Parameter-Dependent Lyapunov Functions for Robust Control of Systems with Real Parametric Uncertainty

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Abstract

This paper is concerned with the robust control problem of plants subject to real parametric uncertainties. The proposed technique builds upon the use of parameter-dependent quadratic Lyapunov functions. Such Lyapunov functions are used to derive sufficient conditions for the existence of controllers ensuring robust performance of the closed-loop system. These conditions lead to a complete synthesis technique, based on a relaxation algorithm reminiscent of μ-synthesis schemes. It alternates analysis phases and synthesis phases both characterized by tractable conditions in the form of Linear Matrix Inequalities (LMIs). The major advantage of the proposed technique is to produce robust controllers whose order is the same as the original plant. It allows to bypass the frequency sampling and curve fitting steps often critical in μ synthesis algorithms. A simple illustrative application demonstrates that the approach in this paper compares favorably to traditional μ-synthesis.

Keywords: Robust stability, Robust $\mathbf{H}_\infty$ Performance, Real parametric uncertainty, Linear matrix inequalities.

Notation and Useful Tools

The notations used throughout the paper are fairly standard. For real symmetric matrices M, $M > 0$ stands for "positive definite" and means that all the eigenvalues of M are positive. Similarly, $M < 0$ means "negative definite" (all the eigenvalues of M are negative) and $M \geq 0$ stands for "nonnegative definite" (the smallest eigenvalue of M is nonnegative). The notation $\mathrm{Ker}(M)$ designates the null space of the linear operator associated with M.

$\mathbf{L}_2(\mathbf{R}^n)$ denotes the Hilbert space of the functions w mapping $\mathbf{R}_+$ into $\mathbf{R}^n$ with bounded $\mathbf{L}_2$ norm, that is,

$$\|w\|_2 := \sqrt{\int_0^\infty w(t)^T w(t)\, dt} < \infty .$$

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The $\mathbf{L}_2$-induced norm of an operator T mapping $\mathbf{L}_2$ into $\mathbf{L}_2$ is noted $\|T\|_\infty$. This norm coincides with the usual $\mathbf{H}_\infty$ norm when T is a linear time-invariant operator. In such a case, it is alternatively defined as

$$\|T(s)\|_\infty = \sup_{\omega \in \mathbf{R}} \sigma_{max}(T(i\omega)),$$

where s is the Laplace variable, $T(s)$ is the transfer function of the operator T and $\sigma_{max}(X)$ stands for the maximum singular value of the matrix X. Given a set of matrices $M_1 \in \mathbf{R}^{n_1 \times p_1}, \ldots, M_L \in \mathbf{R}^{n_L \times p_L}$, and defining $n = \sum_{i=1}^L n_i$, $p = \sum_{i=1}^L p_i$, we define $\mathbf{diag}(M_1, \ldots, M_L)$ as the $n \times p$ block-diagonal matrix

$$\begin{bmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & M_L \end{bmatrix}.$$

Block-diagonal matrices of this form will also be denoted $\mathbf{diag}_{i=1}^L M_i$, hereafter.

The following Lemma is instrumental in our derivation technique.

Lemma 1 (Projection Lemma) *Given a symmetric matrix Ψ and two matrices U and V with suitable dimensions, there exists a matrix K such that*

$$\Psi + U^T K V + V^T K^T U < 0 \tag{1}$$

if and only if

$$\mathcal{N}_U^T \Psi \mathcal{N}_U < 0, \;\; \mathcal{N}_V^T \Psi \mathcal{N}_V < 0\,, \tag{2}$$

where $\mathcal{N}_U$, $\mathcal{N}_V$ are any bases of the nullspaces of U and V, respectively.

Proof: See [1, 2] for instance. ■

Problem Statement and Introduction

In this paper, we consider the uncertain linear system

$$\begin{array}{rcl} \frac{d}{dt}x &=& (A_0 + \sum_{i=1}^{L} \delta_i A_i)x + B_1 w + B_2 u \\ z &=& C_1 x + D_{11} w + D_{12} u \\ y &=& C_2 x + D_{21} w, \end{array} \tag{3}$$

where $x : \mathbf{R}_+ \to \mathbf{R}^n$ is the state vector, $w : \mathbf{R}_+ \to \mathbf{R}^{m_1}$ is an exogenous input, $u : \mathbf{R}_+ \to \mathbf{R}^{m_2}$ is the control input, $z : \mathbf{R}_+ \to \mathbf{R}^{p_1}$ is the controlled output, $y : \mathbf{R}_+ \to \mathbf{R}^{p_2}$ is the measurement vector. The uncertainties δ_i are *fixed in time* but unknown and are *real-valued* in the hypercube of radius $\frac{1}{\alpha}$, that is, $|\delta_i| \leq \frac{1}{\alpha}$, $i = 1, \ldots, L$, for some given positive number α. We address the problem of *robust control with guaranteed* $\mathbf{H}_\infty$ *performance*: does there exist a linear output feedback controller

$$\begin{array}{rcl} \frac{d}{dt}x_K &=& A_K x_K + B_K y \\ u &=& C_K x_K + D_K y\,, \end{array} \tag{4}$$

where $x_K : \mathbf{R}_+ \to \mathbf{R}^k$, such that the $\mathbf{H}_\infty$ norm of the closed-loop system from input w to output z is less than some prescribed value γ, for all possible values of the unknown parameters $\delta_1, \ldots, \delta_L$?

This question is central to many practical problems of control engineering. Indeed, the system dynamics are never perfectly known and designers must account for real parametric uncertainties corrupting the state-space representation of the plant.

Though the work in this paper is reproducible for other performance measures such as the $\mathbf{H}_2$ norm, our choice of the $\mathbf{H}_\infty$ norm has been motivated by practical considerations. Yet, it is widely accepted as a powerful way to formulate loop shaping design objectives and to compromise between conflicting specifications like robustness and performance.

Despite intensive research activity, the treatment of real parametric uncertainties is still a challenging issue both from analysis and synthesis viewpoints. The companion paper [3] is the analysis counterpart of this one and presents a fairly complete bibliography. Recall that proving stability of (3) using parameter-dependent Lyapunov functions consists in finding symmetric matrices P_0, $P_1, \ldots, P_L$ such that for all admissible values of the parameters $\delta_1, \ldots, \delta_L$, the following inequalities hold

$$x^T(P_0 + \delta_1 P_1 + \ldots + \delta_L P_L)x > 0, \quad x \neq 0\,,$$

and

$$2\left((A_0 + \sum_{i=1}^{L} \delta_i A_i)x\right)^T (P_0 + \sum_{i=1}^{L} \delta_i P_i)x < 0, \quad x \neq 0\,.$$

These inequalities involve an infinite number of constraints. Using the concept of "multi-convexity" [4] or by a S-procedure argument [3], these inequalities can be turned into a finite number of simpler conditions. These conditions express in the form of LMIs and therefore fall into the scope of efficient optimization software [5]. It was further shown in [3], how this new stability criterion could be expressed in the frequency domain and compared with Popov's stability criterion and the real μ upper bound by Fan, Tits and Doyle [6]. Computationally attractive, the proposed test was experimentally proved to perform as well as the real μ upperbound, while circumventing frequency sampling problems.

This paper is essentially an extension of the analysis technique developed in [3] to the robust control problem with guaranteed $\mathbf{H}_\infty$ performance. A similar problem was considered in the K_m or μ synthesis techniques [7, 8, 9]. Such techniques take advantage of multiplier theory to derive sufficient conditions for robust performance in the presence of real parametric uncertainties. Moreover, for a fixed scaling or multiplier the controller construction simply reduces to solving an (augmented) $\mathbf{H}_\infty$ control problem. It is therefore possible to alternate analysis phases with synthesis phases to optimize both the performance and the robustness of the closed-loop system. A currently observed weakness of original versions of such approaches was the intermediate step of curve fitting that may lead to numerical difficulties in the optimization procedure. Note that such difficulties are theoretically solved in [8, 9] where it is shown that multipliers with fixed degree can be directly computed, thus circumventing the curve fitting phase. It is our opinion, however, that such approaches raise questions about the necessary order of multipliers to ensure good convergence and, more importantly, about the required complexity of resulting controllers.

This paper exploits a different vein based on the notion of parameter-dependent Lyapunov functions [4, 3]. Such a technique does not require any dynamic augmentation and is demonstrated to produce controllers having the same order as the plant while bypassing the weak phase of curve fitting. A complete synthesis technique is developed. It is reminiscent of μ synthesis schemes in alternating analysis with synthesis phases, both easily implementable on a computer.

The paper is organized as follows. The main results are presented in Section 1. Simple conditions assessing closed-loop $\mathbf{H}_\infty$ performance of the uncertain plant are derived first. The explicit construction of robust controllers is detailed next, with particular emphasis on the practical benefits of the proposed approach. In light of the theoretical results of Section 1, a simple (relaxation) synthesis scheme is presented in Section 2, along with a discussion on computational complexity. Finally, Section 3 gives a simple illustration of the technique and comparison results with the complex μ synthesis.

1 Robust $\mathbf{H}_\infty$ Control with Parameter-Dependent Lyapunov Functions

1.1 Problem setup

Consider the basic system (3) and the controller (4). In order to derive compact solvability conditions and for ease of manipulation, the following notations are introduced

$$\mathcal{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \qquad \mathcal{C}_1 = \begin{bmatrix} C_1 & 0 \end{bmatrix},$$

$$\mathcal{B}_2 = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, \quad \mathcal{C}_2 = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}, \quad \mathcal{D}_{12} = \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \quad \mathcal{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}, \tag{5}$$

$$\mathcal{B}_\delta = \begin{bmatrix} B_\delta \\ 0 \end{bmatrix}, \qquad \mathcal{C}_\delta = \begin{bmatrix} C_\delta & 0 \end{bmatrix},$$

with

$$B_\delta = \begin{bmatrix} A_1 & \cdots & A_L \end{bmatrix}; \qquad C_\delta^T = \begin{bmatrix} I & \cdots & I \end{bmatrix}. \tag{6}$$

Gathering the controller parameters into the single variable

$$K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}, \tag{7}$$

and defining the augmented state variable $x_{\rm cl}^T = \begin{bmatrix} x^T & x_K^T \end{bmatrix}$, the closed-loop system equations reduce to

$$\begin{aligned} \frac{d}{dt}x_{\rm cl} &= (\mathcal{A}_0 + \mathcal{B}_2 K \mathcal{C}_2 + \mathcal{B}_\delta \Delta \mathcal{C}_\delta)x_{\rm cl} + (\mathcal{B}_1 + \mathcal{B}_2 K \mathcal{D}_{21})w, \;\; x_{\rm cl}(0) = x_0, \\ z &= (\mathcal{C}_1 + \mathcal{D}_{12} K \mathcal{C}_2)x_{\rm cl} + (D_{11} + \mathcal{D}_{12} K \mathcal{D}_{21})w, \end{aligned} \tag{8}$$

with $\Delta = \mathbf{diag}_{i=1}^L \delta_i I$.

Defining

$$\begin{aligned} \mathcal{A}_{\rm cl} &= \mathcal{A}_0 + \mathcal{B}_2 K \mathcal{C}_2, & \mathcal{B}_{\rm cl} &= \mathcal{B}_1 + \mathcal{B}_2 K \mathcal{D}_{21}, \\ \mathcal{C}_{\rm cl} &= \mathcal{C}_1 + \mathcal{D}_{12} K \mathcal{C}_2, & \mathcal{D}_{\rm cl} &= D_{11} + \mathcal{D}_{12} K \mathcal{D}_{21}, \end{aligned}$$

the system (8) further simplifies to

$$\begin{aligned} \frac{d}{dt}x_{\rm cl} &= (\mathcal{A}_{\rm cl} + \mathcal{B}_\delta \Delta \mathcal{C}_\delta)x_{\rm cl} + \mathcal{B}_{\rm cl} w \\ z &= \mathcal{C}_{\rm cl} x_{\rm cl} + \mathcal{D}_{\rm cl} w. \end{aligned} \tag{9}$$

Note that the uncertainty block Δ only affects the A matrix in the closed-loop system (9). Extensions to handle the more general case where B_1, C_1 and D_{11} are depending on Δ are easily deduced from the results in this paper at the expense of more intricate notations.

1.2 Closed-loop robust performance assessment

Determining necessary and sufficient conditions for robust performance of the system (9) has high computational complexity. This section focuses on deriving simpler *sufficient* conditions ensuring both stability and an $\mathbf{H}_\infty$ performance bound for the closed-loop system (9). A particularly fruitful technique to check robust performance of uncertain systems uses adequate Lyapunov functions and has been examined in numerous references [10, 11, 12, 13, 14], to cite a few. The derivation here relies on the notion of parameter-dependent Lyapunov functions as discussed in [15, 16, 4, 3].

As mentioned earlier, the uncertainty Δ is real-valued and constrained by

$$-\frac{1}{\alpha} \le \Delta \le \frac{1}{\alpha}. \tag{10}$$

Assume there exists a parameter-dependent quadratic Lyapunov function $V(x_{\rm cl}, \Delta) = x_{\rm cl}^T P(\Delta) x_{\rm cl}$ such that for any nonzero $x_{\rm cl}$ and any input w, the following conditions hold

$$V(x_{\rm cl}, \Delta) > 0, \tag{11}$$

and

$$\frac{d}{dt} V(x_{\rm cl}, \Delta) + z^T z - \gamma^2 w^T w < 0, \tag{12}$$

then it is easily shown that the closed-loop system (9)

- is asymptotically stable,
- has $\mathbf{H}_\infty$ norm less than γ,

for all admissible uncertainties Δ. See, for example [4, 3] for details.

Note however that the inequalities in (11)-(12) involve an infinite number of constraints since all possible values of Δ must be checked. Following the idea introduced in [4] and [3], we are seeking simpler and computable conditions enforcing (11)-(12) with the help of affinely parameter-dependent Lyapunov functions. These functions are of the form

$$V(x_{\rm cl}, \delta) = x_{\rm cl}^T(\mathcal{P}_0 + \sum_{i=1}^{L} \delta_i \mathcal{P}_i) x_{\rm cl},$$

where $\mathcal{P}_0, \ldots \mathcal{P}_L$ are partitioned conformally with the dimensions of the plant and the controller, i.e,

$$\mathcal{P}_0 = \begin{bmatrix} X & N \\ N^T & E \end{bmatrix}, \quad \mathcal{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1, \ldots, L. \tag{13}$$

An alternate expression for $V(x_{\rm cl}, \Delta)$ is

$$V(x_{\rm cl}, \Delta) = x_{\rm cl}^T(\mathcal{P}_0 + \mathcal{C}_\delta^T \Delta P \mathcal{C}_\delta) x_{\rm cl}, \tag{14}$$

with $\mathcal{C}_\delta$ defined in (6) and $P = \mathbf{diag}_{i=1}^L P_i$. The condition (11) then becomes

$$x_{\rm cl}^T(\mathcal{P}_0 + \mathcal{C}_\delta^T \Delta P \mathcal{C}_\delta) x_{\rm cl} > 0, \;\; x_{\rm cl} \ne 0, \;\; -\frac{1}{\alpha} I \le \Delta \le \frac{1}{\alpha} I, \tag{15}$$

and the condition (12) now reads

$$\begin{array}{l} 2((\mathcal{A}_{\rm cl} + \mathcal{B}_\delta \Delta \mathcal{C}_\delta) x_{\rm cl} + \mathcal{B}_{\rm cl} w)^T (\mathcal{P}_0 + \mathcal{C}_\delta^T \Delta P \mathcal{C}_\delta) x_{\rm cl} \\ \quad + (\mathcal{C}_{\rm cl} x_{\rm cl} + \mathcal{D}_{\rm cl} w)^T (\mathcal{C}_{\rm cl} x_{\rm cl} + \mathcal{D}_{\rm cl} w) - \gamma^2 w^T w < 0, \;\; x_{\rm cl} \ne 0, \;\; -\frac{1}{\alpha} I \le \Delta \le \frac{1}{\alpha} I. \end{array} \tag{16}$$

Following a path already presented in the companion paper [3], the positivity condition (15) is easily inferred from the constraints $\mathcal{P}_0 > 0$ and (16). With this in mind and by a $\mathcal{S}$-procedure argument, see [3], sufficient conditions guaranteeing robust $\mathbf{H}_\infty$ performance are obtained. This is presented in the next theorem.

Theorem 1 (Analysis theorem) *Consider the uncertain system governed by (3) and the controller described in (4). The closed-loop system (9) is stable and has $\mathbf{H}_\infty$ norm less than γ for all Δ constrained by (10), if there exist symmetric matrices $\mathcal{P}_0$, $P = \mathbf{diag}_{i=1}^{L} P_i$, $S = \mathbf{diag}_{i=1}^{L} S_i$ and a skew-symmetric matrix $T = \mathbf{diag}_{i=1}^{L} T_i$, where $P_i \in \mathbf{R}^{n\times n}$, $S_i \in \mathbf{R}^{n\times n}$ and $T_i \in \mathbf{R}^{n\times n}$, $i = 1, \ldots, L$ such that*

$$\left[\begin{array}{ccc|c} \mathcal{A}_{\rm cl}^T\mathcal{P}_0 + \mathcal{P}_0\mathcal{A}_{\rm cl} + \mathcal{C}_\delta^T S\mathcal{C}_\delta & \mathcal{P}_0\mathcal{B}_{\rm cl} & \mathcal{C}_{\rm cl}^T & \mathcal{A}_{\rm cl}^T\mathcal{C}_\delta^T P + \mathcal{P}_0\mathcal{B}_\delta + \mathcal{C}_\delta^T T \\ \mathcal{B}_{\rm cl}^T\mathcal{P}_0 & -I_{m_1} & \mathcal{D}_{\rm cl}^T & \mathcal{B}_{\rm cl}^T\mathcal{C}_\delta^T P \\ \mathcal{C}_{\rm cl} & \mathcal{D}_{\rm cl} & -\gamma^2 I_{p_1} & 0 \\ \hline P\mathcal{C}_\delta\mathcal{A}_{\rm cl} + \mathcal{B}_\delta^T\mathcal{P}_0 - T\mathcal{C}_\delta & P\mathcal{C}_\delta\mathcal{B}_{\rm cl} & 0 & \mathcal{B}_\delta^T\mathcal{C}_\delta^T P + P\mathcal{C}_\delta\mathcal{B}_\delta - \alpha^2 S \end{array}\right] < 0 \qquad (17)$$

$$\mathcal{P}_0 > 0, \qquad S > 0\,.$$

Proof: see the companion paper [3]. ■

The inequalities appearing in Theorem 1 are LMIs in the variables $\mathcal{P}_0$, P, T, S and γ^2. Thus, for a fixed controller K, checking robust performance, i.e., for all admissible Δ, of the closed-loop system is a convex feasibility problem. In fact, the best performance level may be obtained by minimizing γ subject to the LMI constraint (17). Note that this test can be used to compute the best performance level for a given size of the parameter hypercube or conversely, the largest parameter hypercube for a given performance level.

The constraint (17) is however not simultaneously convex in the parameters $\mathcal{P}_0$, P, S, T and the controller K. Consequently, it cannot be used directly for synthesis, though little research has been devoted to the problem having this bilinear nature [17]. Using the projection Lemma 1, it is however possible to derive tractable solvability conditions. This is examined in the sequel.

1.3 Robust controller design with guaranteed $\mathbf{H}_\infty$ performance

The previous results are now used to derive simple solvability conditions for the controller construction problem. It is shown that if such a controller exists, then one can always find a controller whose order is at most the order of the plant (3). This is especially important for applications in aeronautics or robotics, where low-order controllers are considered as a desirable feature. This is a new and very attractive advantage of the proposed approach that offers a reliable mean to compute robust controllers having the same complexity as customary $\mathbf{H}_\infty$ controllers.

The particular form (17) suggests that if the variables P, S, T are maintained fixed, solving (17) in K, $\mathcal{P}_0$ with $\mathcal{P}_0 > 0$, can be given a structure similar to that of Riccati- [18] or LMI-based $\mathbf{H}_\infty$ synthesis algorithms [2]. This point is central to the next developments and will allow the conception of a complete synthesis procedure. It also constitutes the main result of the paper and is formalized in the following theorem.

Theorem 2 (Synthesis theorem) *Consider the uncertain system governed by (3). There exists a controller K such that the conditions of Theorem 1 are satisfied if and only if there exist symmetric* *matrices X, Y, $P = \mathbf{diag}_{i=1}^{L} P_i$ and $S = \mathbf{diag}_{i=1}^{L} S_i$ and a skew-symmetric matrix $T = \mathbf{diag}_{i=1}^{L} T_i$ such that*

$$\mathcal{N}_X^T \left[\begin{array}{ccc|c} A_0^T X + X A_0 + C_\delta^T S C_\delta & X B_1 & C_1^T & X B_\delta + A_0^T C_\delta^T P + C_\delta^T T \\ B_1^T X & -I_{m_1} & D_{11}^T & B_1^T C_\delta^T P \\ C_1 & D_{11} & -\gamma^2 I_{p_1} & 0 \\ \hline B_\delta^T X + P C_\delta A_0 - T C_\delta & P C_\delta B_1 & 0 & B_\delta^T C_\delta^T P + P C_\delta B_\delta - \alpha^2 S \end{array}\right] \mathcal{N}_X < 0\,, \tag{18}$$

$$\mathcal{N}_Y^T \left[\begin{array}{ccc|c|c} A_0 Y + Y A_0^T & B_1 & Y C_1^T & Y A_0^T C_\delta^T P + B_\delta + Y C_\delta^T T & Y C_\delta^T \\ B_1^T & -I_{m_1} & D_{11}^T & B_1^T C_\delta^T P & 0 \\ C_1 Y & D_{11} & -\gamma^2 I_{p_1} & 0 & 0 \\ \hline P C_\delta A_0 Y + B_\delta^T - T C_\delta Y & P C_\delta B_1 & 0 & B_\delta^T C_\delta^T P + P C_\delta B_\delta - \alpha^2 S & 0 \\ \hline C_\delta Y & 0 & 0 & 0 & -S \end{array}\right] \mathcal{N}_Y < 0\,, \tag{19}$$

$$\left[\begin{array}{cc} X & I \\ I & Y \end{array}\right] \geq 0\,, \tag{20}$$

where $\mathcal{N}_X$ and $\mathcal{N}_Y$ are any bases of the nullspaces of

$$\left[\begin{array}{cccc} C_2 & D_{21} & 0 & 0 \end{array}\right]; \qquad \left[\begin{array}{ccccc} B_2^T & 0 & D_{12}^T & B_2^T C_\delta P & 0 \end{array}\right], \tag{21}$$

respectively.

Proof: The proof involves the projection Lemma 1. Note first that it is possible to rewrite the inequality (17) in the format (1) by using Schur complements and defining

$$\Psi = \left[\begin{array}{ccc|c} \mathcal{A}_0^T \mathcal{P}_0 + \mathcal{P}_0 \mathcal{A}_0 + \mathcal{C}_\delta^T S \mathcal{C}_\delta & \mathcal{P}_0 \mathcal{B}_1 & \mathcal{C}_1^T & \mathcal{A}_0^T \mathcal{C}_\delta^T P + \mathcal{P}_0 \mathcal{B}_\delta + \mathcal{C}_\delta^T T \\ \mathcal{B}_1^T \mathcal{P}_0 & -I_{m_1} & \mathcal{D}_{11}^T & \mathcal{B}_1^T \mathcal{C}_\delta^T P \\ \mathcal{C}_1 & \mathcal{D}_{11} & -\gamma^2 I_{p_1} & 0 \\ \hline P \mathcal{C}_\delta \mathcal{A}_0 + \mathcal{B}_\delta^T \mathcal{P}_0 - T \mathcal{C}_\delta & P \mathcal{C}_\delta \mathcal{B}_1 & 0 & \mathcal{B}_\delta^T \mathcal{C}_\delta^T P + P \mathcal{C}_\delta \mathcal{B}_\delta - \alpha^2 S \end{array}\right], \tag{22}$$

and

$$V = \left[\begin{array}{cccc} \mathcal{C}_2 & \mathcal{D}_{21} & 0 & 0 \end{array}\right]; \qquad U = \left[\begin{array}{cccc} \mathcal{B}_2^T \mathcal{P}_0 & 0 & \mathcal{D}_{12}^T & \mathcal{B}_2^T \mathcal{C}_\delta^T P \end{array}\right]. \tag{23}$$

The first two inequalities (18)-(19) are obtained by explicitely computing the projections (2) with the notations (5)-(6), (13). For instance, the second inequality (19) is obtained by introducing

$$\mathcal{P}_0^{-1} = \left[\begin{array}{cc} Y & * \\ * & * \end{array}\right], \tag{24}$$

where the $*$'s are entries do not impact the solvability of the problem. The last inequality (20) is equivalent, for $k = n$ (full-order case), to requiring that $\mathcal{P}_0$ be positive-definite subject to the partitioning (13)-(24). See [19, 2] for a detailed discussion. ■

Now the characterization in (18)-(20) is a convex feasibility program in the variables γ^2, X, Y. It is therefore easily tractable using solvers like LMILab [20].

Moreover, paralleling the results of [2], the feasibility of LMIs (18)-(20) ensures the existence of a n-th order controller K solution to the robust control problem with garanteed $\mathbf{H}_\infty$ performance. Such a controller can be constructed as follows:

(1) compute full-column-rank matrices M, $N \in \mathbf{R}^{n \times k}$ such that

$$MN^T = I - YX,$$

(2) an adequate $\mathcal{P}_0$ is then obtained as the unique solution of the linear equation

$$\begin{pmatrix} X & I \\ N^T & 0 \end{pmatrix} = \mathcal{P}_0 \begin{pmatrix} I & Y \\ 0 & M^T \end{pmatrix},$$

(3) finally, K defined in (7), is obtained as any solution to the LMI problem (17), where $\mathcal{P}_0$, P, S and T are all fixed matrices.

The reader is referred to [2, 21] for further discussion on the computation of K. Note that one important consequence of Theorem 2 is that the search for controllers ensuring robust performance via parameter-dependent quadratic Lyapunov functions, if successful, always generates controllers with the same order as the plant (3). We believe this is a significant improvement over current μ synthesis procedures.

1.4 Special cases and extensions

The solvability conditions given in Theorems 1-2 have specializations and extensions in various directions that are briefly mentioned.

- *Nominal plant:* if the uncertain plant (3) is restricted to a single plant, that is, does not depend on Δ, then the LMI characterizations (17), (18)-(19) reduce to their upperleft blocks which respectively correspond to the Bounded Real Lemma [22] and the LMI characterization occurring in $\mathbf{H}_\infty$ control theory [2].
- *Time-varying parameters:* we have assumed throughout that the parameters δ_i's are uncertain but not varying in time. The case of time-varying parameters is easily addressed by setting $P = 0$ in Theorems 1-2. Indeed, for arbitrarily fast time variations, the quadratic Lyapunov function is no longer depending on Δ. Note however that if the δ_i's have known bounded rate of variations, the results in this paper are easily extended to this case, see [3]. Other situations with mixed types of parameters are also easily incorporated.
- *Parameter dependence:* more complex parameter dependence can theoretically be integrated to our technique (multi-affine, multi-affine fractional). Unfortunately, they characterizations may be prohibitive even for small problems in regard to the computational time induced.

2 Synthesis Technique and Computational Issues

Based on the results of Section 1 simple schemes to effectively compute the state-space data K of the controller are now discussed. The proposed synthesis procedure is reminiscent of μ synthesis algorithms as it alternates analysis phases (Theorem 1) with synthesis phases (Theorem 2). Such schemes are guaranteed to converge to a local minimizer only, but have proved efficient in a number of interesting applications [23, 24, 25, 26, 27, 28]. Recall however, that while bypassing the critical phase of curve fitting, the technique in this paper always yields controllers having at most the order of the plant.

Similarly to μ synthesis algorithms, the proposed procedure simultaneously maximize the performance level γ and the parameter margins $\frac{1}{\alpha}$. Namely, we set $\alpha = \gamma$ in the LMI characterizations of Section 1. With this in mind, a possible procedure to compute an optimal controller is as follows.

step 0 *initialize* P, S, T.

step 1 P, S, T *given,*

$$\begin{array}{c} \textit{minimize } \gamma \\ \textit{subject to the LMI constraints (18)-(20)} \end{array}$$

step 2 *deduce a* $\mathcal{P}_0$ *from the pair* (X,Y), *and for* $\mathcal{P}_0$, P, S, T *given, compute the controller state-space data* K *as described in Theorem 1.*

step 3 K *given, perform an analysis test to determine* $\mathcal{P}_0$, P, S, T, *that is*

$$\begin{array}{c} \textit{minimize } \gamma \\ \textit{subject to the LMI constraint (17)} \end{array}$$

step 4 *iterate over steps* 1-3 *until* γ *cannot be decreased further.*

Note that step 1 has the same computational complexity as LMI-based $\mathbf{H}_\infty$ synthesis. Step 3 may be very demanding since it involves a number of scalar variables which is roughly $2n^2 + \frac{3Ln^2}{2}$. This number can be further reduced by noticing that a candidate $\mathcal{P}_0$ for step 3 is provided by step 2. As a result, the number of variables in step 3 is roughly $3\frac{Ln^2}{2}$. This number may be still prohibitive for large systems with available solvers. Note finally that if the procedure ends up with a controller K and a performance level γ then for all values of the parameter Δ in the hypercube of radius $\frac{1}{\gamma}$ the closed-loop system (9) will be stable and have $\mathbf{H}_\infty$ norm less than γ.

3 An Illustrative Example

This section presents a simple illustration of the proposed technique to the control of an aircraft pitch axis. The problem under consideration has been deliberately simplified to facilitate illustration of the technique and give some first comparison results with μ synthesis has defined in [29].

3.1 Problem formulation

The aircraft dynamics are highly depending on the angle of attack α, the air speed V and the altitude H. These three variables completely define the flight condition or operating point of the aircraft. Therefore, based on the linearization of the aircraft equations around its flight conditions, a simplified state-space representation can be developed. It is described in the form

$$\begin{array}{rcl} \begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} & = & \begin{bmatrix} -Z_\alpha & 1 \\ -M_\alpha & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_m \\ \begin{bmatrix} a_z \\ q \end{bmatrix} & = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} \end{array} , \qquad (25)$$

where α denotes the angle of attack, q is the pitch rate, a_z is the vertical acceleration, δ_m denotes the fin deflection and V is the air speed. The parameters Z_α, M_α are functions of the flight condition (α, V, H).

Hereafter, the parameter vector δ of the uncertain plant (25) is defined as

$$\delta := \begin{bmatrix} Z_\alpha \\ M_\alpha \end{bmatrix}.$$

In this application, we consider the classical control structure depicted in Figure 1 which can be associated with a mixed sensitivity optimization problem.

The performance objectives are expressed through the sensitivity function $S(s,\delta)$ while additive robustness is captured by the transfer function $K(s).S(s,\delta)$. Note that denoting $T_{a\rightarrow b}$ as the operator mapping the signal a into the signal b yields the well-known relations:

$$T_{w\rightarrow e}(\delta) = S(\delta) := [I + G(\delta)K]^{-1} \;\; ; T_{w\rightarrow u}(\delta) := K.S(\delta),$$

where all signals w, e and u are defined in Figure 1. The robust control problem consists in finding a controller $K(s)$ which satisfies for all admissible values of the parameter δ the following objectives:

- internal stability of the closed-loop system of Figure 1,
- the $\mathbf{H}_\infty$ norm of the closed-loop operator

  $$\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} .F_l(P(\delta), K)$$

  is minimized.

Here, W_1 and W_2 are frequency weights which are described next.

The synthesis structure containing all necessary ingredients is depicted in Figure 2. It is readily shown that the unweighted uncertain plant $P(s,\delta)$ is completely defined by the state-space relations:

$$\begin{bmatrix} \dot{x} \\ u \\ e \\ w \end{bmatrix} = \begin{bmatrix} A(\delta) & 0 & 0 & B \\ 0 & 0 & 0 & I \\ -C & I & -I & 0 \\ 0 & I & 0 & 0 \end{bmatrix} . \begin{bmatrix} x \\ w \\ d \\ u \end{bmatrix}, \tag{26}$$

The parameter range is described as

$$(Z_\alpha, M_\alpha) \in [0.4 \,;\, 4] \times [-20.0 \,;\, 20.0] .$$

Introducing a normalization of the parameter range, the matrix $A(\delta)$ in (26) can be re-expressed as

$$A(\delta) = \begin{bmatrix} -2.25 & 1.0 \\ 0 & 0 \end{bmatrix} + \delta_1 \begin{bmatrix} -1.75 & 0 \\ 0 & 0 \end{bmatrix} + \delta_2 \begin{bmatrix} 0 & 0 \\ -20 & 0 \end{bmatrix},$$

with now

$$(\delta_1, \delta_2) \in [-1 \,;\, 1] \times [-1 \,;\, 1] .$$

3.2 Weight Selection and Synthesis

Based on preliminary $\mathbf{H}_\infty$ syntheses on the "central" plant ($\delta = 0$), the performance weight $W_1(s)$ and the robustness weight $W_2(s)$ have been selected in order to guarantee adequate time responses and high frequency gain attenuation:

$$W_1(s) = \mathbf{diag}(\frac{0.001\,s + 2.0101}{s + 0.201}, 0.001);\; W_2(s) = \frac{267.37\,s^3 - 0.0035\,s + 78.90}{s^3 + 3.01e5\,s^2 + 3e8\,s + 2.72e11}$$

This problem has also been formulated for μ synthesis and solved using both techniques. Remind that both techniques involve a synthesis phase followed by an analysis phase etc... The evolutions of γ along iterations is depicted in Figure 3. Note first that our technique leads to a better value of γ ($\gamma_{\text{opt}} = 2.66$ for μ synthesis and $\gamma_{\text{opt}} = 1.28$ with our approach). Note also that the μ iteration process starts to increase γ after the 6th iteration. Thus, we have retained the μ controller corresponding to the smallest value of γ for comparison purposes. The allowable parameter ranges provided by both techniques are also presented in Figure 4. It is seen that the proposed technique guarantees a parameter range twice as bigger. More importantly, the μ controller has 18 states (possibly reducible to 14) but our controller requires only 6 states !

The proposed technique has been further tested by time-domain simulations for randomly generated plants in the parameter range $\frac{1}{\gamma_{\text{opt}}}[-1\,;\,1] \times [-1\,;\,1]$. This is shown in Figure 5 where a_z must follow a step reference signal. Here again, the controller provides satisfactory performance on the predicted parameter range.

4 Conclusions

We have presented a new synthesis technique based on the concept of parameter-dependent Lyapunov functions. The technique addresses the problem of robust controller synthesis for plants subject to real parametric uncertainties. Our mains results are Theorem 1 and 2 demonstrating that a synthesis scheme alternating analysis steps with synthesis steps is possible as they both involve easily tractable LMI conditions. The synthesis phase is given a separation structure similar to the Riccati-based characterization of $\mathbf{H}_\infty$ controllers that enables simple construction of controllers. As immediate benefits of our approach, we must point out that

- it does not require dynamic augmentation of the plant and hence produces controllers having the same order as customary $\mathbf{H}_\infty$ controllers,
- it allows to bypass the critical phase of curve fitting that causes numerical troublesome in traditional approaches to μ synthesis.

An illustrative example has demonstrated that our approach compares well to complex μ synthesis and holds promise to account for real parametric uncertainties.

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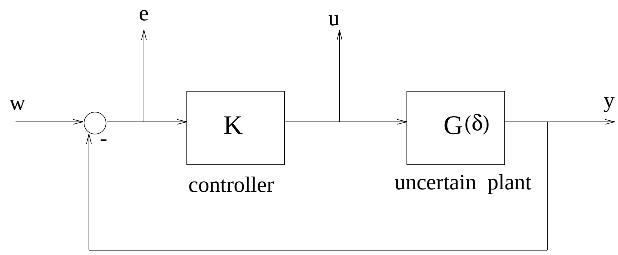

Figure 1: Control structure

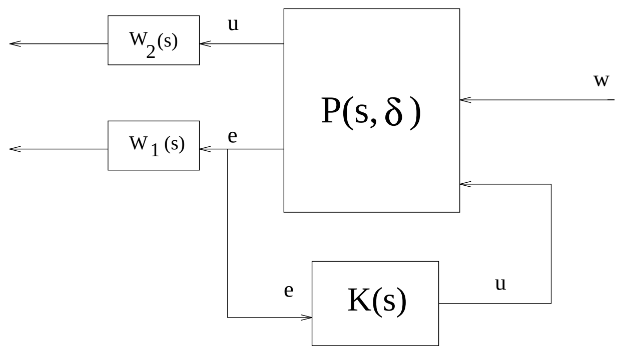

Figure 2: Synthesis interconnection structure

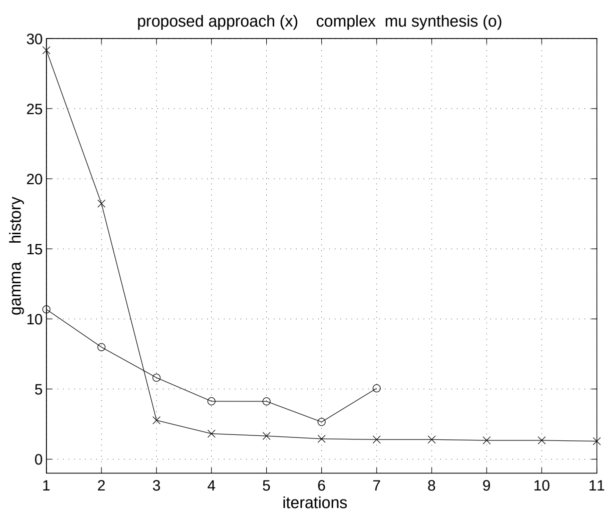

Figure 3: γ histograms for proposed approach and complex μ synthesis

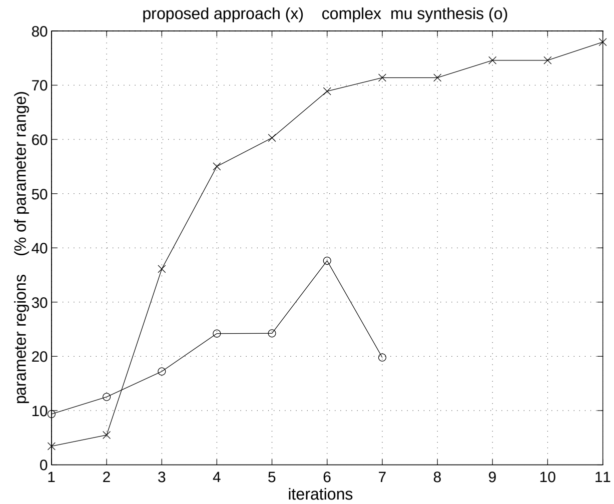

Figure 4: Admissible parameter ranges for proposed technique and μ synthesis

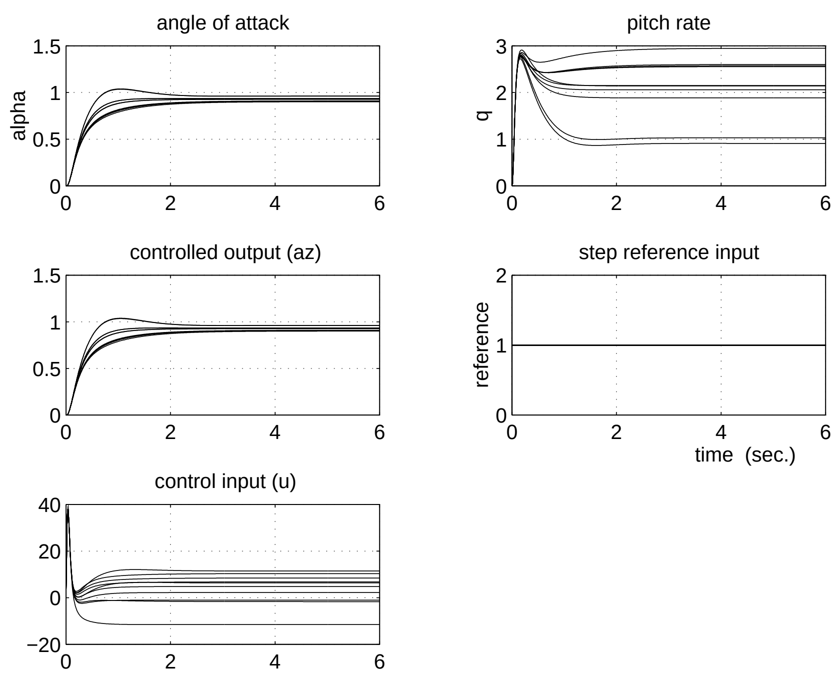

Figure 5: Time-domain simulations (proposed technique)