ means $M - N$ is negative definite or positive definite, respectively. In long matrix expressions, we use the simplification

$$\begin{bmatrix} M_{11} & * \\ M_{12} & M_{22} \end{bmatrix}(\alpha) \equiv \begin{bmatrix} M_{11}(\alpha) & M_{12}^T(\alpha) \\ M_{12}(\alpha) & M_{22}(\alpha) \end{bmatrix}. \quad (6)$$

To avoid ambiguity, we write, for instance, I_n or 0_{nm} to indicate the dimensions of matrices and matrix variables in boldface.

II. Characterizations for Performance Constraints

This section provides LMI-based formulations for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances of filters. This is done with the augmented system formed by (1) and (4) having the estimation error $z(k) - z_F(k)$ as the output to be minimized

$$\begin{bmatrix} x_{cl}(k+1) \\ z_\Delta(k) \\ z_{cl}(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{cl}(\alpha) & \mathcal{B}_{\Delta cl}(\alpha) & \mathcal{B}_{cl}(\alpha) \\ \mathcal{C}_\Delta(\alpha) & D_{\Delta z}(\alpha) & D_z(\alpha) \\ \mathcal{L}_{cl}(\alpha) & \mathcal{D}_{cl}(\alpha) & \mathcal{M}_{cl}(\alpha) \end{bmatrix} \begin{bmatrix} x_{cl}(k) \\ w_\Delta(k) \\ w(k) \end{bmatrix}$$
$$w_\Delta(k) = \Delta(\alpha) z_\Delta(k) \quad (7)$$

where $x_{cl}(k) = \begin{bmatrix} x(k) \\ x_F(k) \end{bmatrix}$, $\mathcal{A}_{cl}(\alpha) = \begin{bmatrix} A(\alpha) & 0 \\ \mathbf{B}_F C(\alpha) & \mathbf{A}_F \end{bmatrix}$, $\mathcal{B}_{\Delta cl}(\alpha) = \begin{bmatrix} B_\Delta(\alpha) \\ \mathbf{B}_F D_\Delta(\alpha) \end{bmatrix}$, $\mathcal{B}_{cl}(\alpha) = \begin{bmatrix} B(\alpha) \\ \mathbf{B}_F D(\alpha) \end{bmatrix}$, $\mathcal{C}_\Delta(\alpha) = [C_\Delta(\alpha)\ \ 0]$, $\mathcal{D}_{cl}(\alpha) = D_{\Delta\Delta}(\alpha) - \mathbf{D}_F D_\Delta(\alpha)$, $\mathcal{L}_{cl}(\alpha) = [L(\alpha) - \mathbf{D}_F C(\alpha)\ \ -\mathbf{L}_F]$, and $\mathcal{M}_{cl}(\alpha) = M(\alpha) - \mathbf{D}_F D(\alpha)$.

For convenience, (7) is temporarily written in the virtual form

$$\begin{bmatrix} x_{cl}(k+1) \\ z_{cl}(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\alpha) & \mathcal{B}(\alpha) \\ \mathcal{L}(\alpha) & \mathcal{M}(\alpha) \end{bmatrix} \begin{bmatrix} x_{cl}(k) \\ w(k) \end{bmatrix}. \quad (8)$$

Then, the $\mathcal{H}_2$-norm (see, e.g., [6]) is defined as

$$\sqrt{\mathrm{Trace}\{\mathcal{L}(\alpha)\mathbf{P}(\alpha)\mathcal{L}^T(\alpha) + \mathcal{M}(\alpha)\mathcal{M}^T(\alpha)\}}$$

where $\mathbf{P}(\alpha)$ satisfies

$$\mathbf{P}(\alpha) = \mathcal{A}(\alpha)\mathbf{P}(\alpha)\mathcal{A}^T(\alpha) + \mathcal{B}(\alpha)\mathcal{B}^T(\alpha), \qquad \mathbf{P}(\alpha) > 0.$$

Actually, when $\mathcal{E}(w(i)w^T(k)) = \delta_{ik} I$ this norm is exactly the standard deviation of the output, $\sqrt{\mathcal{E}\left(z_{cl}^T z_{cl}\right)}$.

Hereafter, we consider the following robust $\mathcal{H}_2$ performance of system (7):

$$\sup_{\alpha\in\Gamma} \mathrm{Trace}\{\mathcal{L}(\alpha)\mathbf{P}(\alpha)\mathcal{L}^T(\alpha) + \mathcal{M}(\alpha)\mathcal{M}^T(\alpha)\} < \nu. \quad (9)$$

In order to compute this upperbound, we take the Lyapunov function candidate

$$V(x_{cl}(k)) = x_{cl}^T(k)\mathbf{X}(\alpha)x_{cl}(k), \quad \mathbf{X}(\alpha) > 0 \qquad \forall\,\alpha \in \Gamma$$

which, for any nonzero $\{w(k)\}$, satisfies the following two inequalities:

$$V(x_{cl}(k+1)) - V(x_{cl}(k)) + z_\Delta^T(k)\mathbf{R}_1(\alpha)z_\Delta(k) + w_\Delta^T(k)\mathbf{S}_1(\alpha)w_\Delta(k) - \|w(k)\|^2 < 0 \quad (10)$$
$$z_{cl}(k)^T\mathbf{Z}^{-1}(\alpha)z_{cl}(k) - V(x_{cl}(k)) + z_\Delta^T(k)\mathbf{R}_2(\alpha)z_\Delta(k) + w_\Delta^T(k)\mathbf{S}_2(\alpha)w_\Delta(k) - \|w(k)\|^2 < 0 \quad (11)$$

where the matrix $\mathbf{Z}(\alpha)$ is such that

$$\mathbf{Z}(\alpha) > 0, \quad \mathrm{Trace}\{\mathbf{Z}(\alpha)\} < \nu \quad (12)$$

and matrices $\mathbf{R}_i(\alpha) > 0, \mathbf{S}_i(\alpha) < 0$ belonging to the symmetric scaling sets used in [1], [12], hence are such that for all $w_\Delta(k)$, $z_\Delta(k)$ in (7)

$$z_\Delta^T(k)\mathbf{R}_i(\alpha)z_\Delta(k) + w_\Delta^T(k)\mathbf{S}_i(\alpha)w_\Delta(k) \geq 0, \quad i = 1, 2. \quad (13)$$

Using (13) and Schur's complement, it follows from (10), (11) and (13) that

$$-\mathbf{X}^{-1}(\alpha) + \mathcal{A}(\alpha)\mathbf{X}^{-1}(\alpha)\mathcal{A}^T(\alpha) + \mathcal{B}(\alpha)\mathcal{B}^T(\alpha) < 0 \quad (14)$$
$$-\mathbf{Z}(\alpha) + \mathcal{L}(\alpha)\mathbf{X}^{-1}\mathcal{L}(\alpha)^T + \mathcal{M}(\alpha)\mathcal{M}(\alpha)^T < 0. \quad (15)$$

The system (8) is stable by (14) and $\mathbf{X}^{-1}(\alpha) > \mathbf{P}(\alpha)$. Further, it is apparent from (12) and (15) that (9) holds. Thus, we conclude that (10), (11) together with (13) secure both the stability and the upper bound $\sqrt{\nu}$ on the $\mathcal{H}_2$ performance of the system (7). Then, the theorem below follows along the line of [1, Th. 1].

Theorem 1: The $\mathcal{H}_2$ norm of the system (7) is less than $\sqrt{\nu}$ if for every $\alpha \in \Gamma$, there are symmetric matrices $\mathbf{X}(\alpha)$, $\mathbf{Z}(\alpha)$, scalings $\mathbf{R}_i(\alpha) > 0$, $\mathbf{S}_i(\alpha) < 0$ and slack matrices $\mathbf{V}(\alpha)$, $\mathbf{H}_i(\alpha)$, $\mathbf{F}_i(\alpha)$ satisfying the following inequalities:

$$\begin{bmatrix} T_{11} & * & * & * \\ 0 & T_{22} & * & * \\ T_{31} & T_{32} & T_{33} & * \\ T_{41} & T_{42} & 0 & T_{44} \end{bmatrix}(\alpha) < 0 \qquad \forall\, \alpha \in \Gamma \tag{16}$$

$$\begin{bmatrix} U_{11} & * & * & * \\ 0 & U_{22} & * & * \\ U_{31} & U_{32} & U_{33} & * \\ U_{41} & U_{42} & 0 & -\mathbf{Z} \end{bmatrix}(\alpha) < 0 \qquad \forall\, \alpha \in \Gamma \tag{17}$$

$$\begin{bmatrix} \mathbf{R}_i & \Delta^T \mathbf{H}_i^T \\ \mathbf{H}_i \Delta & \mathbf{S}_i + \left(\mathbf{H}_i + \mathbf{H}_i^T\right) \end{bmatrix}(\alpha) \geq 0 \qquad \forall\, \alpha \in \Gamma; \quad i = 1,2 \tag{18}$$

$$\operatorname{Trace}\{\mathbf{Z}(\alpha)\} < \nu \tag{19}$$

where $T_{11} = -\mathbf{X}$, $T_{22} = \begin{bmatrix} \mathbf{S}_1 & * \\ 0 & -I \end{bmatrix}$, $T_{31} = \mathbf{V}^T \mathcal{A}_{cl}$, $T_{32} = \mathbf{V}^T [\, \mathcal{B}_{\Delta cl} \quad \mathcal{B}_{cl} \,]$, $T_{33} = \mathbf{X} - (\mathbf{V} + \mathbf{V}^T)$, $T_{41} = \mathbf{F}_1 \mathcal{C}_\Delta$, $T_{42} = \mathbf{F}_1 [\, D_{\Delta z} \quad D_z \,]$, $T_{44} = \mathbf{R}_1 - \left(\mathbf{F}_1 + \mathbf{F}_1^T\right)$, $U_{11} = -\mathbf{X}$, $U_{22} = \begin{bmatrix} \mathbf{S}_2 & * \\ 0 & -I \end{bmatrix}$, $U_{31} = \mathbf{F}_2 \mathcal{C}_\Delta$, $U_{32} = \mathbf{F}_2 [\, D_{\Delta z} \quad D_z \,]$, $U_{33} = \mathbf{R}_2 - \left(\mathbf{F}_2 + \mathbf{F}_2^T\right)$, $U_{41} = \mathcal{L}_{cl}$, $U_{42} = [\, \mathcal{D}_{cl} \quad \mathcal{M}_{cl} \,]$. □

Recall that the $\mathcal{H}_\infty$ norm of the system (7) is

$$\sup_{w,T} \left[\frac{\sum_{k=0}^{T} \|z_{cl}(k)\|^2}{\sum_{k=0}^{T} \|w(k)\|^2} \right]^{1/2}.$$

With a scaling pair $\mathbf{S}(\alpha)$, $\mathbf{R}(\alpha)$

$$z_\Delta^T(k) \mathbf{R}(\alpha) z_\Delta(k) + w_\Delta^T(k) \mathbf{S}(\alpha) w_\Delta(k) \geq 0$$

if we have

$$\begin{aligned} &V(x_{cl}(k+1)) - V(x_{cl}(k)) + z_\Delta^T(t) \mathbf{R}(\alpha) z_\Delta(k) \\ &\quad + w_\Delta^T(k) \mathbf{S}(\alpha) w_\Delta(k) + \frac{1}{\gamma} \|z_{cl}(k)\|^2 - \|w(k)\|^2 < 0 \end{aligned} \tag{20}$$

where $V(x_{cl}(k)) = x_{cl}(k)^T \mathbf{Y}(\alpha) x_{cl}(k)$ with $\mathbf{Y}(\alpha) > 0$ being the Lyapunov function, then

$$V(x_{cl}(k+1)) - V(x_{cl}(k)) + \frac{1}{\gamma} \|z_{cl}(k)\|^2 - \|w(k)\|^2 < 0.$$

So

$$\sum_{k=0}^{T} \|z_{cl}(k)\|^2 < \gamma \sum_{k=0}^{T} \|w(k)\|^2 \tag{21}$$

implying the $\mathcal{H}_\infty$ norm of the system (7) is less than $\sqrt{\gamma}$. Paralleling [1, Th. 2], we have the following theorem.

Theorem 2: The performance condition (21) is satisfied if for every $\alpha \in \Gamma$ there are matrices $\mathbf{Y}(\alpha)$, $\mathbf{V}(\alpha)$, $\mathbf{R}(\alpha) > 0$, $\mathbf{S}(\alpha) < 0$, $\mathbf{H}(\alpha)$ and $\mathbf{F}(\alpha)$ satisfying the following inequalities:

$$\begin{bmatrix} P_{11} & * & * & * & * \\ 0 & P_{22} & * & * & * \\ P_{31} & P_{32} & P_{33} & * & * \\ P_{41} & P_{42} & 0 & P_{44} & * \\ P_{51} & P_{52} & 0 & 0 & -\gamma I \end{bmatrix}(\alpha) < 0 \qquad \forall\, \alpha \in \Gamma \tag{22}$$

$$\begin{bmatrix} \mathbf{R} & \Delta^T \mathbf{H}^T \\ \mathbf{H}\Delta & \mathbf{S} + (\mathbf{H} + \mathbf{H}^T) \end{bmatrix}(\alpha) \geq 0 \qquad \forall\, \alpha \in \Gamma. \tag{23}$$

with $P_{11} = -\mathbf{Y}$, $P_{22} = \begin{bmatrix} \mathbf{S} & * \\ 0 & -I \end{bmatrix}$, $P_{31} = \mathbf{V}^T \mathcal{A}_{cl}$, $P_{32} = \mathbf{V}^T [\, \mathcal{B}_{\Delta cl} \quad \mathcal{B}_{cl} \,]$, $P_{33} = \mathbf{Y} - \left(\mathbf{V} + \mathbf{V}^T\right)$, $P_{41} = \mathbf{F}\mathcal{C}_\Delta$, $P_{42} = \mathbf{F} [\, D_{\Delta z} \quad D_z \,]$, $P_{44} = \mathbf{R} - \left(\mathbf{F} + \mathbf{F}^T\right)$, $P_{51} = \mathcal{L}_{cl}$, $P_{52} = [\, \mathcal{D}_{cl} \quad \mathcal{M}_{cl} \,]$. □

III. Robust Filters for NFT

With the variable $\mathbf{K} = [\, \mathbf{B}_F \quad \mathbf{A}_F \,]$, we write $\mathcal{A}_{cl}(\alpha)$, $\mathcal{B}_{\Delta cl}(\alpha)$ and $\mathcal{B}_{cl}(\alpha)$ in (7) as

$$\begin{aligned} &[\, \mathcal{A}_{cl} \quad [\, \mathcal{B}_{\Delta cl} \quad \mathcal{B}_{cl} \,] \,](\alpha) = \\ &\qquad \sum_{j=1}^{s} \alpha_j \left([\, \Theta A_j \Theta^T \quad \Theta \mathcal{B}_j \,] + \Upsilon \mathbf{K} [\, \mathcal{C}_j \quad \Theta_p \mathcal{D}_j \,] \right) \end{aligned} \tag{24}$$

where $\Upsilon = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}$, $\Theta = \begin{bmatrix} I_n \\ 0_n \end{bmatrix}$, $\Theta_p = \begin{bmatrix} I_p \\ 0_{np} \end{bmatrix}$, $\mathcal{I} = [\, I_n \quad I_n \,]$, $\mathcal{B}_j = [\, B_{\Delta j} \quad B_j \,]$, $\mathcal{C}_j = \begin{bmatrix} C_j & 0_{pn} \\ 0_n & I_n \end{bmatrix}$, $\mathcal{D}_j = [\, D_{\Delta j} \quad D_j \,]$, $\mathcal{D}_{zj} = [\, D_{\Delta zj} \quad D_{zj} \,]$.

To translate (16)–(18) and (22), (23) into LMIs, we have to make some restrictions: $\begin{bmatrix} \mathbf{X}(\alpha) \\ \mathbf{Y}(\alpha) \end{bmatrix} = \sum_{j=1}^{s} \alpha_j \begin{bmatrix} \mathbf{X}_j \\ \mathbf{Y}_j \end{bmatrix}$, $\begin{bmatrix} \mathbf{R}_i(\alpha) \\ \mathbf{S}_i(\alpha) \\ \mathbf{R}_i(\alpha) \\ \mathbf{S}_i(\alpha) \end{bmatrix} = \sum_{j=1}^{s} \alpha_j \begin{bmatrix} \mathbf{R}_{ij} \\ \mathbf{S}_{ij} \\ \mathbf{R}_i \\ \mathbf{S}_i \end{bmatrix}$, $i = 1,2$; $\mathbf{Z}(\alpha) = \sum_{j=1}^{s} \alpha_j \mathbf{Z}_j$, $\mathbf{V}(\alpha) \equiv \mathbf{V}$, $\mathbf{F}_i(\alpha) \equiv \mathbf{F}_i$, $\mathbf{H}_i(\alpha) \equiv \mathbf{H}_i$, $\mathbf{F}(\alpha) \equiv \mathbf{F}$, $\mathbf{H}(\alpha) \equiv \mathbf{H}$ $\forall\, \alpha \in \Gamma$, $i = 1,2$, i.e., the basic variables are linearly parameter-dependent while the slack variables are parameter-independent. As a result, (18), (19), and (23) immediately become LMIs

$$\begin{bmatrix} \mathbf{R}_{ij} & \Delta_j^T \mathbf{H}_i^T \\ \mathbf{H}_i \Delta_j & \mathbf{S}_{ij} + \left(\mathbf{H}_i + \mathbf{H}_i^T\right) \end{bmatrix} \geq 0, \qquad j = 1,2,\ldots,s; \; i = 1,2 \tag{25}$$

$$\operatorname{Trace}\{\mathbf{Z}_j\} < \nu, \qquad j = 1,2,\ldots,s \tag{26}$$

$$\begin{bmatrix} \mathbf{R}_j & \Delta_j^T \mathbf{H}^T \\ \mathbf{H}\Delta_j & \mathbf{S}_j + (\mathbf{H} + \mathbf{H}^T) \end{bmatrix} \geq 0, \qquad j = 1,2,\ldots,s. \tag{27}$$

Section III-A and B will equivalently transform the remaining inequalities (16), (17), and (22) into LMIs via appropriate congruent transformations and variable changes.

A. Robust $\mathcal{H}_2$ Filter

With the partition $\mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix}$, $\mathbf{V}_{ij} \in R^{n\times n}$, it follows from (24) that the only bilinear term in inequalities (16) and (17) is $\mathbf{V}^T \Upsilon \mathbf{K} = \begin{bmatrix} \mathbf{V}_{21}^T \\ \mathbf{V}_{22}^T \end{bmatrix} [\,\mathbf{B}_F \quad \mathbf{A}_F\,]$.

To linearize it let us introduce $\Pi_{\mathbf{V}} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{V}_{22}^{-1}\mathbf{V}_{21} \end{bmatrix}$ and variable changes

$$\hat{\mathbf{X}}_j = \Pi_{\mathbf{V}}^T \mathbf{X}_j \Pi_{\mathbf{V}} \quad \hat{\mathbf{V}} = \begin{bmatrix} \hat{\mathbf{V}}_1 & \hat{\mathbf{V}}_2 \\ \hat{\mathbf{V}}_3 & \hat{\mathbf{V}}_3 \end{bmatrix} = \Pi_{\mathbf{V}}^T \mathbf{V} \Pi_{\mathbf{V}}$$
$$\hat{\mathbf{K}} = [\,\hat{\mathbf{B}}_F \quad \hat{\mathbf{A}}_F\,] = [\,\mathbf{V}_{21}^T \mathbf{B}_F \quad \mathbf{V}_{21}^T \mathbf{A}_F \mathbf{V}_{22}^{-1} \mathbf{V}_{21}\,]. \quad (28)$$

Applying the congruence transformations $\mathrm{diag}\,[\,\Pi_{\mathbf{V}} \quad I \quad I \quad \Pi_{\mathbf{V}} \quad I\,]$ and $\mathrm{diag}\,[\,\Pi_{\mathbf{V}} \quad I \quad I \quad I\,]$ to (16) and (17), respectively, results in

$$\begin{bmatrix} M_{11}^j & * & * & * \\ 0 & M_{22}^j & * & * \\ M_{31}^j & M_{32}^j & M_{33}^j & * \\ M_{41}^j & M_{42}^j & 0 & M_{44}^j \end{bmatrix} < 0, \qquad j = 1, 2, \ldots, s. \quad (29)$$

$$\begin{bmatrix} N_{11}^j & * & * & * \\ 0 & N_{22}^j & * & * \\ N_{31}^j & N_{32}^j & N_{33}^j & * \\ N_{41}^j & N_{42}^j & 0 & -\mathbf{Z}_j \end{bmatrix} < 0, \qquad j = 1, 2, \ldots, s \quad (30)$$

where $\hat{\mathbf{L}}_F = \mathbf{L}_F \mathbf{V}_{22}^{-1} \mathbf{V}_{21}$ and $M_{11}^j = -\hat{\mathbf{X}}_j$, $M_{22}^j = \begin{bmatrix} \mathbf{S}_{1j} & * \\ 0 & -I \end{bmatrix}$, $M_{31}^j = \hat{\mathbf{V}}^T \Theta A_j \Theta^T + \mathcal{I}^T \hat{\mathbf{K}} \mathcal{C}_j$, $M_{32}^j = \hat{\mathbf{V}}^T \Theta \mathcal{B}_j + \mathcal{I}^T \hat{\mathbf{K}} \Theta_p \mathcal{D}_j$, $M_{33}^j = \hat{\mathbf{X}}_j - (\hat{\mathbf{V}} + \hat{\mathbf{V}}^T)$, $M_{41}^j = \mathbf{F}_1 C_{\Delta j} \Theta^T$, $M_{42}^j = \mathbf{F}_1 \mathcal{D}_{zj}$, $M_{44}^j = \mathbf{R}_{1j} - (\mathbf{F}_1 + \mathbf{F}_1^T)$, $N_{11}^j = -\hat{\mathbf{X}}_j$, $N_{22}^j = \begin{bmatrix} \mathbf{S}_{2j} & * \\ 0 & -I \end{bmatrix}$, $N_{31}^j = \mathbf{F}_2 C_{\Delta j} \Theta^T$, $N_{32}^j = \mathbf{F}_2 \mathcal{D}_{zj}$, $N_{33}^j = \mathbf{R}_{2j} - (\mathbf{F}_2 + \mathbf{F}_2^T)$, $N_{41}^j = [\,L_j - \mathbf{D}_F C_j \quad -\hat{\mathbf{L}}_F\,]$, $N_{42}^j = [\,D_{\Delta\Delta j} \quad M_j\,] - \mathbf{D}_F \mathcal{D}_j$ (see [5] for more technical details).

Theorem 3: There is a filter (4) satisfying the estimation criterion (9) whenever the LMIs (25), (26), (29), and (30) are feasible in the decision variables $\hat{\mathbf{V}}, \hat{\mathbf{X}}_j, \mathbf{Z}_j, \mathbf{S}_{ij}, \mathbf{R}_{ij}, \hat{\mathbf{K}}, \hat{\mathbf{L}}_F, \mathbf{D}_F, \mathbf{H}_i, \mathbf{F}_i$. The matrix data $\mathbf{A}_F, \mathbf{B}_F, \mathbf{L}_F, \mathbf{D}_F$ defining the filter (4) can be derived from a solution to matrix inequalities (25), (26), (29) and (30) via the formula

$$\mathbf{A}_F = \hat{\mathbf{A}}_F \hat{\mathbf{V}}_3^{-T} \quad \mathbf{B}_F = \hat{\mathbf{B}}_F \quad \mathbf{L}_F = \hat{\mathbf{L}}_F \hat{\mathbf{V}}_3^{-T}. \quad (31)$$

Proof: Given $\hat{\mathbf{V}}$, $V = \begin{bmatrix} \hat{\mathbf{V}}_1 & \hat{\mathbf{V}}_2 \hat{\mathbf{V}}_3^{-T} \\ I & \hat{\mathbf{V}}_3^{-T} \end{bmatrix}$ satisfies (28), hence, (31) follows easily. □

B. $\mathcal{H}_\infty$ and Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Filters

By arguments similar to those in Section III-A, the theorem below holds.

Theorem 4: There is a filter (4) satisfying the estimation criterion (21) whenever the LMIs (27) and (32) are feasible in $\hat{\mathbf{V}}$, $\hat{\mathbf{Y}}_j, \mathbf{S}_j, \mathbf{R}_j, \hat{\mathbf{K}}, \hat{\mathbf{L}}_F, \mathbf{D}_F, \mathbf{H}, \mathbf{F}$. According to (31), the filter data $\mathbf{A}_F, \mathbf{B}_F, \mathbf{L}_F, \mathbf{D}_F$ defining the filter (4) can be derived from a solution to (27), and the LMI

$$\begin{bmatrix} E_{11}^j & * & * & * & * \\ 0 & E_{22}^j & * & * & * \\ E_{31}^j & E_{32}^j & E_{33}^j & * & * \\ E_{41}^j & E_{42}^j & 0 & E_{44}^j & * \\ E_{51}^j & E_{52}^j & 0 & 0 & -\gamma I \end{bmatrix} < 0, \qquad j = 1, 2, \ldots, s \quad (32)$$

here, $E_{11}^j = -\hat{\mathbf{Y}}_j$, $E_{22}^j = \begin{bmatrix} \mathbf{S}_j & * \\ 0 & -I \end{bmatrix}$, $E_{31}^j = \hat{\mathbf{V}}^T \Theta A_j \Theta^T + \mathcal{I}^T \hat{\mathbf{K}} \mathcal{C}_j$, $E_{32}^j = \hat{\mathbf{V}}^T \Theta B_j + \mathcal{I}^T \hat{\mathbf{K}} \Theta_p \mathcal{D}_j$, $E_{33}^j = \hat{\mathbf{Y}}_j - (\hat{\mathbf{V}} + \hat{\mathbf{V}}^T)$, $E_{41}^j = \mathbf{F} C_{\Delta j} \Theta^T$, $E_{42}^j = \mathbf{F} \mathcal{D}_{zj}$, $E_{44}^j = \mathbf{R}_j - (\mathbf{F} + \mathbf{F}^T)$, $E_{51}^j = [\,L_j - \mathbf{D}_F C_j \quad -\hat{\mathbf{L}}_F\,]$, $E_{52}^j = [\,D_{\Delta\Delta j} \quad M_j\,] - \mathbf{D}_F \mathcal{D}_j$. Consequently, a suboptimal robust filter (4) that solve problem (5) is obtained from the solution to the optimization problem

$$\min_{\rho} \nu + [(1-\rho)\gamma] : (25), (26), (29), (30), (27), (32) \quad (33)$$

with decision variables $\hat{\mathbf{V}}$, $\hat{\mathbf{X}}_j$, $\mathbf{Z}_j$, $\hat{\mathbf{Y}}_j$, $\mathbf{S}_{ij}$, $\mathbf{R}_{ij}$, $\mathbf{S}_j$, $\mathbf{R}_j$, $\hat{\mathbf{K}}$, $\hat{\mathbf{L}}_F$, $\mathbf{D}_F$, $\mathbf{H}_i$, $\mathbf{F}_i$, $\mathbf{H}$, $\mathbf{F}$, ν, and γ via formula (31). □

TABLE I
$\mathcal{H}_2$ PERFORMANCES BY DIFFERENT LMI FORMULATIONS FOR EXAMPLE 1 IN [11]

Formulations	$\mathcal{H}_2$ per.	Imp. ratio
Proposed ones in [11]	7.22 ($\sqrt{52.17}$)	1.173
Ones in [4]	7.22 ($\sqrt{52.17}$)	1.173
Ours/Str. proper filter	7.22 ($\sqrt{52.17}$)	1.173
Ours/Proper filter	0.7870 ($\sqrt{0.6194}$)	10.762

IV. NUMERICAL EXAMPLES WITH MATLAB LMI CONTROL TOOLBOX [2]

A. Polytopic Case

The effectiveness of our LMI formulations and how filter structure (4) can be better than (3) are demonstrated via the solutions to the two plants used as examples in [11]. The simplification of Theorem 3 is used as scaling pairs are no longer needed in this case.

First, consider Example 1 in [11] the upper bound on the $\mathcal{H}_2$ norm of the corresponding plant is 8.47 ($\sqrt{71.68}$). Results are in Table I where improvement ratios correspond to the ratios of the $\mathcal{H}_2$ norm of the plant and the $\mathcal{H}_2$ performances achieved by each robust filter. Clearly, the strictly proper filter structure (3) used in [4], [11] give almost no improvement in comparison with the zero filter (the filter that takes zero as the estimate) while in contrast, a significant improvement is observed with proposed filter structure (4).

Next, we move to Example 2 in [11], accordingly, the upper bound on the $\mathcal{H}_2$ norm of the plant under our considerration is 18.11 ($\sqrt{328.11}$). Results are listed in Table II. Once again, the improvement ratio greater than 29 obtained by our formulation shows the effectiveness of the proper filter structure over the strictly proper one in [11], [4].

TABLE II
$\mathcal{H}_2$ PERFORMANCES BY DIFFERENT LMI FORMULATIONS FOR EXAMPLE 2 IN [11]

Formulations	$\mathcal{H}_2$ per.	Imp. ratio
Ones in [11]	8.39 ($\sqrt{70.40}$)	2.16
Ones in [4]	8.39 ($\sqrt{70.40}$)	2.16
Ours/Str. proper filter	8.39 ($\sqrt{70.40}$)	2.16
Ours/Proper filter	0.6238 ($\sqrt{0.3891}$)	29.03

B. NFT and LFT Cases

Our example demonstrates that different representations and different filter structures (4), (3) may result in dramatically different estimation performances. The plant is

$$\begin{bmatrix} x(k+1) \\ y(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} \tilde{A}(\alpha) & B \\ C & D \\ L & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{34}$$

where $\tilde{A}(\alpha) = Q_0 + \alpha_1^3 Q_1 + \alpha_2^3 Q_2 + \alpha_1\alpha_2^2 Q_3 + \alpha_1 Q_4 + \alpha_2 Q_5$, $Q_0 = \begin{bmatrix} -0.3 & 0.5 \\ 0.2 & -0.1 \end{bmatrix}$, $Q_1 = \begin{bmatrix} 0.1 & 0.15 \\ 0.1 & 0.15 \end{bmatrix}$, $Q_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.25 & 0.25 \end{bmatrix}$, $Q_3 = \begin{bmatrix} 0.2 & 0.15 \\ 0.2 & 0.15 \end{bmatrix}$, $Q_4 = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}$, $Q_5 = \begin{bmatrix} 0.1 & 0 \\ 0.25 & 0.1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 \\ 1.5 & 0 \end{bmatrix}$, $C = [-10 \quad 10]$, $D = [0 \quad 3]$, $L = [1 \quad 0]$.

The LFT representation of plant (34) is in the form (1) with

$$\begin{aligned} A &= Q_0 \quad B_\Delta = [I_2 \quad 0_2 \quad 0_2 \quad I_2 \quad 0_2 \quad 0_2] \\ D_{\Delta z} &= \begin{bmatrix} 0_2 & Q_1 & 0_2 & 0_2 & Q_3 & 0_2 \\ 0_2 & 0_2 & I_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & Q_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & I_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \end{bmatrix} \\ C_\Delta &= \begin{bmatrix} Q_4 \\ 0_2 \\ I_2 \\ Q_5 \\ 0_2 \\ I_2 \end{bmatrix} \\ D_z &= 0 \\ D_\Delta &= 0 \\ D_{\Delta\Delta} &= 0 \\ \Delta(\alpha) &= \begin{bmatrix} \alpha_1 I_6 & 0_6 \\ 0_6 & \alpha_2 I_6 \end{bmatrix}. \end{aligned} \tag{35}$$

Alternatively, its NFT is in the form (1) with

$$\begin{aligned} A(\alpha) &= \alpha_1(Q_0 + Q_4) + \alpha_2(Q_0 + Q_5) \\ B_\Delta(\alpha) &= [\alpha_1 I_2 \quad \alpha_2 I_2] \begin{bmatrix} Q_1 & Q_3 \\ O & Q_2 \end{bmatrix} \\ \Delta(\alpha) &= \begin{bmatrix} \alpha_1 I_2 & 0_2 \\ 0_2 & \alpha_2 I_2 \end{bmatrix} \\ D_{\Delta z} &= 0 \\ C_\Delta(\alpha) &= \begin{bmatrix} \alpha_1 I_2 \\ \alpha_2 I_2 \end{bmatrix} \\ D_z &= 0 \\ D_\Delta &= 0 \\ D_{\Delta\Delta} &= 0. \end{aligned} \tag{36}$$

TABLE III
$\mathcal{H}_2$ PERFORMANCES OF FILTERS WITH FILTER STRUCTURES AND SYSTEM REPRESENTATIONS

Model/Filter	$\mathcal{H}_2$ per.	Com. time	Imp. ratio
NFT/Proper	0.3904	7.67 sec	6.7649
LFT/Proper	1.0995	19140 sec	2.402
NFT/Str. proper	2.038	4.93 sec	1.2959

TABLE IV
$\mathcal{H}_\infty$ PERFORMANCES OF FILTERS WITH FILTER STRUCTURES AND SYSTEM REPRESENTATIONS

Model/Filter	$\mathcal{H}_\infty$ per.	Comp. time	Imp. ratio
NFT/Proper	0.7853	3.2 sec	7.1193
LFT/Proper	2.2715	2755 sec	2.4613
NFT/Str. proper	2.2556	2.9 sec	2.4786

TABLE V
PERFORMANCES OF MIXED FILTERS FOR NFT MODEL (1), (36) BY TRADE-OFF CONSTANTS (ρ)

ρ	mixed per.	$\mathcal{H}_2$	$\mathcal{H}_\infty$
0.1	0.5758	0.4555	0.7855
0.3	0.4925	0.4390	0.7880
0.5	0.4042	0.4205	0.7947
0.7	0.3103	0.4063	0.8056
0.9	0.2098	0.3963	0.8274

The dimension 12 of z_Δ in LFT (1), (35) is three times greater than that of the NFT (1), (36), severely affecting the computational efficiency and the estimation performances of the filters as described in Tables III and IV. Table V lists mixed performances as well as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances of mixed filters corresponding to different tradeoff constants ρ. We also consider strictly proper filters (3) with the NFT (1), (36). Computed performances are also shown in Tables III and IV. Tables all reveal the performance improvements due to the proper filter structure. Note that improvement ratios are defined as before and the upper bounds on the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of this plant are 2.641 and 5.5908, respectively.

V. CONCLUSION

In this paper, we have developed new techniques to design robust filters which minimize the estimation error in the sense of the $\mathcal{H}_2$ norm, the $\mathcal{H}_\infty$ norm or a prescribed combination of these norms. The proposed techniques are applicable to a wide range of uncertain systems admitting an NFT representation. The resulting design procedure reduces to solving LMIs; thus, it is highly practical. Finally, the validity and power of this procedure have been demonstrated on a number of numerical examples.

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