Gain-Scheduled $\mathcal{H}_\infty$ Control of a Missile via Linear Matrix Inequalities

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Abstract

This paper is concerned with the application of advanced Linear Parameter-Varying (LPV) techniques to the global control of a missile. The LPV technique considered in this paper is an extension of the standard $\mathcal{H}_\infty$ synthesis technique to the case where the plant depends affinely on a time-varying vector $\theta(t)$. Working in the class of LPV plants, the proposed methodology produces an LPV controller. That is, a controller which is automatically "gain-scheduled" along the trajectories of the plant. LPV controllers solutions to the problem are characterized via a set of Riccati Linear Matrix Inequalities (LMI) which can be solved using convex programming.

The missile under consideration is highly non-linear and non-stationary. Its LPV model exhibits brutal parameter variations as functions of the flight conditions (angle of attack, speed, altitude). Additionally, some measurements are corrupted by flexible modes and the performance requirements are very strong. Since they obliterate the LPV nature of the plant, usual Linear Time Invariant techniques may be helpless for this problem. The power and advantages of the proposed methodology as an efficient tool to handle the global performances and robustness of the missile on its whole operating range are demonstrated.

1 Introduction and Motivations

Three important classes of linear systems can be investigated:

LTI Linear Time-Invariant systems which is the more familiar class in the control system literature and give rise to the most successful and self-content theories. LTI plant are described in state-space form as

$$\begin{aligned}\dot{x} &= A\,x + B\,u \\ y &= C\,x + D\,u.\end{aligned}$$

LTV Linear Time-Varying systems for which the theory is not so mature as for LTI systems and consequently potential applications are more restricted to special problems. The state-space description of LTV systems is completely defined by the functional time-dependence of the state-space data $A(t)$, $B(t)$, $C(t)$ and $D(t)$ whose associated differential equations are

$$\begin{aligned}\dot{x} &= A(t)\,x + B(t)\,u \\ y &= C(t)\,x + D(t)\,u.\end{aligned}$$

LPV Linear Parameter-Varying systems are linear systems where the state-space entries $A(.)$, $B(.)$, $C(.)$ and $D(.)$ are now explicit functions of a time-varying parameter $\theta(t)$. They have therefore a differential representation in the form

$$\dot{x} = A(\theta(t))\,x + B(\theta(t))\,u \tag{1.1}$$

$$y = C(\theta(t))\,x + D(\theta(t))\,u. \tag{1.2}$$

It also must be pointed out that an LPV system reduces to an LTV system for a given trajectory $\theta := \theta(t)$ and is transformed into an LTI system on constant trajectories $\theta := \theta_0 \;\; \forall t \geq 0$. It follows that for frozen value of the parameter, LPV systems can be analyzed as LTI systems. Though the LPV properties (i.e. with θ time-varying) of an LPV system cannot generally be inferred from its underlying LTI properties, they provide useful tools that are part of the engineering practice.

Despite formal analogies, LTI, LTV and LPV systems have fundamental differences. One of the most important is the "off-line" or "in-line" natures of such systems. Specifically, the LTI and LTV classes are composed of "off-line" systems in the sense that the state-space data A, B, C and D or $A(t)$, $B(t)$, $C(t)$ and $D(t)$ must be known beforehand. In contrast, LPV systems are "in-line" systems as they are completely known when the trajectory $\theta := \theta(t)$ is known. That is, when the plant is operating and experiences a particular trajectory in its domain. In other words, the study of LPV systems require the anticipation of the future behavior of the plant and thus has immediate advantages for real-world applications. Another notion intimately related to LPV systems is that of operating domain. The operating domain Θ of an LPV system (1.1)-(1.2) is the parameter range of the system, that is the domain of the parameter space where it takes its trajectories $\theta(t)$. Summing up the above remarks, an LPV system is well-defined whenever its parameter-dependence and its operating domain are fixed. This can be expressed more formally by the set of relations

$$\begin{aligned}\dot{x} &= A(\theta(t))\,x + B(\theta(t))\,u \\ y &= C(\theta(t))\,x + D(\theta(t))\,u \\ \theta(t) &\in \Theta, \quad \forall t \geq 0.\end{aligned}$$

Moreover, LPV systems can be given at least three interesting interpretations:

- they can be viewed as linear systems subject to uncertainties $\theta(t)$. As a consequence an adequate controller must be robust in the face of parameter uncertainties. The controller $K(s)$ is generally LTI and one has to solve a **robust control problem** (see Figure 1).

- they can be viewed as a family of linear systems issues from the linearization of a non-linear plant or more simply as parameter-dependent systems where the parameter values are directly exploitable in the control structure. This leads to a **scheduling problem** in a general sense (see Figure 2).

- mixing the above two situations one comes up to the more general case where the parameter vector θ can be partitioned as $\theta := (\theta_1; \theta_2)^T$ where θ_1 is known in real-time and θ_2 is an uncertain parameter. It must be emphasized that most practical control problems enters this last situation (see Figure 3).

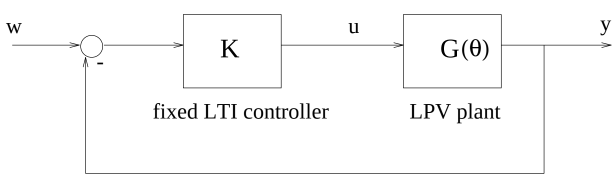

Figure 1: LTI Control of a LPV System

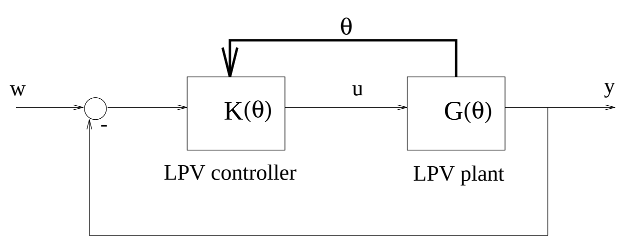

Figure 2: LPV Control of a LPV System

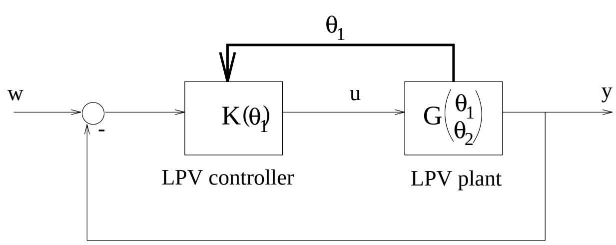

Figure 3: LPV Control of a mixed LPV System

Therefore, the notion of LPV system provides a formalization of various important issues of the control design problem. In our opinion, one of the most significant potentiality of the LPV framework is the derivation of LPV or "self-scheduled" controllers (see Figure 2). Such controllers have the same parameter-dependence as the plant and can be described in state-space form as

$$\begin{aligned} \dot{x} &= A_K(\theta(t))\, x + B_K(\theta(t))\, y \\ u &= C_K(\theta(t))\, x + D_K(\theta(t))\, y \end{aligned}$$

The improvements over classical LTI designs that can be expected from using this new control structure are the following

Stability and Performance. Some LPV systems are not even stabilizable via a single LTI controller. The LPV control structure is potentially more powerful for solving such problems since it incorporates the parameter measurements. The same remarks apply to the performance objectives. Higher performances can generally be achieved with a controller that adjusts to the true plant dynamics.

Robustness. It is widely accepted that the design problem is primary a tradeoff between performance and robustness objectives. Since the nominal performance objectives are easier to satisfy with an LPV controller, it turns out that the compromise between performance and robustness is made more feasible with this class of controllers.

Globality. The classical approach to the control of LPV systems proceeds by

1. selecting a number of operating conditions of the LPV plant
2. designing LTI controllers for the selected points
3. scheduling or interpolating the LTI controllers to derive the global control law of the original LPV plant.

It must pointed out that the steps 1 and 2 are critical. As a result, the final control law, though working locally, does not provide any guarantee concerning the LPV behavior of the system. That is, when the parameter θ is really varying in time. The advantages of a synthesis technique working in the larger class of LPV systems appear clearly. It allows to bypass the critical phases of the classical approach. Indeed, the whole operating range is handled in "one-shot" without requiring repeated designs and scheduling.

From a practical point of view, the missile control design remains a very challenging task. Indeed, a missile is a nonlinear and rapidly parameter-varying plant operating in a large range of aerodynamic conditions. In addition to strong performance requirements, the design methodology has to meet two different kinds of objectives.

- It must provide adequate margins (gain, phase, delay) and flexible modes attenuation when the system is analyzed as an LTI system.
- It must handle a large operating domain and therefore must provide some kind of adjustment to the current plant dynamics.

Modern syntheses methodologies as those considered in references [1, 2, 3] give a systematic answer to the first type of objectives. However, the gain scheduling task becomes obscure because of the complexity (number of states) of the resulting controllers. In opposite, adaptive techniques [4, 5] integrate the gain scheduling aspects but are often inadequate to address LTI robustness specifications. In short, these techniques captures one particular design concern but fail to integrate both robust control and gain scheduling. The central idea leading to the development of LPV control techniques [6, 7, 8, 9, 10, 11] is to address simultaneously both problems within the same design methodology.

The remainder of the paper is organized as follows. Section 2 gives some fairly standard notations and definitions used throughout the paper. Section 3 is a brief review of the $\mathcal{H}_\infty$ synthesis technique for LPV systems. LMI characterizations of solutions as well as the LPV controller reconstruction are recapped in this section. Finally, the application of the LPV design technique to the missile control problem is detailed in Section **??**. Specifically, LPV simulations are presented to highlight the power and performance of the proposed methodology.

2 Notations and Definitions

Throughout the paper, matrix transfer functions will be denoted $P(s)$ where s stands for the Laplace variable.

For a stable real-rational transfer function $P(s)$, the $\mathcal{H}_\infty$ norm is defined in the usual way:

$$\|P(s)\|_\infty = \sup_{\omega \in \mathbf{R}} \sigma_{max}(P(j\omega))$$

where $\sigma_{max}(M)$ stands for the largest singular value of a matrix M. For real symmetric matrices M, the notation $M > 0$ stands for "positive definite" and indicates that all the eigenvalues of M are positive. Similarly, $M < 0$ means "negative definite", that is, all the eigenvalues of M are negative.

Matrix polytopes are defined as the convex hull of a finite number of matrices N_i with the same dimensions. That is,

$$Co\,\{N_i \;:\; i = 1, ..., r\} := \{\sum_{i=1}^{r} \alpha_i N_i :\; \alpha_i \geq 0,\; \sum_{i=1}^{r} \alpha_i = 1\;\}$$

3 $\mathcal{H}_\infty$ Control of LPV Systems: A Brief Review

This section is a brief review of the main results of the $\mathcal{H}_\infty$ synthesis technique for LPV systems. The reader is referred to [11] for further details and complete solutions of both the continuous- and discrete-time problems.

3.1 Preliminary Notions

We are interested in a particular class of LPV systems where

(a) the parameter-dependence is affine, that is, the state-space matrices $A(\theta(t))$, $B(\theta(t))$, $C(\theta(t))$, $D(\theta(t))$ depend affinely on $\theta(t)$,

(b) the time-varying parameter $\theta(t)$ varies in a polytope Θ of vertices $\theta_1, \theta_2, \ldots, \theta_r$. That is,

$$\theta(t) \in \ \Theta := Co\,\{\theta_1, \theta_2, \ldots, \theta_r\}.$$

These vertices represent the extremal values of the parameters.

Though not fully general, this description encompasses many practical situations. From this characterization, it is clear that the state-space matrices $A(\theta(t))$, $B(\theta(t))$, $C(\theta(t))$, $D(\theta(t))$ evolve in a polytope of matrices whose vertices are the images of the vertices $\theta_1, \theta_2, \ldots, \theta_r$. In other words,

$$\begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} \in Co\left\{\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} : \ i = 1, ..., r\right\} \tag{3.1}$$

where

$$\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} := \begin{pmatrix} A(\theta_i) & B(\theta_i) \\ C(\theta_i) & D(\theta_i) \end{pmatrix}.$$

Because of this property, and with a slight abuse of language, we will refer to such LPV plants as "polytopic" in the sequel.

With these notations in mind, the $\mathcal{H}_\infty$ control problem for LPV systems parallels the customary $\mathcal{H}_\infty$ synthesis where the $\mathcal{H}_\infty$-norm bound is replaced by the more suitable notion of *Quadratic* $\mathcal{H}_\infty$ *Performance* and the plant under consideration is now LPV.

Definition 3.1 (Quadratic $\mathcal{H}_\infty$ Performance) *The LPV system*

$$\dot{x} = A(\theta(t))x + B(\theta(t))u \tag{3.2}$$
$$y = C(\theta(t))x + D(\theta(t))u \tag{3.3}$$

has Quadratic $\mathcal{H}_\infty$ Performance γ if and only if there exists a Lyapunov function $V(x) = x^T X x$ with $X > 0$ that establishes global stability and the $\mathcal{L}_2$ gain of the input/output map is bounded by γ. That is,

$$\|y\|_2 < \gamma \|u\|_2$$

along all possible parameter trajectories $\theta(t)$ in Θ.

■

3.2 Problem Description

We consider an augmented LPV plant mapping exogenous inputs w and control inputs u to controlled outputs z and measured outputs y, i.e.,

$$\begin{array}{rcl} \dot{x} & = & A(\theta(t))x + B_1(\theta(t))w + B_2(\theta(t))u \\ z & = & C_1(\theta(t))x + D_{11}(\theta(t))w + D_{12}(\theta(t))u \\ y & = & C_2(\theta(t))x + D_{21}(\theta(t))w + D_{22}(\theta(t))u \\ \theta(t) & \in & \Theta := Co\,\{\theta_1, \theta_2, \ldots, \theta_r\}, \quad \forall t \geq 0 \end{array} \tag{3.4}$$

The plant is further assumed to be polytopic, i.e

$$\begin{pmatrix} A(\theta(t)) & B_1(\theta(t)) & B_2(\theta(t)) \\ C_1(\theta(t)) & D_{11}(\theta(t)) & D_{12}(\theta(t)) \\ C_2(\theta(t)) & D_{21}(\theta(t)) & D_{22}(\theta(t)) \end{pmatrix} \in \mathcal{P} := Co \left\{ \begin{pmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & D_{11i} & D_{12i} \\ C_{2i} & D_{21i} & D_{22i} \end{pmatrix}, \; i = 1, 2, \ldots, r \right\} \tag{3.5}$$

where $A_i, B_{1i}, \ldots$ denote the values of $A(\theta(t)), B_1(\theta(t)), \ldots$ at the vertices $\theta(t) = \theta_i$ of the parameter polytope. The problem dimensions are given by

$$A(\theta(t)) \in \mathbf{R}^{n \times n}, \; D_{11}(\theta(t)) \in \mathbf{R}^{p_1 \times m_1}, \; D_{22}(\theta(t)) \in \mathbf{R}^{p_2 \times m_2} \tag{3.6}$$

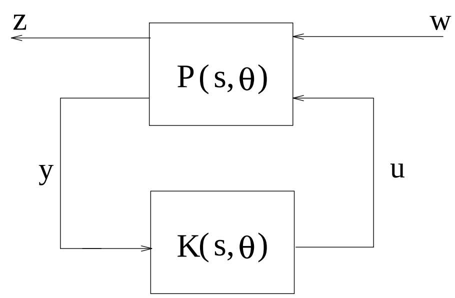

Figure 4: $\mathcal{H}_\infty$ Synthesis Structure for LPV systems

The assumptions on the plant are as follows:

(A1) $D_{22}(\theta(t)) = 0$ or equivalently $D_{22i} = 0$ for $i = 1, 2, \ldots, r$,

(A2) $B_2(\theta(t)), C_2(\theta(t)), D_{12}(\theta(t)), D_{21}(\theta(t))$ are parameter-independent or equivalently,

$$B_{2i} = B_2, \; C_{2i} = C_2, \; D_{12i} = D_{12}, \; D_{21i} = D_{21} \qquad (i = 1, 2, \ldots, r). \tag{3.7}$$

(A3) The pairs $(A(\theta), B_2)$ and $(A(\theta), C_2)$ are quadratically stabilizable and quadratically detectable over Θ, respectively.

It must be noted that assumptions (A1)-(A2) can be alleviated using very simple manipulations. See [11] for more details.

With this notations and assumptions in mind, the $\mathcal{H}_\infty$ control problem for LPV systems can be stated as follows.

Problem Statement 3.2 ($\mathcal{H}_\infty$ Control Problem of LPV Systems) *Find a LPV Controller of the form*

$$\begin{aligned} \dot{x} &= A_K(\theta(t))\, x + B_K(\theta(t))\, y \\ u &= C_K(\theta(t))\, x + D_K(\theta(t))\, y \end{aligned}$$

which guarantees Quadratic $\mathcal{H}_\infty$ Performance γ for the closed-loop system of Figure 4 (see Definition 3.1). This will ensure that

- *the closed-loop system is quadratically stable over Θ,*
- *the $\mathcal{L}_2$-induced of the operator mapping w into z is bounded by γ for all possible trajectories $\theta(t)$ in Θ.*

■

3.3 Characterization of Solutions

The following Theorem taken from [11] presents complete solvability conditions for the $\mathcal{H}_\infty$ control problem of LPV systems in the form of a set of LMI's.

Theorem 3.3 (Existence Conditions)
Consider a continuous LPV polytopic plant (3.4) and assume **(A1)-(A3)**. *Let $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null space of $\left(B_2^T, D_{12}^T\right)$ and (C_2, D_{21}), respectively.*

There exists an LPV controller guaranteeing Quadratic $\mathcal{H}_\infty$ Performance γ along all parameter trajectories in the polytope Θ if and only if there exist two symmetric matrices (R, S) in $\mathbf{R}^{n\times n}$ satisfying the system of $2r+1$ LMI's:

$$\left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} A_i R + R A_i^T & R C_{1i}^T & B_{1i} \\ C_{1i} R & -\gamma I & D_{11i} \\ \hline B_{1i}^T & D_{11i}^T & -\gamma I \end{array}\right) \left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right) < 0 \qquad (i = 1, \dots, r) \quad (3.8)$$

$$\left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} A_i^T S + S A_i & S B_{1i} & C_{1i}^T \\ B_{1i}^T S & -\gamma I & D_{11i}^T \\ \hline C_{1i} & D_{11i} & -\gamma I \end{array}\right) \left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right) < 0 \qquad (i = 1, \dots, r) \quad (3.9)$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0. \qquad (3.10)$$

■

This Theorem only gives existence conditions. Since these conditions are LMI's in the variables R and S, they are convex and fall into the scope of efficient convex optimization techniques. For γ, R and S solutions to the LMI's (3.8)-(3.10), there always exist LPV polytopic controllers solving the problem. In turn, such controllers are described by a system of LMI's from which one can extract a particular solution by simple algebraic manipulations [11]. More precisely, along some trajectory $\theta(t)$ in the polytope Θ, i.e.

$$\theta(t) = \sum_{i=1}^{r} \alpha_i(t) \theta_i \qquad (3.11)$$

the state-space matrices $A_K(\theta(t)), B_K(\theta(t)), C_K(\theta(t)), D_K(\theta(t))$ of the LPV polytopic controllers read

$$\begin{pmatrix} A_K(\theta(t)) & B_K(\theta(t)) \\ C_K(\theta(t)) & D_K(\theta(t)) \end{pmatrix} := \sum_{i=1}^{r} \alpha_i(t) \begin{pmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{pmatrix},$$

where the α_i's are computed according to the convex decomposition (3.11).
It is important to note that while the state-space data A_{Ki}, B_{Ki}, C_{Ki}, D_{Ki} can be computed off-line, the LPV controller matrices $A_K(\theta), B_K(\theta), C_K(\theta), D_K(\theta)$ must be updated in real time depending on the parameter measurement $\theta(t)$.

4 Gain-Scheduled $\mathcal{H}_\infty$ Control of a Missile

This section presents a complete application of the LPV synthesis technique developed in Section 3 to the control of a missile pitch axis.

The missile dynamics are highly depending on the angle of attack α, the velocity V and the altitude H. These three variables completely define the flight condition or operating point of the missile. They are assumed to be measured in real-time. Therefore, based on the linearization of the missile equations around its flight conditions, a LPV representation can be developed. The gain-scheduled $\mathcal{H}_\infty$ control presented above is then immediately applicable to the problem under consideration.

Before going further, an open-loop analysis of the missile pitch channel is presented. As a result of this preliminary analysis, the motivations for using LPV control appear clearly. Then follow the problem description, the control law development and assessment.

4.1 Open-loop analysis

It is important to note that it is implicitly assumed that the pitch, yaw and roll channels are decoupled. Though this assumption ignores some coupling phenomenons of the missile, it greatly simplifies the design task and retains all the central (LPV) difficulties of the problem. The pitch axis model of the missile is depicted in the block-diagram of Figure 5.

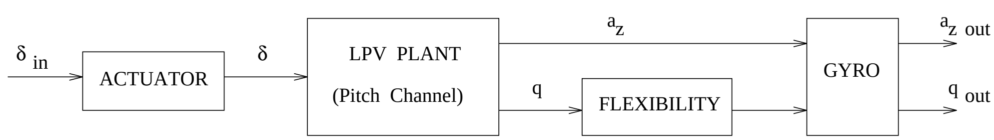

Figure 5: Block-Diagram General Description

The associated linearized dynamics of the missile (LPV part) are described by a state-space representation in the form

$$\left\{ \begin{array}{rcl} \begin{pmatrix} \dot{\alpha} \\ \dot{q} \end{pmatrix} & = & \begin{pmatrix} -Z_\alpha & 1 \\ -M_\alpha & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ M_{\delta m} \end{pmatrix} .\delta_l \\ \begin{pmatrix} a_z \\ q \end{pmatrix} & = & \begin{pmatrix} -Z_\alpha V & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ q \end{pmatrix} \end{array} \right. , \qquad (4.12)$$

where α denotes the angle of attack, q is the pitch rate, a_z is the vertical acceleration, δ_m denotes the fin deflection and V is the air speed.
The varying parameters Z_α, M_α, $M_{\delta m}$, and $Z_{\delta m}$ are functions of the flight condition (α, V, H)

and are therefore available to measurement in real-time. It turns out that the problem can be further simplified by incorporating the parameter-dependence of the input and measurement matrices into the LPV controller. This yields a simplified state-space LPV representation of the missile where the input and measurement matrices are no longer parameter-dependent:

$$\left\{ \begin{array}{lcl} \begin{pmatrix} \dot{\alpha} \\ \dot{q} \end{pmatrix} & = & \begin{pmatrix} -Z_\alpha & 1 \\ -M_\alpha & 0 \end{pmatrix} . \begin{pmatrix} \alpha \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} .\delta_l \\ \begin{pmatrix} a_{zv} \\ q \end{pmatrix} & = & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} \alpha \\ q \end{pmatrix} \end{array} \right. \tag{4.13}$$

where a_{zv} is the normalized acceleration, i.e.

$$a_{zv} = \frac{a_z}{\mathbf{Z}_{alpha}.V}$$

. In this simpler form the LPV description of the missile satisfies assumptions (A1)-(A2) and the proposed synthesis technique is directly applicable.

Though the parameter-dependence of the original plant has been simplified, the control of the missile dynamics remains a hard task. Indeed, the parameter M_α and Z_α abruptly change as functions of the flight condition and range over a large operating domain where the stability properties of the missile are greatly influenced. Moreover, the system can switch between stability and high unstability depending on the sign of the parameter M_α. Analyzed as an LTI plant, the characteristic polynomial of the plant 4.13 reads $p^2 + Z_\alpha p + M_\alpha$. It follows that the plant is LTI unstable whenever M_α is negative. While the parameter Z_α is less critical and influences the damping.

The missile speed varies between $Mach$ 0.5 and $Mach$ 4. The altitude belongs to the interval [0, 18000] ($m.$) between while the angle of attack evolves between 0 and 40 degrees. A large parameter range obviously results from this large operating domain. Moreover, a small increase in the angle of attack may induce large parameters variations. The parameter range is defined by $[-365,\ 380]$ for M_α and $[0.35,\ 4.35]$ for Z_α. An immediate stability analysis reveals that the operating domain can be separated into three regions with different stability properties. See Figure 6.

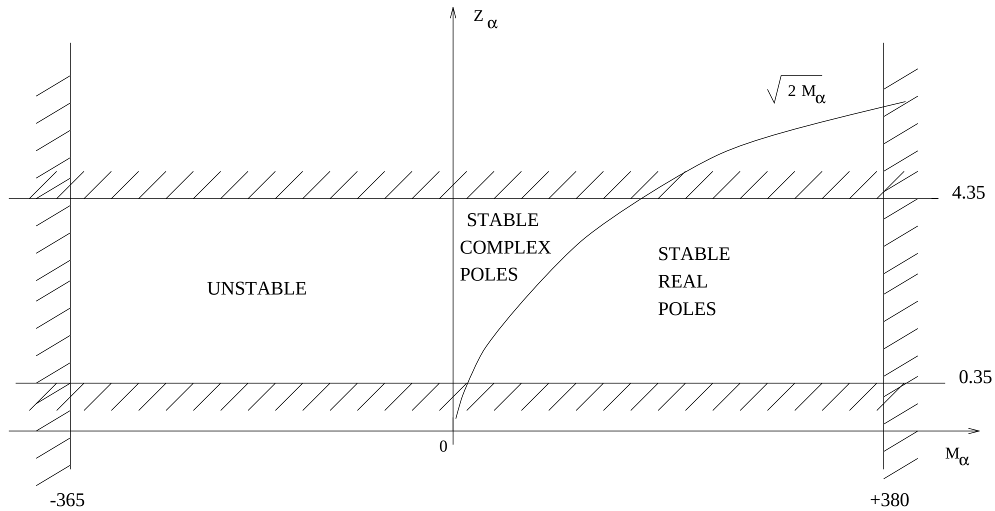

Figure 6: Stability Regions of the LPV Plant as an LTI System

For future validations of the final LPV controller 25 operating points have been selected in the parameter range. These points are regularly distributed in each of the three regions. Figure 7 displays the poles of the associated 25 LTI models.

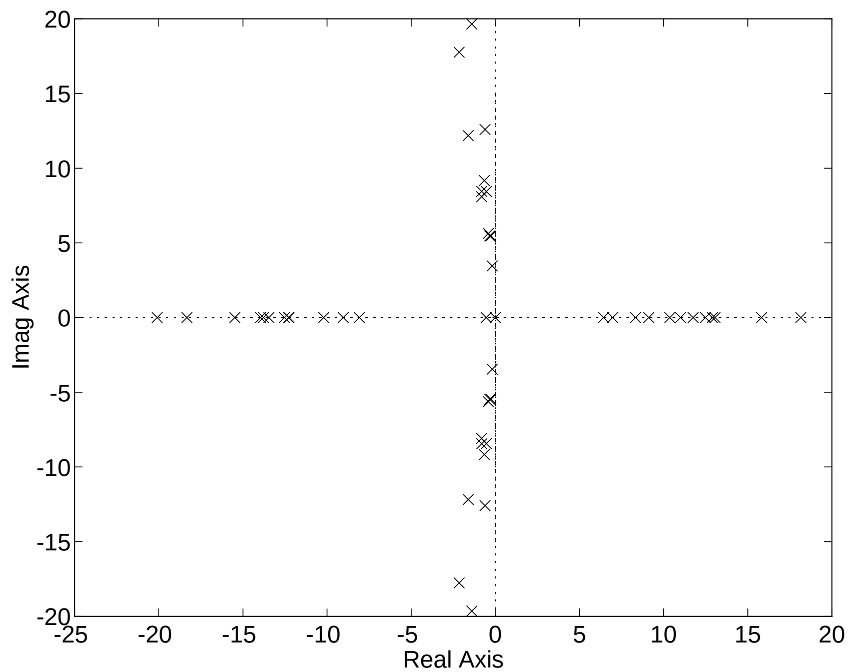

Figure 7: Evolution of the Poles

It appears clearly from Figure 7 that even in the pure LTI case the missile control remains difficult. Indeed, some flight conditions are highly unstable, some others correspond to an oscillating behaviour of the system. Hence, the search of a single LTI controller providing adequate robustness and performances for all these operating conditions appears rather unfeasible. Actually, $\mathcal{H}_\infty$ or μ syntheses techniques fail to give an answer to that kind of problem. These difficulties are enforced by the LPV or non-stationary nature of the missile. Eventhough an adequate LTI controller exist, it would not necessarily guarantee satisfying robustness and performances when the parameters are rapidly varying.

Actuator, Gyro and Flexibility

As already mentioned, in addition to the LPV plant dynamics, tail-deflection actuators, gyros and bending flexible modes must be integrated into the overall model of the missile. The gyros and actuators are adequately represented by second-order and third-order transfer functions, respectively. Besides, the flexible modes are modelized as an additive LTI perturbation affecting the measurement of the pitch rate q. In fact, the flexible should be modelized as a LPV system, since their frequency is varying during the evolution of the missile. To further simplified the problem, the worst case situation has been considered. That is, the greatest pic at the lowest frequency. The resulting frequency response is shown in Figure 8.

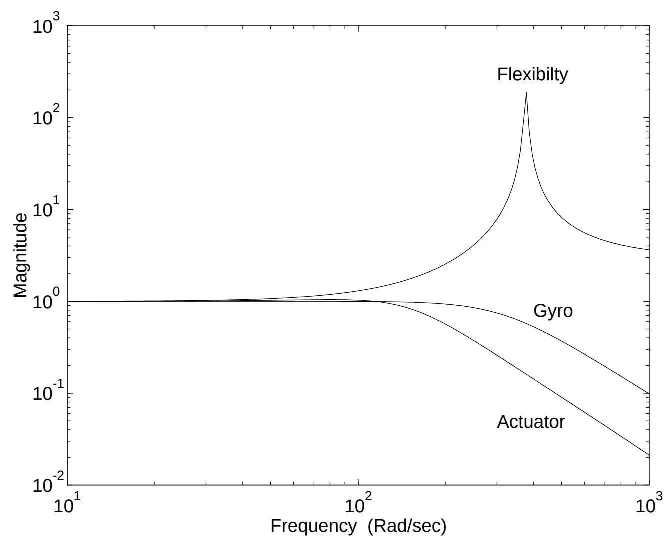

Figure 8: Actuator,gyro and flexibility transfer functions

4.1.1 LPV Synthesis Model

In this application, a two-degree-of-freedom synthesis structure is considered. This structure includes a feedforward K_2 and a feedback K_1 and is potentially more powerful to achieve strong performance requirements.

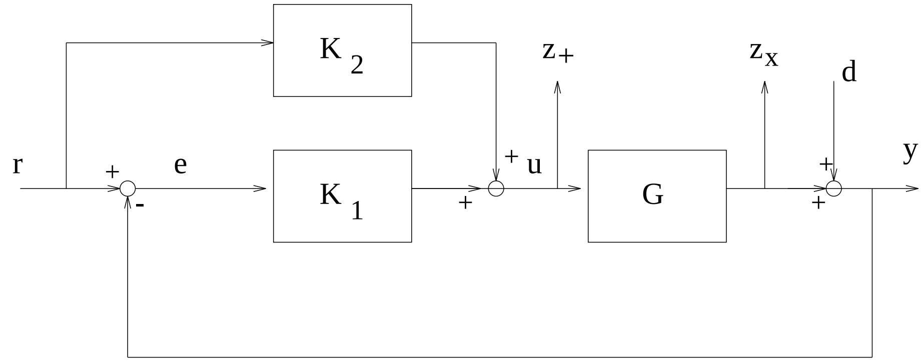

Figure 9: Feedback and feedforward control structure

The design process completely parallels usual $\mathcal{H}_\infty$ syntheses except that the operators to be minimized are now parameter-dependent. Thus, the minimization must handle all possible trajectories in the operating domain. Here, we considere a mixed sensitivity problem suitably adapted to our particular control structure Figure 9. The performance objectives are expressed through the sensitivity operator S while additive and multiplicative robustness objectives are captured by the operators $K.S$ and T, respectively.

- $T_{re}(s,\theta(t)) := S = (I+GK)^{-1}$
- $T_{dz_x}(s,\theta(t)) := T = GK(I+GK)^{-1}$
- $T_{rz_+}(s,\theta(t)) := KS = K(I+GK)^{-1}$

with:
$K := (K1 \;\; K2)$
$u = K \begin{pmatrix} e \\ r \end{pmatrix}$

and the $\mathcal{H}_\infty$ LPV control problem consists of finding a controller $K(s,\theta(t)) := (K1(s,\theta(t)) \;\; K2(s,\theta(t)))$ which satisfies for any admissible parameter trajectory $\theta(t)$ the following two conditions:

- Internal stability of the closed-loop system
- γ minimization such that: $\left\| \begin{pmatrix} W_1(s).T_{re}(s,\theta(t) \\ W_2(s).T_{dz_x}(s,\theta(t) \\ W_3(s).T_{rz_+}(s,\theta(t) \end{pmatrix} \right\|_\infty < \gamma.$

where W_1, W_2 and W_3 are weighting functions, the choice of which requires several trials.

The second condition can be re-written using the augmented LPV plant $P(s,\theta(t))$ mapping exogenous and control inputs to controlled and measured outputs represented on figure 10.

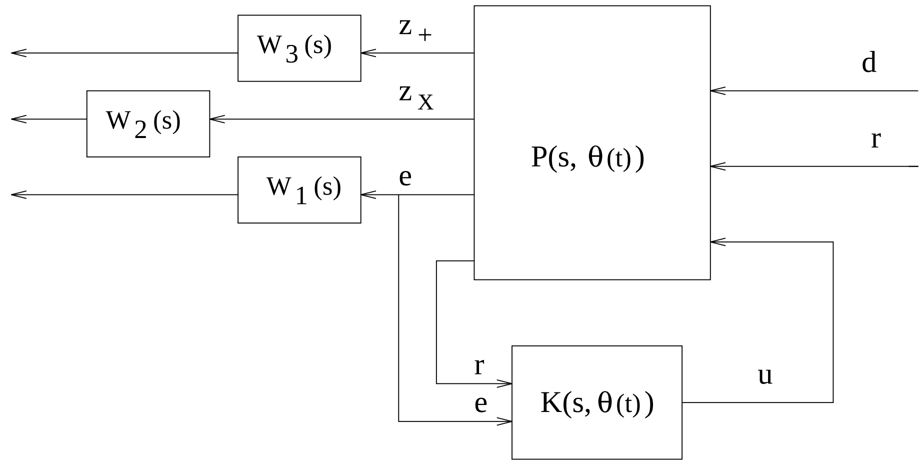

Figure 10: Feedback and feedforward control structure

From figure 9 the following equations are straightforward:

$$
\begin{array}{rcl}
\dot{x} &=& A.x + 0.r + 0.d + B.u \\
z_{+} &=& 0.x + 0.r + 0.d + D.u \\
z_x &=& C.x + 0.r + 0.d + D.u \\
e &=& -C.x + I.r - I.d - D.u \\
r &=& 0.x + I.r + 0.d + 0.u
\end{array}
\tag{4.14}
$$

Then, the interconnexion matrix $P(s, \theta(t))$ is given by:

$$
P(s, \theta(t)) = \begin{pmatrix} A(\theta(t)) & 0 & 0 & B \\ 0 & 0 & 0 & I \\ C & 0 & 0 & D \\ -C & I & -I & -D \\ 0 & I & 0 & 0 \end{pmatrix}
\tag{4.15}
$$

which may be partitioned as follows:

$$
P(s, \theta(t)) = \begin{pmatrix} A(\theta(t)) & B1 & B2 \\ C1 & D11 & D12 \\ C2 & D21 & D22 \end{pmatrix}
\tag{4.16}
$$

with:

$$
A(\theta(t)) = \begin{pmatrix} -Z_\alpha(t) & 1 \\ -M_\alpha(t) & 0 \end{pmatrix}
\tag{4.17}
$$

$$
B_1 = \begin{pmatrix} 0_{2\times 2} & 0_{2\times 2} \end{pmatrix}, \; B2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; C_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \; C_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}
\tag{4.18}
$$

$$D_{11} = \begin{pmatrix} 0_{1\times2} & 0_{1\times2} \\ 0_{2\times2} & 0_{2\times2} \end{pmatrix},\ D_{12} = \begin{pmatrix} 1 \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ D_{21} = \begin{pmatrix} I_{2\times2} & -I_{2\times2} \\ I_{2\times2} & 0_{2\times2} \end{pmatrix},\ D_{22} = \begin{pmatrix} -D \\ 0_{2\times1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.19}$$

Convex Polytope

Assuming that the augmented plant $\P(s,\theta(t))$ is polytopic, the LPV control problem including weighting functions can be described as the convex hull of four matrices. Each matrix corresponds to extremal values of the two varying parameters (M_α and Z_α).

$$P_W(s,\theta(t)) = Co\left\{ \begin{pmatrix} W_1(s) & & \\ & W_2(s) & \\ & & W_3(s) \end{pmatrix} \cdot \begin{pmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & D_{11i} & D_{12i} \\ B_{2i} & D_{21i} & D_{22i} \end{pmatrix}, i = 1,2,3,4 \right\} \tag{4.20}$$

with : $A_1 = A(Z_{\alpha min}, M_{\alpha min}) = \begin{pmatrix} -0.35 & 1 \\ 365 & 0 \end{pmatrix}$

$A_2 = A(Z_{\alpha min}, M_{\alpha max}) = \begin{pmatrix} -0.35 & 1 \\ -385 & 0 \end{pmatrix}$

$A_3 = A(Z_{\alpha max}, M_{\alpha min}) = \begin{pmatrix} -4.35 & 1 \\ 365 & 0 \end{pmatrix}$

$A_4 = A(Z_{\alpha max}, M_{\alpha max}) = \begin{pmatrix} -4.35 & 1 \\ -385 & 0 \end{pmatrix}$

and, $\forall i = 1,2,3,4$:

$B_{1i} = B_1, \qquad B_{2i} = B_2$

$C_{1i} = C_1, \qquad C_{2i} = C_2$

$D_{11i} = D_{11}, \qquad D_{12i} = D_{12}$

$D_{21i} = D_{21}, \qquad D_{22i} = D_{22}$

Given a set of weighting functions $\{W_1(s), W_2(s), W_3(s)\}$, there exists (see [???????]) an LPV polytopic controller minimizing the quadratic $\mathcal{H}_\infty$ performance γ. The choice of these frequency weightings is then all the more important that it determines the required performance and robustness goals. However, these specified goals have to be reachable, which means that the optimized γ must remains in a neighborhood of 1. Therefore, a compromise has to be found.

Weighting functions

After many trials, it has appeared that γ-minimization could not be efficiently achieved as long as $W_2(s)$ would not be null. Only $W_1(s)$ (for performance achievements) and $W_3(s)$ (for high frequency gain limitation) were kept to perform synthesis. It cannot be hidden that the structure choice and coefficients adjustments of these two weighting functions unfortunately remains long and difficult. However, a satisfying compromise could finally be obtained:

- Response time ≤ 0.2 sec

- Overshoot limitation ($\leq$ 10

- High frequency control gain limitation

$W_1(s)$ is a two inputs-two outputs diagonal transfer matrix. Its first diagonal term is a second order low-pass filter, whereas its second term remains constant overall frequencies. $W_3(s)$ is a sixth order Chebycheff high-pass filter.

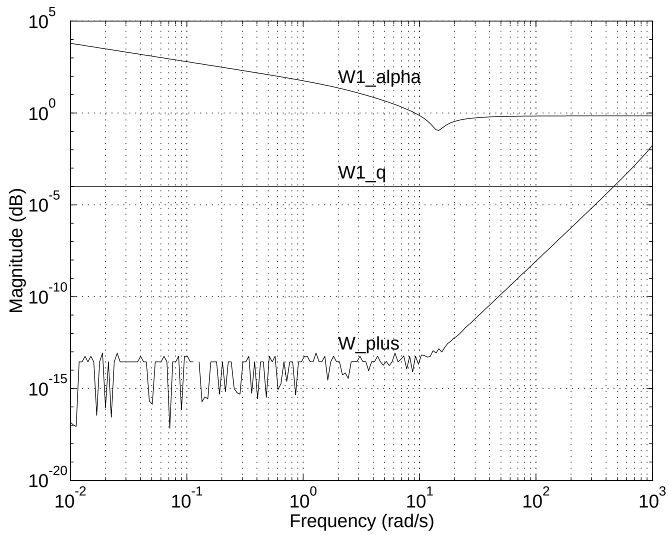

Figure 11: Weighting functions

For this set of weighting functions, the convex optimization algorithm converged and produced a polytopic LPV controller verifying the conditions of theorem 3.3 with $\gamma_o pt = 1.2$. Its structure is described below:

$$K(s, \theta(t)) = Co\{K(\theta(i)), i = 1, 2, 3, 4\} \tag{4.21}$$

4.1.2 Final controller construction

The final controller has to take actuator, gyros and flexibility modelization into account. It must also integrates $M_{\delta m}$ and Z_α interpolation. As mentioned before, actuator and gyros are supposed to be well known and not varying. Therefore, they can be easily inverted within the controller. In order to obtain strictly proper inverted systems , highly stable poles have been added. As for flexibility, it could not be robustly invert. Nevertheless, an efficient attenuation on a sufficiently large frequency band may be achieved thanks to a pseudo-inversion. These transformations are applied to each of the four matrices $K_{\theta(i)}$ of the controller convex hull. Thus a new LPV polytopic controller of higher dimension is obtained.

$$K_{new}(s, \theta(t)) = Co\{S_1(s).K(\theta(i)).S_2(s), i = 1, 2, 3, 4\} \tag{4.22}$$

4.2 Simulations

In this last point, the previously described controller will now be tested and analyzed. Pointwise and then real LPV simulations will be presented.

Superposition of time responses corresponding to 25 different testing points is firstly considered (see Figure 12). For each of these points, one determines its polytopic coordinated, then the corresponding system and controller. These operations only require matrices additions and multiplications, therefore, "on-line" implementation should be easy.

Superposition of Nichols plots is also presented on Figure 13. These plots are usefull to analyse local robustness properties which are, by the way, really satisfying. In the worse case, the gain margin remains greater than 5 dB, and phase margin greater than 34 degrees. As for delay margin, it varies between 10 ms and 15 ms which is 5 times the sampling period.

However, this tool does not permit to analyse global stability along parameter trajectories. It should be recalled, by the way, that LPV system analysis theory is not mature yet and therfore, no tools is at present available. There remains one solution to status on stability, which consists of LPV simulations.

In order to perform these tests, three different parameter trajectories have been examined.

- The first varies continuously in the operating domain and converges after about 1s (see Figure 14). The corresponding simulation represented on Figure 15 shows that both stability and performance level are achieved.
- The second trajectory also converges but does not vary continuously. Every 0.2s, the system "jumps" from stability to instability or the contrary (see Figure 16). Here again, stability and perforance requirements are satisfied.
- In the third case, parameter measurements are supposed to be noised. Thus, the LPV controller "follows" the trajectory depicted on Figure 18, whereas the system still "follows" the one of Figure 16. Despite this measurements perturbations, stability is still preserved (see Figure 19).

4.2.1 Local simulations

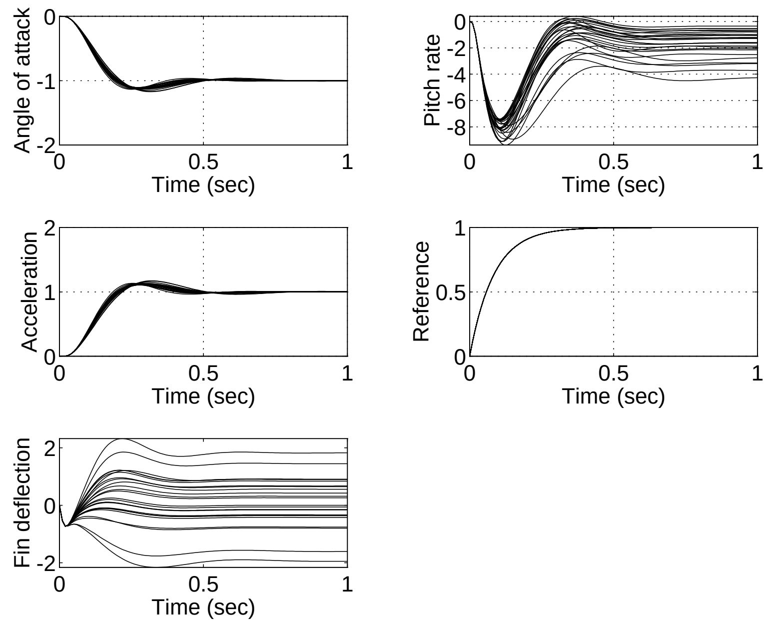

Figure 12: Local Simulations on 25 points

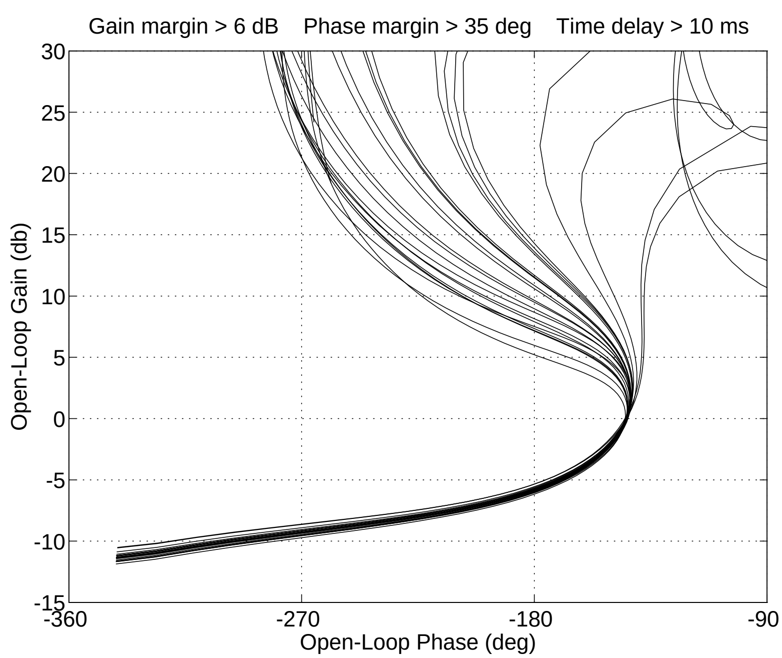

Figure 13: Superposition of Nichols Plots

4.2.2 LPV simulations

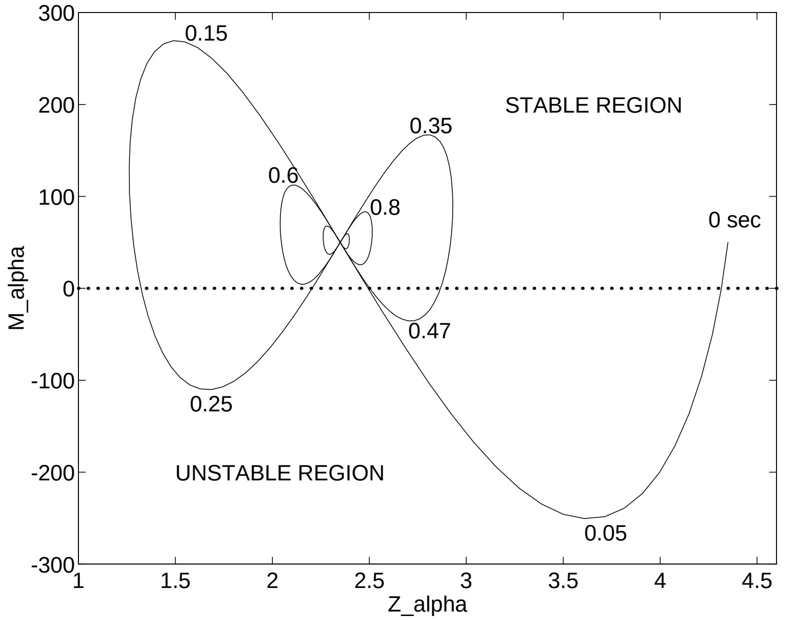

Figure 14: Parameter trajectory 1

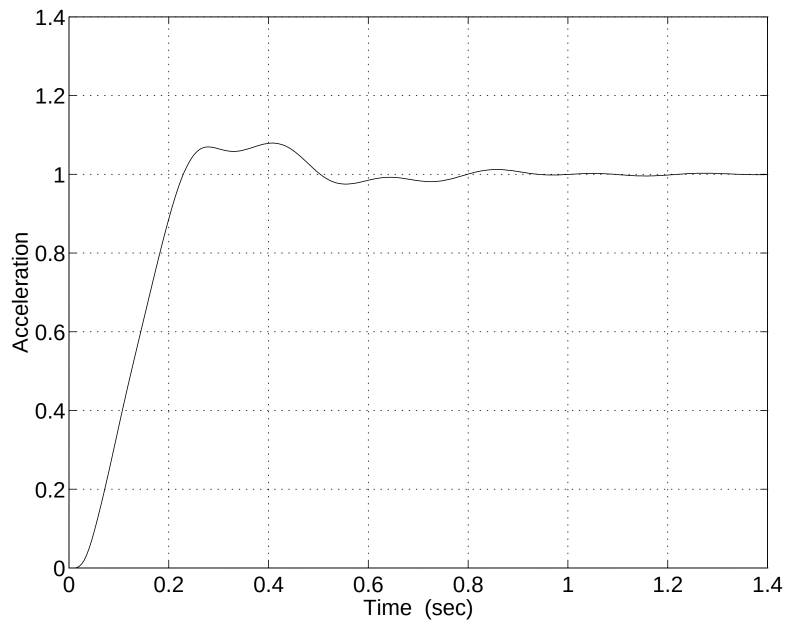

Figure 15: LPV Simulation 1

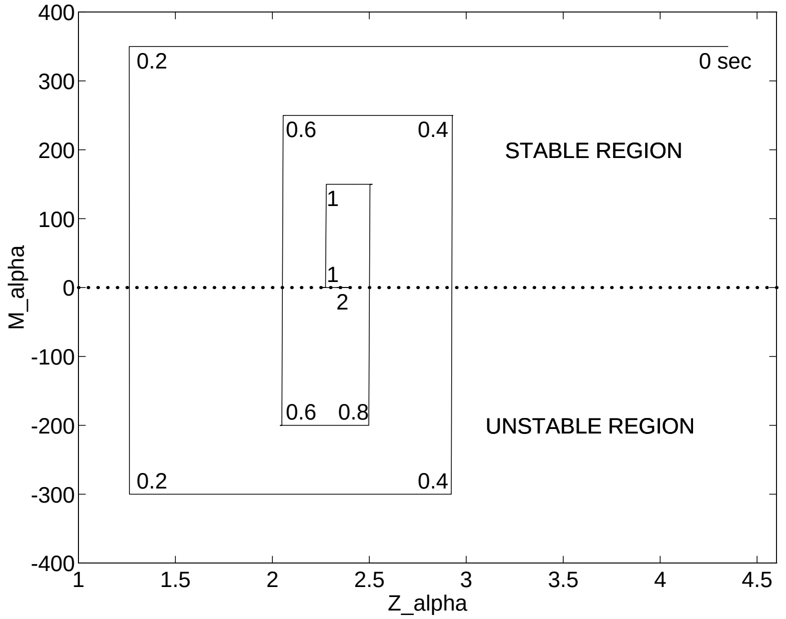

Figure 16: Parameter trajectory 2

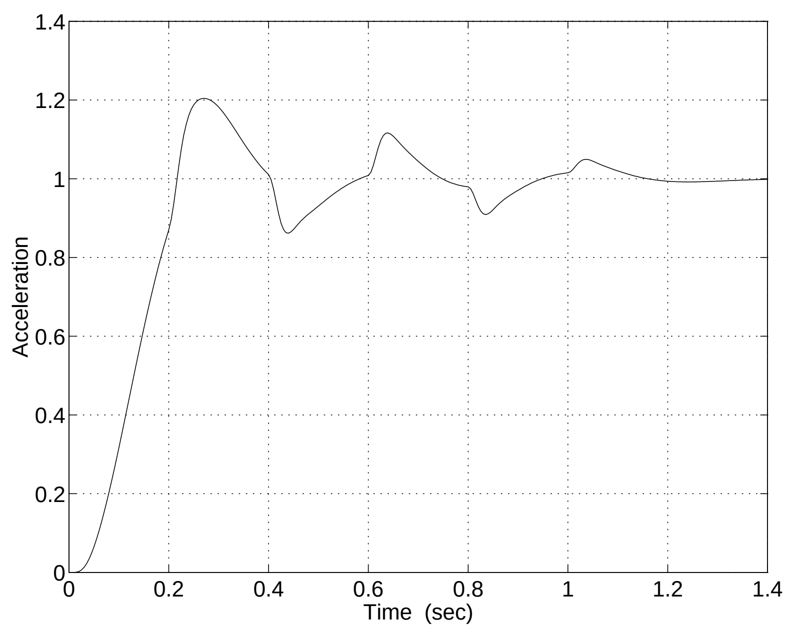

Figure 17: LPV Simulation 2

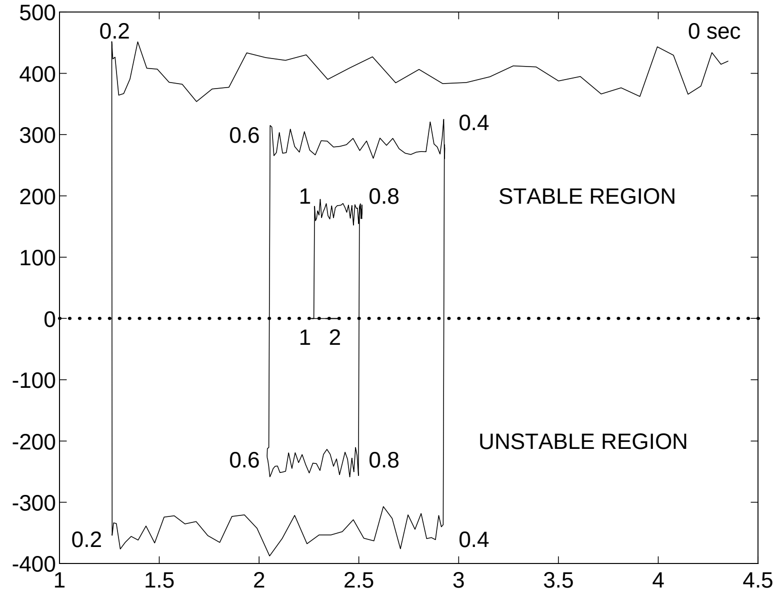

Figure 18: Parameter trajectory 3

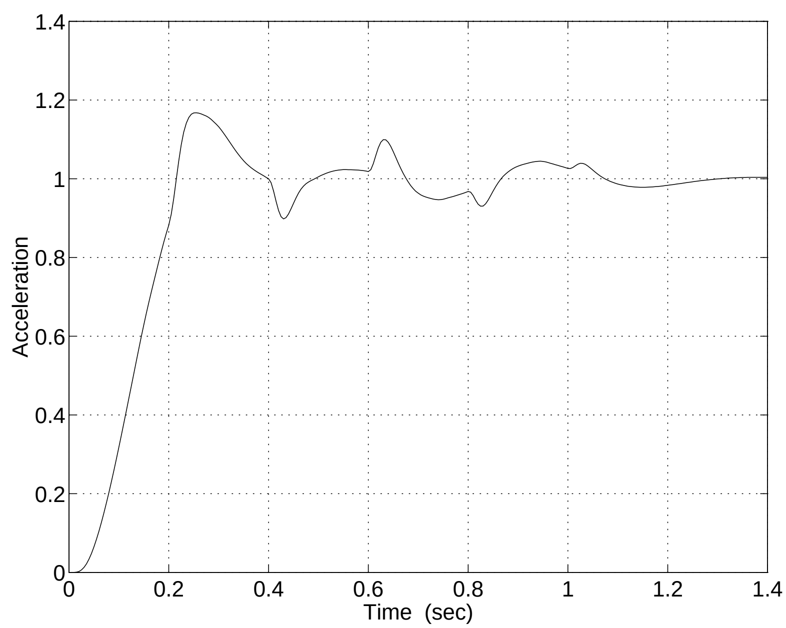

Figure 19: LPV Simulation 3

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