# Advanced Gain-Scheduling Techniques For Uncertain Systems

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**Abstract**

This paper is concerned with the design of gain-scheduled controllers for uncertain Linear Parameter-Varying systems. Two alternative design techniques for constructing such controllers are discussed. Both techniques are amenable to Linear Matrix Inequality problems via a gridding of the parameter space and a selection of basis functions. These problems are then readily solvable using available tools in convex semi-definite programming. When used together, these techniques provide complementary advantages of reduced computational burden and ease of controller implementation. The problem of synthesis for robust performance is then addressed by a new scaling approach for gain-scheduled control. The validity of the theoretical results are demonstrated through a two-link flexible manipulator design example. This is a challenging problem that requires scheduling of the controller in the manipulator geometry and robustness in face of uncertainty in the high frequency range.

## 1 Introduction

The gain-scheduling problem has been the subject of a great deal of research over recent years, both from theoretical and practical viewpoints. This renewed interest probably stems from the development of new techniques and software which allow for a more rigorous and systematic treatment of the gain-scheduling problem. The classical approach to this problem essentially consists of repeated design syntheses associated with some scheduling strategy connecting locally designed controllers. Such schemes, however, lack supporting theories that guarantee the behavior of the scheduled controller. A significant contribution toward the elimination of such weaknesses is the formulation of the gain-scheduling problem in the context of convex semi-definite programming [1], an elegant and solidly based branch of optimization theory [2, 3, 4]. Expressed in terms of Linear Matrix Inequalities (LMIs), the gain-scheduling problem is readily and globally solved using currently available efficient optimization software [5]. LMI techniques now appear as very natural mechanisms for the formulation of gain-scheduling problems as well as for a vast array of other problems in the control field. Reference [6] gives an overview of the scope of application of such techniques.

As emphasized in $\mathbf{H}_\infty$ control theory, a key stage in the characterization of gain-scheduled controllers is the search for adequate Lyapunov functions that establish stability and a performance bound for the closed-loop system. The Linear Fractional Transformation (LFT) gain-scheduling techniques in [7, 8, 9, 10] or the so-called quadratic gain-scheduled techniques in [11, 12] make use of a fixed Lyapunov function, as opposed to one which depends on the scheduled variables, to characterize stability and performance. According to [13], such approaches are potentially very conservative because they allow for arbitrary rates of variation in the scheduled variables. More dramatically, it has been shown in [13] that some systems are

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not even quadratically stabilizable, that is, are not stabilizable on the basis of a single Lyapunov function. A significant improvement over such techniques can be obtained by exploiting the concept of parameter-dependent Lyapunov functions. This is discussed in the context of robustness analysis and synthesis in [14, 15, 16] and for the gain-scheduling problem in [13, 16]. Parameter-dependent Lyapunov functions allow the incorporation of knowledge on the rate of variation in the analysis or synthesis technique, and therefore lead to much less conservative answers. The reader is referred to [17, 13] for earlier work related to the approaches considered here. The discretization of continuous-time gain-scheduled controllers is considered in [18].

In this paper we investigate two different techniques: [19, 20, 21] and an extension of [22, 23] to the gain-scheduling problem. These techniques impose no restriction on the plant and provide a simple and streamlined treatment of the gain-scheduling problem. Moreover, the technique in [19, 20] allows the incorporation of multiple specifications into the design problem such as $\mathbf{H}_2 - \mathbf{H}_\infty$, pole clustering, or control effort constraints. The second technique is more restrictive but offers computational advantages. The focus of this work is on the computational effort for controller calculation and on the practical issues of controller implementation. A special emphasis is placed on the development of scaling techniques which take advantage of the problem's structural properties and thus reduce the conservatism of the gain-scheduling approach. It is further shown that combining the capabilities of both techniques provides a comprehensive and effective methodology, encompassing all components of the gain-scheduling task from theoretical constructions to real-time implementations.

The paper is structured as follows. Sections 2 to 4 give a thorough discussion of gain-scheduling synthesis techniques together with some refinements and improvements in Section 5. Finally, the validity and applicability of concepts and techniques are demonstrated for a two-link flexible manipulator application in Section 6.

The notation used in the paper is fairly conventional. For real symmetric matrices $M$, $M > 0$ stands for "positive definite" and means that all the eigenvalues of $M$ are positive. Similarly, $M < 0$ means "negative definite" (all the eigenvalues of $M$ are negative) and $M \geq 0$ stands for "nonnegative definite" (the smallest eigenvalue of $M$ is nonnegative). In large symmetric matrix expressions, terms denoted $\star$ will be induced by symmetry. For instance, with $S$ symmetric

$$\begin{bmatrix} S + M + N + (\star) & \star \\ Q & P \end{bmatrix} := \begin{bmatrix} S + M + M^T + N + N^T & Q^T \\ Q & P \end{bmatrix}.$$

We shall also use the matrix notations

$$\operatorname*{\mathbf{diag}}_{i=1}^{N} X_i := \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_N \end{bmatrix}, \quad [L_{jk}]_{j,k} := \begin{bmatrix} L_{11} & \cdots & L_{1m} \\ \vdots & \ddots & \vdots \\ L_{m1} & \cdots & L_{mm} \end{bmatrix}.$$

For appropriately dimensioned matrices $K$ and $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and assuming the inverse exists, the upper LFT is defined as

$$F_u(M, K) = M_{22} + M_{21}K(I - M_{11}K)^{-1}M_{12}. \tag{1}$$

## 2 Output-Feedback Synthesis with guaranteed $L_2$-gain Performance

In this section we recap some known results on the gain-scheduling technique with bounded parameter variations rates and point out connections between different approaches. We first give a general characterization of gain-scheduled controllers, the solution to which involves both intermediate controller matrices and Lyapunov variables $X$ and $Y$. This formulation will be referred as the *basic* characterization, emphasizing the fact that it can be easily extended to multiple objective problems, pole clustering problems, etc... [19, 20]. Next, a second formulation of gain-scheduled controllers is presented. It will be referred as the *projected* characterization, as the intermediate controller matrices have been eliminated through projections [22]. Reconstructing the controller state-space data from the projected conditions has been addressed in [22, 23] for the customary $\mathbf{H}_\infty$ control problem. The reconstruction procedure is again described here, in the case of the gain-scheduling problem, for completeness of the discussion. The reader is referred to [17, 24, 13] for details, insights and applications of analogous gain-scheduling techniques.

The problem addressed throughout the paper is the following. Suppose we are given a Linear Parameter-Varying (LPV) plant $G(\theta)$ with state-space realization

$$\begin{array}{rcl} \dot{x} &=& A(\theta)x + B_1(\theta)w + B_2(\theta)u \\ z &=& C_1(\theta)x + D_{11}(\theta)w + D_{12}(\theta)u \\ y &=& C_2(\theta)x + D_{21}(\theta)w \; , \end{array} \tag{2}$$

where

$$A \in \mathbb{R}^{n\times n}, \quad D_{12} \in \mathbb{R}^{p_1\times m_2}, \text{and} \quad D_{21} \in \mathbb{R}^{p_2\times m_1}$$

define the problem dimension. The time-varying parameter $\theta := (\theta_1, \cdots, \theta_L)^T$ as well as its rates of variation $\dot{\theta}$ are assumed bounded as follows,

**(a)** each parameter $\theta_i$ ranges between known extremal values $\underline{\theta}_i$ and $\bar{\theta}_i$:

$$\theta_i(t) \in [\underline{\theta}_i, \bar{\theta}_i], \quad \forall t \geq 0 \tag{3}$$

**(b)** the rate of variation $\dot{\theta}_i$ is assumed well-defined at all times and satisfies

$$\dot{\theta}_i(t) \in [\underline{\nu}_i, \bar{\nu}_i], \quad \forall t \geq 0 \tag{4}$$

where $\underline{\nu}_i \leq \bar{\nu}_i$ are known lower and upper bounds on $\dot{\theta}_i$.

The first assumption means that the parameter vector $\theta$ is valued in a hypercube $\Theta$. Similarly, (4) defines a hypercube $\Theta_d$ of $\mathbb{R}^L$ with vertices in

$$\mathcal{T} := \left\{ (\tau_1, \ldots, \tau_L)^T \;:\; \tau_i \in \{\underline{\nu}_i, \bar{\nu}_i\} \right\} . \tag{5}$$

The gain-scheduled output-feedback control problem consists of finding a dynamic LPV controller, $K(\theta)$, with state-space equations

$$\begin{array}{rcl} \dot{x}_K &=& A_K(\theta,\dot{\theta})x_K + B_K(\theta,\dot{\theta})y \\ u &=& C_K(\theta,\dot{\theta})x_K + D_K(\theta,\dot{\theta})y \; , \end{array} \tag{6}$$

which ensures internal stability and a guaranteed $L_2$-gain bound $\gamma$ for the closed-loop operator (2)-(6) from the disturbance signal $w$ to the error signal $z$, that is

$$\int_0^T z^T z \, d\tau \leq \gamma^2 \int_0^T w^T w \, d\tau, \quad \forall T \geq 0$$

and all admissible trajectories $(\theta, \dot{\theta})$ and zero state initial conditions. Note that $A$ and $A_K$ have the same dimensions, since we restrict the discussion to the full-order case. The formulation of such controllers can be handled via an extension of the Bounded Real Lemma with quadratic parameter-dependent Lyapunov functions $V(x_{\rm cl}, \theta) = x_{\rm cl}^T P(\theta) x_{\rm cl}$ where $x_{\rm cl}$ stands for the state vector of the closed-loop system. See [13, 14, 15, 19] for details. Note that the controller state-space matrices are allowed to depend explicitly on the derivative of the time-varying parameter $\theta$. Different techniques to remove the dependence on $\dot{\theta}$ will be extensively discussed in Section 3, see also [24].

Except the usual smoothness assumptions on the dependence on $\theta$, the problem data and variables will be unrestricted in the subsequent derivations. The basic characterization of gain-scheduled controllers with guaranteed $L_2$-gain performance is presented in the next theorem where the dependence of data and variables on $\theta$ and $\dot{\theta}$ has been dropped for simplicity.

**Theorem 2.1 (Basic Characterization)** *Consider the LPV plant governed by (2), with parameter trajectories constrained by (3), (4). There exists a gain-scheduled output-feedback controller (6) enforcing internal stability and a bound $\gamma$ on the $L_2$ gain of the closed-loop system (2) and (6), whenever there exist parameter-dependent symmetric matrices $Y$ and $X$ and a parameter-dependent quadruple of state-space data $(\widehat{A}_K, \widehat{B}_K, \widehat{C}_K, D_K)$ such that for all pairs $(\theta, \dot{\theta})$ in $\Theta \times \Theta_d$ the following infinite-dimensional LMI problem holds,*

$$\begin{bmatrix} \dot{X} + XA + \widehat{B}_K C_2 + (\star) & * & \star & \star \\ \widehat{A}_K^T + A + B_2 D_K C_2 & -\dot{Y} + AY + B_2 \widehat{C}_K + (\star) & \star & \star \\ (XB_1 + \widehat{B}_K D_{21})^T & (B_1 + B_2 D_K D_{21})^T & -\gamma I & \star \\ C_1 + D_{12} D_K C_2 & C_1 Y + D_{12} \widehat{C}_K & D_{11} + D_{12} D_K D_{21} & -\gamma I \end{bmatrix} < 0 \tag{7}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{8}$$

*In such case, a gain-scheduled controller of the form (6) is readily obtained with the following two-step scheme:*

- *solve for $N$, $M$, the factorization problem*

$$I - XY = NM^T .$$

- *compute $A_K$, $B_K$, $C_K$ with*

$$\begin{aligned} A_K &= N^{-1}(X\dot{Y} + N\dot{M}^T + \widehat{A}_K - X(A - B_2 D_K C_2)Y \\ &\quad -\widehat{B}_K C_2 Y - XB_2 \widehat{C}_K)M^{-T} && (9) \\ B_K &= N^{-1}(\widehat{B}_K - XB_2 D_K) && (10) \\ C_K &= (\widehat{C}_K - D_K C_2 Y)M^{-T} . && (11) \end{aligned}$$

**Proof:** See [19, 20]. ■

Note that since all variables are involved linearly, the constraints (7) and (8) constitute an LMI system. This system is, however, infinite due to its dependence on $(\theta, \dot{\theta})$ ranging over $\Theta \times \Theta_d$. Using the Projection Lemma, detailed in [22], the controller variables can be eliminated, leading to a characterization involving the variables $X$ and $Y$, only. This is presented in the next theorem.

**Theorem 2.2 (Projected Solvability Conditions)** *Consider the LPV plant governed by (2), with parameter trajectories constrained by (3) and (4). There exists a gain-scheduled output-feedback controller (6) enforcing internal stability and a bound $\gamma$ on the $L_2$ gain of the closed-loop system (2) and (6),* *whenever there exist parameter-dependent symmetric matrices $Y(\theta)$ and $X(\theta)$ such that for all pairs $(\theta, \dot{\theta})$ in $\Theta \times \Theta_d$ the following infinite-dimensional LMI problem holds,*

$$\left[\begin{array}{c|c} \mathcal{N}_X & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} \dot{X} + XA + A^T X & XB_1 & C_1^T \\ B_1^T X & -\gamma I & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma I \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_X & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{12}$$

$$\left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} -\dot{Y} + YA^T + AY & YC_1^T & B_1 \\ C_1 Y & -\gamma I & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma I \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{13}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{14}$$

*where $\mathcal{N}_X$ and $\mathcal{N}_Y$ designate any bases of the null spaces of $[\, C_2 \quad D_{21}\,]$ and $[\, B_2^T \quad D_{12}^T\,]$, respectively.*

**Proof:** This is a straightforward application of the Projection Lemma [22] to the LMI (7), with respect to the matrix variable

$$\begin{bmatrix} \widehat{A}_K + (A + B_2 D_K C_2)^T & \widehat{B}_K \\ \widehat{C}_K & D_K \end{bmatrix}.$$

■

Theorem 2.2 only provides existence conditions for controllers of the form (6). These conditions become necessary and sufficient if we confine the involved Lyapunov functions to the set of quadratic forms

$$V(x_{\mathrm{cl}}, \theta) := x_{\mathrm{cl}}^T P(\theta) x_{\mathrm{cl}}, \qquad \text{with} \qquad x_{\mathrm{cl}} := \begin{bmatrix} x \\ x_K \end{bmatrix}.$$

As an immediate extension of the results in [23], the next theorem provides for controller construction. Once again, the dependence on $\theta$ has been dropped to facilitate manipulations. It is further assumed that

- **(H1)** $D_{12}$ and $D_{21}$ are full-column and full-row rank, respectively.

This assumption is without restriction and greatly simplifies the presentation. The construction is easily extended to the singular case along the lines of [23].

**Theorem 2.3 (Controller Construction from Projections)** *Assume the conditions of Theorem 2.2 hold for a pair $(X, Y)$ and some performance level $\gamma$. Then a gain-scheduled controller can be constructed for any pair $(\theta, \dot{\theta})$ in $\Theta \times \Theta_d$ by the following sequential scheme:*

- *compute $D_K$ solution to*

$$\sigma_{max}(D_{11} + D_{12} D_K D_{21}) < \gamma, \tag{15}$$

  *and set $D_{c\ell} := D_{11} + D_{12} D_K D_{21}$.*

- *compute $\widehat{B}_K$ and $\widehat{C}_K$ solutions to the linear matrix equations*

$$\begin{bmatrix} 0 & D_{21} & 0 \\ D_{21}^T & -\gamma I & D_{c\ell}^T \\ 0 & D_{c\ell} & -\gamma I \end{bmatrix} \begin{bmatrix} \widehat{B}_K^T \\ \star \end{bmatrix} = - \begin{bmatrix} C_2 \\ B_1^T X \\ C_1 + D_{12} D_K C_2 \end{bmatrix}; \tag{16}$$

$$\begin{bmatrix} 0 & D_{12}^T & 0 \\ D_{12} & -\gamma I & D_{c\ell} \\ 0 & D_{c\ell}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \widehat{C}_K \\ \star \end{bmatrix} = - \begin{bmatrix} B_2^T \\ C_1 Y \\ (B_1 + B_2 D_K D_{21})^T \end{bmatrix}. \tag{17}$$

- *compute*

$$\begin{aligned}\widehat{A}_K &= -(A + B_2 D_K C_2)^T + \\ &\quad [\, XB_1 + \widehat{B}_K D_{21} \quad (C_1 + D_{12} D_K C_2)^T \,] \begin{bmatrix} -\gamma I & D_{c\ell}^T \\ D_{c\ell} & -\gamma I \end{bmatrix}^{-1} \begin{bmatrix} (B_1 + B_2 D_K D_{21})^T \\ C_1 Y + D_{12} \widehat{C}_K \end{bmatrix} .\end{aligned} \tag{18}$$

- *solve for $N$, $M$, the factorization problem*

$$I - XY = NM^T .$$

- *finally, compute $A_K$, $B_K$ and $C_K$ with the help of (9)-(11)*

■

It should be noted that in spite of their different structures, the characterizations given in Theorems 2.1 and 2.2-2.3 are equivalent and can virtually be used interchangeably for controller synthesis. In contrast, when the focus is on computational complexity or practical implementation, these techniques exhibit significant differences. This is discussed in Section 4. Finally, the case where only some parameters $\theta_i$ are subject to constraints on their derivatives is easily handled by removing the unconstrained parameters from the matrix functions $X(.)$ and $Y(.)$.

### 2.1 Extensions to Multi-Objective Problems

A useful practical advantage of the basic technique is that it easily extends to multi-objective problems. Various channels of the closed-loop system can be specified independently with a rich list of specifications. See [25, 21] for a thorough discussion. As an example, it is possible to specify an $L_2$ gain bound with regional pole constraints on the closed-loop dynamics of the underlying LTI systems ($\theta$ frozen). Such constraints consist of vertical and horizontal strips, disks, conic sectors, parabolas, ..., or intersections of such regions. The LMIs (7)-(8) must then be complemented with

$$\left[ \lambda_{jk} \begin{bmatrix} Y & I \\ I & X \end{bmatrix} + \mu_{jk} \begin{bmatrix} AY + B_2 \widehat{C}_K & A + B_2 D_K C_2 \\ \widehat{A}_K & XA + \widehat{B}_K C_2 \end{bmatrix} + \mu_{kj} \begin{bmatrix} AY + B_2 \widehat{C}_K & A + B_2 D_K C_2 \\ \widehat{A}_K & XA + \widehat{B}_K C_2 \end{bmatrix}^T \right]_{j,k} < 0. \tag{19}$$

where the data $\lambda_{jk}$ and $\mu_{kj}$ defines the geometry of the region.

## 3 Practical Validity of Gain-Scheduled Controllers

It must be stressed that an LPV controller derived from Theorem 2.1 or Theorems 2.2-2.3 is not gain-scheduled in the usual sense of the term. Its implementation requires not only the real-time measurement of the parameter $\theta$, but also of its time-derivative $\dot{\theta}$. This is generally prohibitive, since parameter derivatives either are not available or are difficult to estimate during system operation. Gain-scheduled controllers that do not require a measurement of $\dot{\theta}$ will be called *practically valid* hereafter. As discussed in [17], there is no systematic and tractable approach for removing the dependence on $\dot{\theta}$ while maintaining the generality of Theorems 2.1 or 2.2-2.3. As suggested by the controller formula (9), a simple but conservative approach has been proposed in [24]. It consists of restricting the variable $Y(\theta)$ to $\dot{Y} = 0$, that is, $Y$ not depending on $\theta$. This operation amounts to using a fixed Lyapunov function for the parameter-dependent control problem described in (13). It thereby sacrifices some performance, resulting in a higher $\gamma$.

Keeping in mind that the dependence of the controller data on $\dot{\theta}$ stems from the term $X\dot{Y} + N\dot{M}^T$, (9), the general characterization of Theorem 2.3 offers additional freedom that is worth pointing out. The discussion is summarized in the next table.

| | Variables $X$, $Y$ | Variables $N$, $M$ | Practical Validity |
|---|---|---|---|
| $\frac{d\theta}{dt} = 0$ | $X := X(\theta)$ ,$Y := Y(\theta)$ | $NM^T = I - X(\theta)Y(\theta)$ | Yes |
| $\frac{d\theta}{dt} \in \Theta_d$ | $X := X(\theta)$ ,$Y := Y(\theta)$ | $NM^T = I - X(\theta)Y(\theta)$ | No |
| $\frac{d\theta}{dt} \in \Theta_d$ | $X := X(\theta)$ ,$Y := Y_0$ | $N := I - X(\theta)Y_0$, $M := I$ | Yes |
| $\frac{d\theta}{dt} \in \Theta_d$ | $X := X_0$ ,$Y := Y(\theta)$ | $N := I$, $M := I - Y(\theta)X_0$ | Yes |
| $\frac{d\theta}{dt}$ unbounded | $X := X_0$ ,$Y := Y_0$ | $NM^T = I - X_0Y_0$ | Yes |

Table 1: Selection of variables in the gain-scheduled control problem

Row #1 of the table simply says that if the scheduled variable is assumed constant in time, a practically valid gain-scheduled controller can theoretically be constructed using Theorem 2.1 or alternatively Theorems 2.2-2.3, for any matrix functions $X(.)$ and $Y(.)$ of $\theta$. Such an approach ignores possible time variations of $\theta$ and provides neither performance nor stability guarantees for the closed-loop system in the face of time-variations. With the same choice of matrix functions $X(.)$ and $Y(.)$, but the rate of variations of $\theta$ being confined to a compact $\Theta_d$, row #2 says that there is no known techniques to compute a practically valid gain-scheduled controller. In rows #3 and #4, we have assumed the conservative choices that $X$ or $Y$ are constant matrix variables. In both cases, the gain-scheduling problem with bounded rate of variations admits practically valid controller solutions, provided the variables $N$ and $M$ are adequately selected in Theorems 2.1 and 2.3. With further conservatism, that is, $\dot{\theta}$ is unbounded, row #5 says that the problem is again tractable and solvable using the same techniques. The case of time-varying parameters with bounds on the rate of variation can be constructively handled by the choices of rows #3 and #4. However, due to the loss of duality in the variables $X$ and $Y$, such choices are not equivalent. As a consequence, there are some problems for which it is better to take a parameter-dependent $X$ and a constant $Y$ while others will require the converse. Hence, both alternatives must be tried to get a less conservative design . In the controller construction scheme, the variables $N$ and $M$ are subject to the algebraic constraint $I - XY = NM^T$ from which one easily infers the identity

$$\dot{X}Y + \dot{N}M^T = -(X\dot{Y} + N\dot{M}^T).$$

In light of this identity, a practically valid gain-scheduled controller in the cases of rows #3 and #4 can be derived using the same formulas (10) and (11), but with $A_K$ suitably updated to

$$A_K = N^{-1}(\widehat{A}_K - X(A - B_2D_KC_2)Y - \widehat{B}_KC_2Y - XB_2\widehat{C}_K)M^{-T} . \tag{20}$$

The same formulas are still valid for the case of frozen-in-time parameters, row #1, and for arbitrarily varying parameters, row #5, the variables $X$ and $Y$ being replaced by their constant values $X_0$ and $Y_0$, in the latter case. Summing up, Table 1 displays all options to handle any situations from the frozen-in-time parameters to arbitrarily time-varying parameters. However, the case in which both $X$ and $Y$ depend on $\theta$ with a bounded $\dot{\theta}$ still resists a convex formulation for a practically valid controller.

# 4 Reduction to Finite-Dimensional Problems

Even with the simplifications of Table 1 in place, the characterizations of Theorems 2.1 or 2.2-2.3 involve the solution of a convex but infinite-dimensional and infinitely constrained problem. This is the price to pay for allowing a general parameter dependence in the plant (2). Generally speaking, there is no systematic rule for selecting the functional dependence of the matrix functions $X$ and $Y$ on $\theta$. We are therefore led to some simple heuristics in order to simplify the computation of solutions to the LMI problems (7)-(8) or (12)-(14). A simple but practical technique has been proposed in [13]. The key idea is to "mimic" the parameter dependence of the plant in the Lyapunov function variables $X$ and $Y$. Interestingly, the same idea can be used in the more general context of the basic characterization of Theorem 2.1. In return, this offers new potential approaches for the synthesis of gain-scheduled controllers with multiple objective constraints (mixed $\mathbf{H}_2 - \mathbf{H}_\infty$, pole clustering, and others still to find). To be more specific, consider the class of plants (2) having an LFT dependence on nonlinear functions of the scheduled variable, that is, whose state-space data further satisfy

$$G(\theta) := F_u \left\{ \begin{bmatrix} A_\rho & B_\rho & B_1 & B_2 \\ C_\rho & D_{\rho\rho} & D_{\rho 1} & D_{\rho 2} \\ C_1 & D_{1\rho} & D_{11} & D_{12} \\ C_2 & D_{2\rho} & D_{21} & 0 \end{bmatrix}, \mathop{\mathbf{diag}}_{i=1}^{N}(\rho_i(\theta) I_{r_i}) \right\}, \tag{21}$$

where $\rho_i(.), i = 1, \ldots, N$ are differentiable functions of $\theta$. Note that such a description encompasses many practical situations, since most systems in aeronautics and robotics can be represented as an LFT in nonlinear functions of the time-varying parameters. Copies of the plant's nonlinear functions, $\rho_i(.)$, can be introduced into the quadruple $(\widehat{A}_K(.), \widehat{B}_K(.), \widehat{C}_K(.), D_K(.))$ and the pair $(X(.), Y(.))$ in an affine fashion,

$$\begin{aligned} \widehat{A}_K(\theta) &:= \widehat{A}_{K,0} + \sum_{i=1}^{N} \rho_i(\theta) \widehat{A}_{K,i}; & \widehat{B}_K(\theta) &:= \widehat{B}_{K,0} + \sum_{i=1}^{N} \rho_i(\theta) \widehat{B}_{K,i} \\ \widehat{C}_K(\theta) &:= \widehat{C}_{K,0} + \sum_{i=1}^{N} \rho_i(\theta) \widehat{C}_{K,i}; & D_K(\theta) &:= D_{K,0} + \sum_{i=1}^{N} \rho_i(\theta) D_{K,i} \end{aligned} \tag{22}$$

and

$$X(\theta) := X_0 + \sum_{i=1}^{N} \rho_i(\theta) X_i; \qquad Y(\theta) := Y_0 + \sum_{i=1}^{N} \rho_i(\theta) Y_i. \tag{23}$$

The functional dependence of $X$ and $Y$ being fixed, the matrices $\widehat{A}_{K,0}$, $\widehat{A}_{K,i}$,..., play the role of decision variables in the infinitely constrained LMI problems (7)-(8) or (12)-(14). A simple remedy for turning such problems into a finite set of LMIs is to grid the value set of $\theta$ [13]. Since the derivative $\dot{\theta}$ appears linearly in the LMIs (7) and (12)-(13), there is only need to check the extreme points of the set $\Theta_d$, denoted $\mathcal{T}$, for all admissible values of $\theta$. The overall procedure can be described as follows.

**step 1** define a grid $\mathcal{G}$ for the value set of $\theta$,

**step 2** minimize $\gamma$ subject to the LMI constraints associated with $\mathcal{G} \times \mathcal{T}$ ,

**step 3** check the constraints with a denser grid,

**step 4** if step 3 fails, increase the grid density and return to **step 2**.

Computing solutions (22) and (23) to the LMI system associated with $\mathcal{G} \times \mathcal{T}$ is a convex optimization that can be solved by polynomial-time algorithms [2, 26] and the software [5]. Such problems generally require a large number of variables and constraints that today limit the scope of application of such techniques. With this considered, available solvers are still efficient for problems of reasonable size, say for up to 15 states and 2 or 3 scheduled variables. LMI-based gain-scheduling techniques have proven very powerful in a number of delicate applications [13, 17, 24, 12].

When restricted to the parameterization (22) and (23), the basic and projected characterizations are no longer equivalent. In the first one, we have further restrictions on the structure of the quadruple $(\widehat{A}_K(.), \widehat{B}_K(.), \widehat{C}_K(.), D_K(.))$. As a result, the first approach is generally more conservative, although we have observed very little difference in practice. See the application Section 6 for comparisons. From a complexity viewpoint, the first technique requires a larger number of scalar variables to be optimized; the number of additional variables being approximately $n(n + m_2 + p_2)L$, where $L$ is number of blocks in the LFT (21). Its scope of application is therefore more restricted. In contrast, the controller equations resulting from the basic characterization are significantly less complex than those resulting from the projected characterization. Note that such controller constructions are essentially dominated by matrix inversions and QR decompositions in (9)-(11) and (16)-(18). At each sampling time, the basic characterization essentially requires

- 1 matrix inversion,

whereas the projected characterization will require

- 2 QR decompositions and 3 matrix inversions for problem (15),
- 2 matrix inversions for the computation of $A_K$, $B_K$, $C_K$ by exploiting partitioning.

Thus, in the light of these comments and because the expressions (22) essentially reduce to scalar-by-matrix multiplications, controllers resulting from the first technique are more easily implemented for rapidly varying LPV systems. In addition, these controllers have an LFT representation in terms of the nonlinear functions, $\rho_i(.)$, and hence are computationally comparable to those of the LFT gain-scheduling approaches in [7, 8]. Note also that for both techniques, the most computationally demanding step comes from the inversion of the term $I - X(\theta)Y(\theta)$, typically a large matrix. It is sometimes possible to exploit rank deficiencies in the $X_i$'s and the $Y_i$'s to further reduce computational efforts, for instance, by using inversion with rank correction formulas. A simple case that does not require the inversion of $I - X(\theta)Y(\theta)$ is when the LFT system (21) depends on a single nonlinear function $\rho_1(\theta)$. Noting that

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix}^{-1} = \begin{bmatrix} (X - Y^{-1})^{-1} & (I - YX)^{-1} \\ (I - XY)^{-1} & (Y - X^{-1})^{-1} \end{bmatrix},$$

the inverse of $I - X(\theta)Y(\theta)$ can be computed from the lower-left block in the above expression. Moreover, if we assume without restriction that 0 is in the image set of $\rho_1(.)$, then

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} = \begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix} + \rho_1 \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix} > 0\,. \tag{24}$$

The positive condition in (24) ensures that the matrices

$$\begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix}; \qquad \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix}$$

are simultaneously diagonalizable. Hence there exists a congruence transformation $T$ and a diagonal matrix $\Lambda_1$, computed off-line, such that for any value of the map $\rho_1(.)$,

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix}^{-1} = T(I + \rho_1 \Lambda_1)^{-1} T^T\,. \tag{25}$$

Therefore, a cheap way of computing $(I - XY)^{-1}$ at each sample of time is simply to invert the diagonal matrix in (25) and perform multiplications of corresponding blocks.

Since they offer complementary advantages, the techniques described above can be used together to yield a more effective methodology. Confirmed by practical experience, the following rules have proven useful.

1. All necessary tunings, requiring repeated computations should be based on the less costly projected technique.
2. The procedure is completed by running the basic technique, for controller implementation purposes.

Though the last phase may be very slow, it is run only once in the whole design process.

### 4.1 Bypassing the gridding phase

As discussed earlier, there is no direct technique to bypass the gridding phase, hence making the design more direct. Under special circumstances, LFT, affine or polynomial parameter dependence of data and variables or polytopic approximation of the original plant, techniques such as the $\mathcal{S}$-procedure or multi-convexity concepts can be used repeatedly to get a *finite* number of (sufficient) LMI conditions. See the conference version of the present paper [27] and also [15, 14] and references therein.

## 5 Reducing Conservatism by Scaling

As is common in robust control theory, it is possible to further enhance the design procedure by exploiting structural informations on the operator relating the signals $w$ and $z$. The conditions of Theorems 2.1 or 2.2 also provides robust stability conditions in face of

- multiplicative memoryless time-varying uncertainties

$$w(t) = \Delta(t)z(t), \qquad \sigma_{max}(\Delta(t)) \leq \frac{1}{\gamma},\ t \geq 0$$

- non-expansive dynamic uncertainty (for instance LTI) [6]

$$w = \Delta(z), \qquad \gamma \int_0^T w^T w \, d\tau \leq \frac{1}{\gamma} \int_0^T z^T z \, d\tau,\ T \geq 0\,.$$

This description, however, ignores potential structures of the operator $\Delta$. We therefore assume, hereafter, that the plant is governed by (2) with $w$ and $z$ subject to

$$[w_1(t)^T, \ldots, w_N(t)^T]^T = \Delta(t) \quad [z_1(t)^T, \ldots, z_N(t)^T]^T\,, \tag{26}$$

where $\Delta$ is a multiplicative memoryless time-varying operator with structure

$$\Delta(t) := \mathbf{diag}(\Delta_1(t), \ldots, \Delta_N(t))\,, \tag{27}$$

and confined to the compact set

$$\sigma_{max}(\Delta(t)) \leq \frac{1}{\gamma}\,, \ \ \forall t \geq 0. \tag{28}$$

The set of scalings associated with the structure (27) is defined as

$$\mathcal{S}_\Delta := \{S : S > 0, \quad S\Delta(t) = \Delta(t)S, \quad \forall t \geq 0\} \tag{29}$$

We have implicitly assumed, without restriction, that the problem has been squared so that $m_1 = p_1$ in the subsequent derivations. With these notations in mind, a scaled version of the Bounded Real Lemma can be established.

**Lemma 5.1 (Structured Robust Stability)** *The LPV system governed by*

$$\begin{array}{rcl} \dot{x} &=& A(\theta)x + B(\theta)w \\ z &=& C(\theta)x + D(\theta)w \end{array} \tag{30}$$

*and (26)-(28), with parameter trajectories $\theta(t)$ constrained with (3) and (4), is internally stable whenever there exists a parameter-dependent symmetric matrix $P(\theta)$ and a parameter-dependent scaling $S(\theta)$ in $\mathcal{S}_\Delta$ such that $P(\theta) > 0$ and*

$$\begin{bmatrix} \dot{P}(\theta) + A(\theta)^T P(\theta) + P(\theta)A(\theta) & P(\theta)B(\theta) & C(\theta)^T \\ B(\theta)^T P(\theta) & -\gamma S(\theta) & D(\theta)^T \\ C(\theta) & D(\theta) & -\gamma S(\theta)^{-1} \end{bmatrix} \ < \ 0 \tag{31}$$

*holds for all admissible values of the parameter vector $\theta$ and of its time derivative $\dot{\theta}$.*

**Proof:** See Appendix A. ■

Note that Lemma 5.1 provides robust stability conditions for non-expansive uncertain operators $\Delta_i$. These conditions also guarantee a robust $L_2$-gain performance bound $\gamma$ with respect to any input/output channel $(w_i, z_i)$, with the remaining channels corresponding to uncertainties of the form described earlier. See for instance [6] for details. Other Extensions to repeated-scalar uncertainties $\Delta_i = \delta_i I_{s_i}$ are straigt-forward. Also important is the fact that the scaling matrix $S(\theta)$ is allowed to vary with $\theta$. Hence, the conservatism of the gain-scheduling technique can potentially be reduced for all values of $\theta$ independently. As an immediate consequence of the above Lemma, the gain-scheduling techniques in Theorems 2.1 and 2.2-2.3 are readily extended to handle the structural constraint (26)-(27). Such extensions simply follow from the following substitutions

$$\begin{bmatrix} -\gamma I & \star \\ D_{11} + D_{12}D_K D_{21} & -\gamma I \end{bmatrix} \to \begin{bmatrix} -\gamma S & \star \\ D_{11} + D_{12}D_K D_{21} & -\gamma S^{-1} \end{bmatrix}$$

in Theorems 2.1 and 2.3 and

$$\begin{bmatrix} -\gamma I & D_{11}^T \\ D_{11} & -\gamma I \end{bmatrix} \to \begin{bmatrix} -\gamma S & D_{11}^T \\ D_{11} & -\gamma S^{-1} \end{bmatrix}; \quad \begin{bmatrix} -\gamma I & D_{11} \\ D_{11}^T & -\gamma I \end{bmatrix} \to \begin{bmatrix} -\gamma S^{-1} & D_{11} \\ D_{11}^T & -\gamma S \end{bmatrix},$$

in Theorem 2.2.

Owing to the dependence of the modified characterizations on both $S$ and $S^{-1}$, they are no longer standard LMI problems. Such problems are in the class of LMI problems with rank reduction constraints hence difficult to solve. Here, we adopted a simple computational scheme in the spirit of $\mu$ synthesis techniques.

Recall that the LMI (7), with the scaling $S(\theta)$ in place, can take the form

$$\begin{bmatrix} \dot{X} + XA + \widehat{B}_K C_2 + (\star) & \star & \star & \star \\ \widehat{A}_K^T + A + B_2 D_K C_2 & -\dot{Y} + AY + B_2\widehat{C}_K + (\star) & \star & \star \\ S^{-1}(XB_1 + \widehat{B}_K D_{21})^T & S^{-1}(B_1 + B_2 D_K D_{21})^T & -\gamma S^{-1} & \star \\ C_1 + D_{12}D_K C_2 & C_1 Y + D_{12}\widehat{C}_K & (D_{11} + D_{12}D_K D_{21})S^{-1} & -\gamma S^{-1} \end{bmatrix} < 0. \quad (32)$$

Note that this is an LMI in the variables $S^{-1}$, $Y$, $\widehat{A}_K$ and $\widehat{C}_K$ provided that the variables $X$, $\widehat{B}_K$ and $D_K$ are maintained fixed. Therefore, coupled with Theorem 2.1, this result suggests a first scheme to compute a best possible $S^{-1}$. Now, our goal is to obtain the counterpart of the projected characterization Theorem 2.2 for computing $S^{-1}$. In view of the discussion of Section 4, this will be at a reduced computational cost. Interestingly, such a scheme cannot be directly derived from the LMIs (12)-(13). We shall make use of a partially projected characterization, as follows.

Applying the Projection Lemma [22], with respect to the $(2,1)$ free term in (32), equivalent LMI conditions are

$$\begin{bmatrix} \dot{X} + XA + \widehat{B}_K C_2 + (\star) & \star & \star \\ S^{-1}(XB_1 + \widehat{B}_K D_{21})^T & -\gamma S^{-1} & \star \\ C_1 + D_{12}D_K C_2 & (D_{11} + D_{12}D_K D_{21})S^{-1} & -\gamma S^{-1} \end{bmatrix} < 0; \quad (33)$$

$$\begin{bmatrix} -\dot{Y} + AY + B_2\widehat{C}_K + (\star) & \star & \star \\ S^{-1}(B_1 + B_2 D_K D_{21})^T & -\gamma S^{-1} & \star \\ C_1 Y + D_{12}\widehat{C}_K & (D_{11} + D_{12}D_K D_{21})S^{-1} & -\gamma S^{-1} \end{bmatrix} < 0. \quad (34)$$

Assuming now that (33) holds and projecting again with respect to the controller variable $\widehat{C}_K$ in (34), the following equivalent LMI condition is obtained

$$\left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} -\dot{Y} + YA^T + AY & YC_1^T & \star \\ C_1 Y & -\gamma S^{-1} & \\ \hline S^{-1}(B_1 + B_2 D_K D_{21})^T & S^{-1}(D_{11} + D_{12} D_K D_{21})^T & -\gamma S^{-1} \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array}\right] < 0 . \quad (35)$$

Gathered together with (14), the matrix inequalities (33) and (35) form an LMI system in the variable $Y$ and $S^{-1}$ with fixed $X$, $\widehat{B}_K$ and $D_K$. It turns out that the conditions (32) and (8) on one side or the conditions (14), (33), and (35) on the other side form the basis of two possible schemes for iteratively reducing the conservatism of the gain scheduling techniques of Section 2. Such schemes proceed as follows.

- **step 0** setting $S(\theta) = I$, minimize $\gamma$ with the basic or the projected technique,
- **step 1** compute a scaling $S(\theta)^{-1}$ minimizing $\gamma$ with the help of (32) and (8) or alternatively (33), (35), and (14),
- **step 2** with $S(\theta)$ being fixed, perform a gain-scheduling synthesis with the basic or projected technique,
- **step 3** iterate over steps 1 to 3 until convergence.

As before, we are using a grid of the set $\Theta$ to perform the optimizations and a user-defined functional dependence of $S^{-1}$ on $\theta$. For LFT plants of the form (21), a practical choice is the affine expansion

$$S^{-1}(\theta) := \Sigma_0 + \sum_{i=1}^{L} \rho_i(\theta)\Sigma_i \, .$$

It is important to mention that the previously described iterative procedure, though not giving a global solution to the problem, has proven very efficient in practice. A demonstration is given in the following section. Unlike the standard $D - K$ iteration procedure, it involves not only scalings and Lyapunov variables, but also some controller variables in the same optimization step. In our opinion, this is a central factor favoring convergence, both in speed and accuracy.

# 6 Control of a Two-Link Flexible Manipulator

## 6.1 Problem description

The gain-scheduled control of a two-link flexible manipulator is a nontrivial problem. The dynamics of such a system include both rigid body and lightly damped structural modes. The problem is complicated by uncertainty in the high frequency dynamics of the system and by the variation of dynamics with manipulator geometry. The first of these complications drives the requirement for closed-loop robust stability while the second drives the requirement for gain-scheduling. In addition, a rapid closed-loop response to position commands is desired. The ability of a control synthesis approach to handle the trade-offs between robustness, performance, and gain scheduling with the least possible conservatism is thus critical for such a system. For example, if the gain-scheduling parameters are allowed to vary infinitely quickly, closed-loop performance and robustness will suffer. If the uncertainty structure of the design model is not considered, it will be impossible to find a controller which meets design objectives. These trade-offs are studied and addressed in this section.

SECAFLEX is a two-link flexible planar manipulator driven by geared DC motors, used as a laboratory platform for control-structure interaction experiments at CERT-ONERA in Toulouse, France. The two flexible members are homogeneous beams. There is a concentrated mass at the elbow due to the DC motor and a concentrated mass at the tip of the second beam which is the payload. The modeling of the manipulator has been studied extensively [28]. A simple drawing of the two-link manipulator is shown in Figure 1.

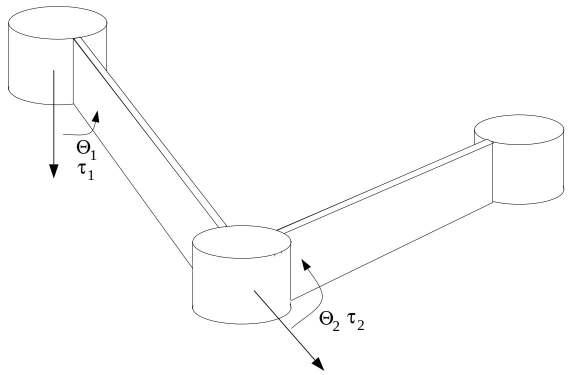


Figure 1: Two-Link Flexible Manipulator

$\theta_1$ and $\theta_2$ are the shoulder and elbow joint angles respectively. $\tau_1$ and $\tau_2$ are the corresponding control torques. The second-order form of the manipulator equations of motion are,

$$M(\theta_2)\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Fu(t) \tag{36}$$

where $M$ is the inertia matrix, $D$ is the damping matrix, and $K$ is the stiffness matrix. $u(t)$ is the input vector, $u = (\tau_1\ \tau_2)^T$. Due to the variable geometry of the system, the inertia matrix is a function of the second joint angle, $\theta_2$. This dependence causes significant changes in the response of the system to input torques over the range of possible configurations, $\theta_2 \in [0, \pi]$ (rad.). If we consider an output vector, $y = (\theta_1\ \theta_2)^T$, then we can define the transfer function from $u$ to $y$ as $G(s, \theta_2)$. With the parameter $\theta_2$ frozen in time, Figure 2 illustrates the variation of the manipulator dynamics with geometry by showing the singular values of $G(s, \theta_2)$ at three different values of $\theta_2$.

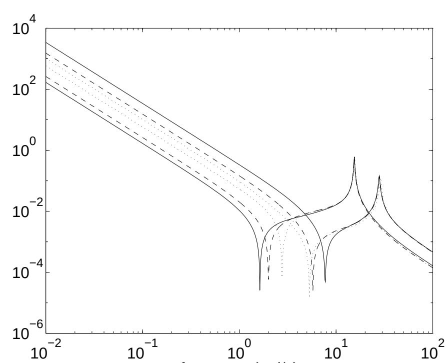


Figure 2: $\sigma_i(G(j\omega, \theta_2))$ for Different Manipulator Geometries
$\theta_2 = 0$: solid; $\theta_2 = \frac{\pi}{2}$: dashed; $\theta_2 = \pi$: dotted .

The numerical values which define this system are as follows. The dependence of the inertia matrix on $\theta_2$ can be expressed as,

$$M(\theta_2) = M(\pi/2) + cos(\theta_2)[M(\pi/2) - M(\pi)], \tag{37}$$

where

$$M(\pi/2) = \begin{bmatrix} 34.7077 & 9.7246 & 23.6398 & 5.9114 \\ 9.7246 & 9.8783 & 9.7246 & 5.9114 \\ 23.6398 & 9.7246 & 17.5711 & 5.9114 \\ 5.9114 & 5.9114 & 5.9114 & 3.7233 \end{bmatrix}, \quad M(\pi) = \begin{bmatrix} 17.0296 & 0.8856 & 9.7776 & 0.8430 \\ 0.8856 & 9.8783 & 4.7016 & 5.9114 \\ 9.7776 & 4.7016 & 7.5249 & 3.0311 \\ 0.8430 & 5.9114 & 3.0311 & 3.7233 \end{bmatrix}, \tag{38}$$

and the damping, stiffness, and control effectiveness matrices are respectively

$$D = \mathbf{diag}\{0, 0, 0.09, 0.05\}, \quad K = \mathbf{diag}\{0, 0, 89.1473, 45.6434\}, \quad F = \begin{bmatrix} I_{2\times2} \\ 0_{2\times2} \end{bmatrix}. \tag{39}$$

The manipulator equations of motion can be rewritten in first order LFT form in $\theta_2$ [29],

$$\dot{x}(t) = \begin{bmatrix} 0 & I \\ -M(\theta_2)^{-1}K & -M(\theta_2)^{-1}D \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ M(\theta_2)^{-1}F \end{bmatrix} u(t) \tag{40}$$

To provide closed-loop command tracking, a simple weighted minimization of the sensitivity function is used. A frequency dependent weight, $W_p$, penalizes the error, $e$, between angular position commands, $w_1$, $w_2$, and the system response, $y$. By forcing this weighted sensitivity function to be less than unity, the complementary sensitivity function approaches identity at low frequencies, thus providing good command tracking.

In order to account for uncertainties in high frequency dynamics, an additive uncertainty model is incorporated into the synthesis model. The additive uncertainty weight is formulated by considering the difference between the full-order geometry dependent model, $G(s, \theta_2)$, and some reduced order design model, $G_r(s, \theta_2)$, of lower order but still dependent on manipulator geometry. Consider $W_f$, the additive uncertainty weighting function and $\Delta_f$, a complex uncertainty block, scaled such that $||\Delta_f||_\infty < 1$. We can define the error between the full-order and reduced-order models as $E(s, \theta_2)$,

$$E(s, \theta_2) = G(s, \theta_2) - G_r(s, \theta_2) \tag{41}$$

The additive uncertainty weight must then provide a frequency domain bound on this error, that is,

$$|W_f(j\omega)| \geq \max_{\theta_2} \sigma_{max}(E(j\omega, \theta_2)), \qquad \forall\omega \in [0, \infty] \tag{42}$$

The LPV design model has the form (2) and is built by combining the above performance and robustness formulations into a single multi-objective synthesis model, Figure 3.

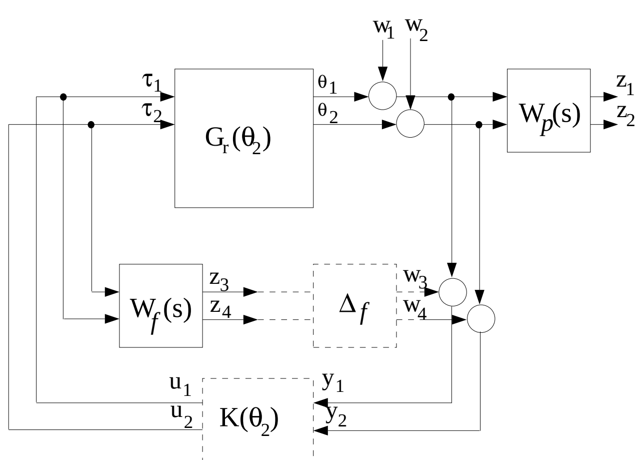


Figure 3: Synthesis Model

The design weights used for the following examples are

$$W_f = \frac{4(s+.1)^2}{(s+100)^2} I_{2\times2}, \quad W_p = \frac{0.1075(s+1.066)}{(s+0.03)} I_{2\times2}\,. \tag{43}$$

It should be noted here that, in the manipulator example, $\theta = \theta_2$ is not an independently evolving external parameter, it is a state of the system. By treating $\theta_2$ as an external parameter, we are actually immersing the "quasi-linear" dynamics (40) into the larger class of LPV dynamics (30). Therefore, in guarantying stability and performance for a class of parameter trajectories (through bounds on $\theta_2$ and $\dot\theta_2$), we ignore the fact that the trajectories themselves are defined by the plant dynamics. The result is thus a degree of conservatism in the design. The resulting controller is in the class of nonlinear controllers since $\theta := \theta_2$ is here a sub-vector of the measurement $y := (\theta_1, \theta_2)^T$. It is usually referred to as a *scheduled on the plant output* controller [30]. It captures the nonlinearity of the plant (40) in $\theta_2$. The controller matrices $(A_K, B_K, C_K, D_K)$ are instantaneously updated with respect to the plant's state $\theta_2$, as described in Section 2.

## 6.2 Numerical examples

Different cases have been investigated which highlight the merits and some of the concerns in using the gain-scheduling methodologies detailed in Sections 2 to 5. Comparisons with existing gain-scheduling techniques are also given.

**Case #1:** In this first case, the two-block structure associated with robustness and performance is ignored. The basic and projected techniques are applied with both $X$ and $Y$ depending on $\theta_2$.

$$X(\theta_2) := X_0 + \cos(\theta_2) X_1\,, \quad Y(\theta_2) := Y_0 + \cos(\theta_2) Y_1\,.$$

As discussed earlier, this amounts to assuming that the scheduled variable $\theta_2$ is frozen in time. The corresponding $\gamma$ levels are compared with the LFT gain-scheduling technique in [8], a technique that puts no bound on the parameter variation rates. The results are presented in Table 2. Surprisingly, all techniques give the same result. The achieved value of $\gamma$ is actually a lower bound that can be checked by performing an $\mathbf{H}_\infty$ synthesis on a nominal model ($\theta_2 = \frac{\pi}{2}$). Therefore, if the structure of the problem is ignored, the techniques in this paper or [7, 8] may offer no advantages over existing LFT gain-scheduling techniques.

| | |
|---|---|
| Basic Technique with $X(\theta_2)$ and $Y(\theta_2)$ | 3.82 |
| Projected Technique with $X(\theta_2)$ and $Y(\theta_2)$ | 3.82 |
| LFT Technique in [8] | 3.82 |

Table 2: Performance comparisons ($\gamma$) with ignored structure.

**Case #2:** In this second case, the problem's uncertainty structure is explicitly taken into account. The performance and robustness objectives are relaxed by introducing a fixed scaling $S := \mathbf{diag}(0.5e\text{-}3\, I_{2\times2}, I_{2\times2})$, found by performing a standard $\mu$ synthesis with constant scaling on the nominal plant ($\theta_2 = \frac{\pi}{2}$). The application of gain-scheduling techniques to this scaled problem leads to the bound, $\gamma = 0.58$. Results are presented in Table 3 for different selections of the Lyapunov variables $X$ and $Y$ and assuming $\theta_2$ fixed in time ($\dot\theta_2 = 0$).

Both the basic and projected techniques guarantee the same performance bound for all admissible values of $\theta_2$, provided that both $X$ and $Y$ depend on $\theta_2$. A slight degradation occurs when the dependence on $\theta_2$ is restricted to $X$, but the design is still acceptable. Conversely, when $Y$ only depends on $\theta_2$, both techniques appear very conservative. As predicted in Section 3, the selections of columns 2 and 3 are not equivalent and both must be tested to reduce conservatism. The designs in column 4 use fixed $X$ and $Y$ matrices and thus yield much more conservative answers. The corresponding value, $\gamma = 380.60$, was also obtained using the LFT technique in [8]. The consequences of this discussion is threefold.

- Ignoring the uncertainty structure introduces undesirable limitations on the power of advanced gain-scheduling techniques. One must compromise between robustness/performance requirements and gain-scheduling objectives in order to fully benefit from such techniques.
- Tradeoffs can be systematically handled through the use of scalings. Good initial guesses for such scalings can often be found on the basis of a nominal system issued from the LPV plant.
- The basic and the projected techniques give about the same results, the first one being preferable in a real-time implementation perspective.

| $\gamma$ | $X(\theta_2)$, $Y(\theta_2)$ | $X(\theta_2)$, $Y_0$ | $X_0$, $Y(\theta_2)$ | $X_0$, $Y_0$ |
|---|---|---|---|---|
| Basic Technique | 0.58 | 0.95 | 25.51 | 380.60 |
| Projected Technique | 0.58 | 0.95 | 25.35 | 380.60 |

Table 3: Performance Comparisons of Basic and Projected Techniques ($\dot{\theta} = 0$)

**Case #3:** We investigate here the effects of a bound on the rate of variation of $\theta_2$ on the achievable robustness/performance level with the projected synthesis technique. The scaling $S$ is as defined previously, and we assume symmetric bounds,

$$|\dot{\theta}_2| < \alpha\,.$$

It is intuitively clear that increasing the bound on the variation of $\theta_2$ degrades both robustness and performance. For our manipulator system a realistic bound is 100 deg./sec. As can be observed in Figure 4 which describes $\gamma$ as a function of $\alpha$, the proposed techniques perform well for variations of up to an order of magnitude greater than those expected.

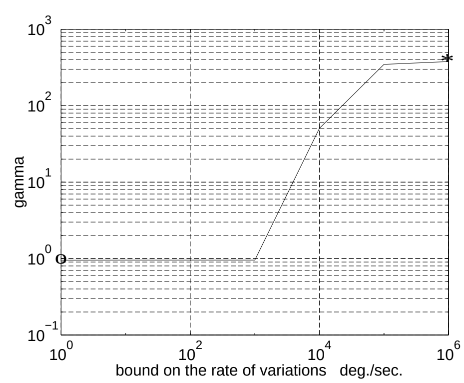


Figure 4: Performance level $\gamma$ vs. bound on the rate of variations
$\star$: value of $\gamma$ for unbounded derivatives,
$\circ$: value of $\gamma$ for frozen parameter.

**Case #4:** In this last case, the iterative schemes proposed in Section 5 are used to further improve robustness and performance. We exploit the scheme based on the matrix inequalities (33), (35), and (14).

Recall that such schemes take advantage of parameter-dependent scalings. According to the definition of $\mathcal{S}_\Delta$ in (29), the scaling assumes here the special form

$$S^{-1}(\theta_2) := \Sigma_0 + \cos(\theta_2)\Sigma_1 \, .$$

The Lyapunov variables have been selected in the form $X = X_0 + cos(\theta_2)X_1$ and $Y = Y_0$. Practical validity of the controller is thus ensured from the analysis in Section 3. The evolution of $\gamma$ during the alternate iterations is depicted in Figure 5. Convergence required 12 elementary steps as described in Section 5 and led to a best $\gamma$ value of 0.30. This is obviously the best design among those attempted. The result in Figure 5 was achieved with the projected technique and the characterizations given in (33) and (35) for the scaling computation. The resulting scaling matrix is

$$S^{-1}(\theta_2) := \begin{bmatrix} 3.363 I_2 & 0 \\ 0 & 0.020 I_2 \end{bmatrix} + \cos(\theta_2) \begin{bmatrix} 0.074 I_2 & 0 \\ 0 & 2e\text{-}4 I_2 \end{bmatrix} \, .$$

Finally, using the above scaling, we recomputed the gain-scheduled controller with the basic technique. The same value, $\gamma = 0.30$, was obtained. Since it is of a simpler form and provides satisfactory performance, this last gain-scheduled controller is used in the subsequent analysis and simulations.

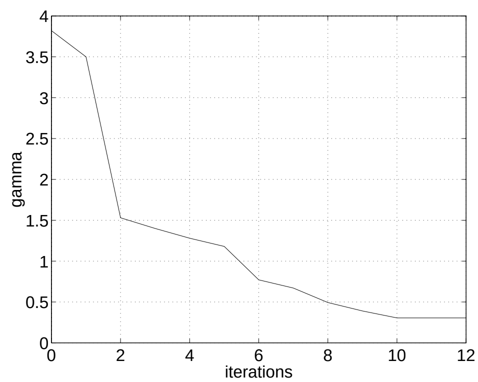


Figure 5: Evolution of $\gamma$ vs. alternate iterations

## 6.3 Frequency and time-domain validations

In this section, the best-$\gamma$ controller from case #4 is analyzed. In Figure 2 we saw how the manipulator dynamics changed with $\theta_2$. It is now interesting to examine how the resulting gain-scheduled controller varies with manipulator geometry. Figure 6 shows the singular values of the underlying LTI controllers $K(s, \theta_2)$ at three different values of $\theta_2$.

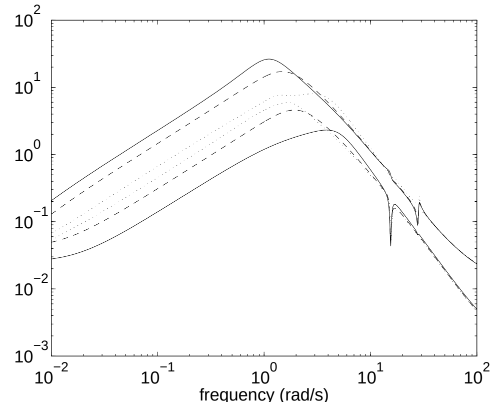


Figure 6: $\sigma_i(K(j\omega, \theta_2))$ for Different Manipulator Geometries
$\theta_2 = 0$: solid; $\theta_2 = \frac{\pi}{2}$: dashed; $\theta_2 = \pi$: dotted .

We see that the gain-scheduled controller evolves as physically expected, applying higher gains when the manipulator inertias are greater ($\theta_2 = 0$) and reduced gains when the inertias are smaller ($\theta_2 = \pi$). Notice that at frequencies above 10 rad/sec, both the manipulator system and the controller are relatively independent of the parameter.

Extensive nonlinear simulations of the response of the closed-loop system to various commands have been performed. For the sake of brevity, we will only present one of these. The high frequency flexible modes that were removed from the synthesis model are reintroduced in the simulation model. With initial conditions of $\theta_1(0) = 0$ and $\theta_2(0) = 0$, a step command of 180 degrees in both angles is given. This manoeuver was chosen to take the manipulator through the entire range of possible dynamics as quickly as possible. The angular responses and corresponding control inputs are depicted in Figure 7. The rise time (3.5 seconds), settling time (5 seconds), and overshoot (< %5) are markedly superior to those observed using either robust LTI controllers or heuristically motivated gain-scheduling approaches [29]. The manoeuver is completed without violating the limits on control authority, $|\tau_1| \leq 100N/m$ and $|\tau_2| \leq 20N/m$.

# 7 Conclusions

Advanced gain-scheduling design approaches for LPV systems have been presented with emphasis on the practical goals of reduced computational burden and ease of implementation. Two complementary techniques for the calculation of such controllers have been investigated which, when used together, achieve these two objectives. The methodology is completed with a new scaling technique that takes into account the uncertainty structure of multi-objective synthesis problems. The challenging problem of the control of a two-link flexible manipulator is introduced in this context and used to demonstrate the validity of the theoretical solutions.

# Appendix A

Proof of Lemma 5.1: The system governed by (30) and (26)-(28) is stable whenever there exists a quadratic Lyapunov function $V(x, \theta) = x^T P(\theta) x$ such that $P(\theta) > 0$ and

$$\frac{d}{dt} V(x, \theta) < 0 \,, \tag{44}$$

for all admissible trajectories $x(t)$, $w(t)$, $z(t)$, $\theta(t)$ and $\Delta(t)$.

The signals $w$ and $z$ related by (26)-(28) satisfy the quadratic constraints

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} \gamma^{-1} S(\theta) & 0 \\ 0 & -\gamma S(\theta) \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \geq 0, \qquad \forall t \geq 0, \tag{45}$$

where $S$ is in $\mathcal{S}_\Delta$ and may depend on $\theta$. Therefore, stability is ensured whenever (44) holds for all trajectories $x(t)$, $\theta(t)$ and any pair $(w(t), z(t))$ characterized by (45). By a $\mathcal{S}$-procedure argument [31, 32], this is guaranteed whenever

$$\frac{d}{dt} V(x,\theta) - \gamma w^T S w + \gamma^{-1} z^T S z \; < \; 0 \, .$$

This expression can be rewritten in the form

$$2(Ax + Bw)^T P x + x^T \dot{P} x + \gamma^{-1} (Cx + Dw)^T S (Cx + Dw) - \gamma w^T S w \; < \; 0,$$

which is nothing else than the quadratic form condition associated with the matrix inequality in (31). This completes the proof of the lemma. ■

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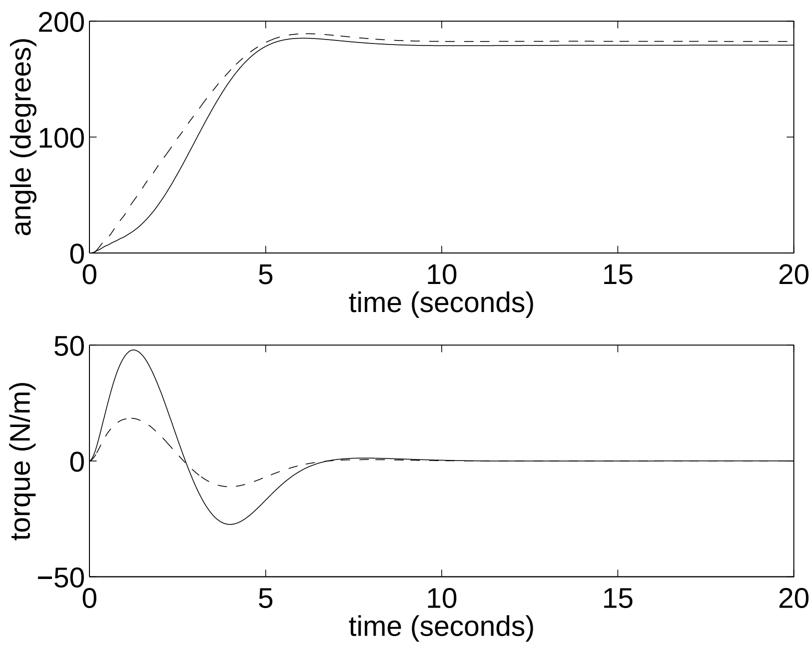


Figure 7: Nonlinear Simulation Results
solid: $\theta_1$, $\tau_1$; dashed: $\theta_2$, $\tau_2$.

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