Monotonic Relaxation for Robust Control: New Characterizations

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Abstract

Parameterized Linear matrix inequalities (PLMIs), that is LMIs depending on a parameter confined to a compact set frequently arise in both analysis and synthesis problems in robust control. As a major difficulty, PLMIs are equivalent to an infinite family of LMI constraints and consequently are very hard to solve numerically. Known approaches to find solutions exploit relaxations inferred from convexity arguments. These relaxations involve a finite family of LMIs the number of which grows exponentially with the number of scalar parameters. In this paper, we propose a novel approach based on monotonic optimization which allows us to solve PLMIs via a finite and of polynomial order family of LMIs. The effectiveness and viability of our approach are demonstrated by numerical examples such as robust stability analysis and Linear Parameter Varying synthesis for which we clearly show that no additional conservatism is entailed as compared to earlier techniques.

1 Introduction

A central problem in robust control theory is to check the quadratic relation (see e.g. [5, 1, 12])

$$f(\alpha) = \alpha^T Q\alpha + q\alpha + p < 0, \ \forall\ \alpha \in \Gamma := [a, b] \in R^L_+, \tag{1}$$

where Q is a $L \times L$ symmetric matrix, $q^T \in (R^L)$ and $p \in R$. Γ denotes a hyper-rectangle in the positive orthant of R^L, i.e.

$$\Gamma := \{\alpha \in R^L : 0 \leq a_i \leq \alpha_i \leq b_i\}.$$

As it is known, a natural relaxation approach for checking (1) is to enforce some additional conditions on f so that it can be checked only by verifying the inequality at the vertices of the hypercube Γ that is on vertΓ. This way, an exponential number (2^L) of LMI constraints have to be solved. This is easily derived via convexity concepts since one of the most fundamental property of convex functions is that their maximum over the convex set Γ is attained on vertΓ. As an instance, the convexification result of [5, 1, 12] states that (1) holds whenever there exist scalar $\mu_i \geq 0,\ i = 1, 2, ..., L$ such that

$$f(\alpha) \leq \bar{f}(\alpha) := f(\alpha) + \sum_{i=1}^{L} \mu_i \alpha_i^2 < 0,\ \forall \alpha \in \text{vert}\Gamma \tag{2}$$

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and function $\bar{f}$ is multi-convex, that is, is convex along all directions $e^i = (0, 0, .., 1, 0, ..., 0)$ (canonical unit vector in R^L). More formally, this is expressed as

$$\frac{\partial^2 \bar{f}(\alpha)}{\partial^2 \alpha_i} = Q_{ii} + \mu_i \geq 0, \qquad i = 1, 2, ..., L. \tag{3}$$

In the same vein, one can take advantage of other properties of the box Γ beyond convexity. The box Γ is not only convex but is characterized by a specific ordering: its two vertices a and b are smallest and largest, meaning that $a_i \leq \alpha_i \leq b_i,\ \forall \alpha \in \Gamma,\ i = 1, 2, ..., L$. This condition is useful in the sense that not only convex functions attain their maximum at the vertices of Γ but there are other functions not characterized by any kind of convexity attribute which also meet the extreme point property. They are *monotonic functions.* Moreover monotonic functions attain their maximum/minimum on one of the smallest or largest vertices a or b. Therefore, it seems that for considering a function f on a box it is of practical interest to explore its monotonicity properties. In this paper, we shall see how the monotonic concept is useful and maybe more natural for solving (1). In particular, it allows us to reduce the feasibility of (1) to the feasibility of a finite and of *polynomial order* family of LMI constraints.

The structure of the paper is as follows. Basic results on monotonic optimization for (1) are considered in Section 2. Derivation techniques for various problems in robust control are discussed in Section 3. Finally, the viability of the proposed approach is demonstrated through some numerical examples in Section 4.

The notations used in the paper is rather standard. Particularly, $M > 0$ ($M \geq 0$, resp.) for a symmetric matrix M means that M is positive definite (positive semidefinite, resp.). Analogously, $M < 0$ ($M \leq 0$, resp.) means that M is negative definite (negative semidefinite, resp.). The symbol $*$ is used as an ellipsis for terms in matrix expressions that are induced by symmetry, e.g., $[M+N]+(*) = [M+N]+[M+N]^T$. For $x \in R^L, y \in R^L$, $x \leq y$ must be understood componentwise, i.e. $x_i \leq y_i,\ i = 1, 2, ..., L$. Let us also recall that a function $f : \Gamma \to R$ is increasing if and only if $f(x) \leq f(y)$ whenever $a \leq x \leq y \leq b$, and is decreasing if and only if $f(x) \geq f(y)$ whenever $a \leq x \leq y \leq b$.

2 Basic monotonic optimization result

For simplication, we introduce the positive scalar:

$$\tau = \sum_{i=1}^{L} (b_i - a_i).$$

The following lemma will be our main instrument in the next derivations.

Lemma 2.1 *The quadratic inequality (1) holds true whenever one of the following conditions is satisfied*

(i) $f(b) < 0$ *and either*

$$\begin{cases} \dfrac{\partial f(b)}{\partial \alpha_i} = 2e^{iT} Q b + q_i = 2\displaystyle\sum_{j=1}^{L} Q_{ij} b_j + q_i \geq 0, \qquad i = 1, 2, .., L \\ \dfrac{\partial^2 f(\alpha)}{\partial \alpha_i \partial \alpha_j} = e^{iT} Q e^j = Q_{ij} \leq 0, \qquad 1 \leq i \leq j \leq L, \end{cases} \tag{4}$$

or

$$\begin{cases} \dfrac{\partial f(a)}{\partial \alpha_i} = 2e^{iT}Qa + q_i = 2\sum_{j=1}^{L} Q_{ij}a_j + q_i \geq 0, & i = 1,2,..,L \\ \dfrac{\partial f(a+\tau e^{\ell})}{\partial \alpha_i} = 2\sum_{j=1}^{L} Q_{ij}a_j + q_i + 2\tau Q_{i\ell} \geq 0, & i,\ell = 1,2,..,L \end{cases} \tag{5}$$

(ii) $f(a) < 0$ *and either*

$$\begin{cases} \dfrac{\partial f(b)}{\partial \alpha_i} = 2e^{iT}Qb + q_i = 2\sum_{j=1}^{L} Q_{ij}b_j + q_i \leq 0, & i = 1,2,..,L \\ \dfrac{\partial^2 f(\alpha)}{\partial \alpha_i \partial \alpha_j} = e^{iT}Qe^j = Q_{ij} \geq 0, & 1 \leq i \leq j \leq L. \end{cases} \tag{6}$$

or

$$\begin{cases} \dfrac{\partial f(a)}{\partial \alpha_i} = 2e^{iT}Qa + q_i = 2\sum_{j=1}^{L} Q_{ij}a_j + q_i \leq 0, & i = 1,2,..,L \\ \dfrac{\partial f(a+\tau e^{\ell})}{\partial \alpha_i} = 2\sum_{j=1}^{L} Q_{ij}a_j + q_i + 2\tau Q_{i\ell} \leq 0, & i,\ell = 1,2,..,L. \end{cases} \tag{7}$$

If $f(a) < 0$ *and*

$$\begin{cases} \dfrac{\partial f(a)}{\partial \alpha_i} = 2e^{iT}Qa + q_i = 2\sum_{j=1}^{L} Q_{ij}a_j + q_i \leq 0, & i = 1,2,..,L \\ \dfrac{\partial^2 f(\alpha)}{\partial \alpha_i \partial \alpha_j} = e^{iT}Qe^j = Q_{ij} \leq 0, & 1 \leq i \leq j \leq L \end{cases} \tag{8}$$

then $f(\alpha) < 0 \ \forall \alpha > a$.

Proof. To prove the first statement we shall show that f is increasing on Γ. By the mean value theorem it suffices to show that $\frac{\partial f(\alpha)}{\partial \alpha_i} \geq 0, \ \forall \ \alpha \in \Gamma$ under conditions (4) or (5). Note that

$$\frac{\partial f(\alpha)}{\partial \alpha_i} = 2e^{iT}Q\alpha + q_i \tag{9}$$

and thus $\frac{\partial f(\alpha)}{\partial \alpha_i}$ is decreasing and $\frac{\partial f(b)}{\partial \alpha_i} \geq 0$ under condition (4). Hence, $\frac{\partial f(\alpha)}{\partial \alpha_i} \geq 0, \ \forall \ \alpha \in \Gamma$.

Under condition (5), $\frac{\partial f(\alpha)}{\partial \alpha_i}$ is nonnegative on the simplex with the vertices $\{a, \ a + \tau e^{\ell}, \ \ell = 1,2,...,L\}$ that contains the box $[a,b]$. Therefore $\frac{\partial f(\alpha)}{\partial \alpha_i} \geq 0 \ \forall \alpha \in [a,b]$ showing that f is increasing on $[a,b]$.

For the proof of the second statement note by an analogous argument that both (6) and (7) guarantee that f is decreasing on Γ, while (8) makes f decreasing on $[a,+\infty]$. □

Now we "monotonize" $f(\alpha)$ on Γ, i.e. we find a monotonic function

$$\bar{f}(\alpha) = \alpha^T(Q + Q^M)\alpha + (q + q^M)\alpha + p + p^M \tag{10}$$

with $Q^M \in R^{L \times L}$ a symmetric matrix, $q^M \in (R^L)^T$ and $p^M \in R$, such that

$$f(\alpha) \leq \bar{f}(\alpha), \qquad \forall \ \alpha \in \Gamma \tag{11}$$

and thus (1) is implied by $\bar{f}(\alpha) < 0,\ \forall\ \alpha \in \Gamma$. Obviously, (11) holds true whenever either

$$Q_{ji}^M = Q_{ij}^M \geq 0,\ q_i^M \geq 0,\ p^M \geq 0\,, \tag{12}$$

or

$$\begin{aligned} & \alpha^T Q^M \alpha + q^M \alpha + p^M \geq \epsilon \sum_{i=1}^{L} (\alpha_i - a_i)(b_i - \alpha_i),\ \forall \alpha \in R_+^L \\ \Leftarrow\quad & Q^M + \epsilon I \geq 0,\ q^M - \epsilon(a+b) \geq 0,\ p^M + \epsilon \sum_{i=1}^{L} a_i b_i \geq 0,\ \epsilon \geq 0 \end{aligned} \tag{13}$$

and an improved version of Lemma 2.1 is the following.

Theorem 2.2 *(1) holds true if there are Q^M, q^M, p^M satisfying one of conditions (12), (13) and moreover one of the following conditions is additionally fulfilled.*

(i) $\bar{f}(b) < 0$ and either

$$\left\{ \begin{array}{l} 2\sum_{j=1}^{L} (Q_{ij} + Q_{ij}^M) b_j + (q_i + q_i^M) \geq 0,\ i = 1, 2, ..., L \\ Q_{ij} + Q_{ij}^M \leq 0, \qquad 1 \leq i \leq j \leq L, \end{array} \right. \tag{14}$$

or

$$\left\{ \begin{array}{l} 2\sum_{j=1}^{L} (Q_{ij} + Q_{ij}^M) a_j + (q_i + q_i^M) \geq 0,\ i = 1, 2, ..., L \\ 2\sum_{j=1}^{L} (Q_{ij} + Q_{ij}^M) b_j + (q_i + q_i^M) + 2\tau(Q_{i\ell} + Q_{i\ell}^M) \geq 0, \qquad i, \ell = 1, 2, ..., L \end{array} \right. \tag{15}$$

(ii) $\bar{f}(a) < 0$ and either

$$\left\{ \begin{array}{l} 2\sum_{j=1}^{L} (Q_{ij} + Q_{ij}^M) b_j + q_i + q_i^M \leq 0, \qquad i = 1, 2, ..., L \\ Q_{ij} + Q_{ij}^M \geq 0, 1 \leq i \leq j \leq L \end{array} \right. \tag{16}$$

or

$$\left\{ \begin{array}{l} 2\sum_{j=1}^{L} (Q_{ij} + Q_{ij}^M) a_j + (q_i + q_i^M) \leq 0, \qquad i = 1, 2, ..., L \\ 2\sum_{j=1}^{L} (Q_{ij} + Q_{ij}^M) a_j + (q_i + q_i^M) + 2\tau(Q_{i\ell} + Q_{i\ell}^M) \leq 0 \end{array} \right. \tag{17}$$

If $\bar{f}(a) < 0$ with (12) are in force and

$$\left\{ \begin{array}{l} 2\sum_{j=1}^{L} (Q_{ij} + Q_{ij}^M) a_j + (q_i + q_i^M) \leq 0,\ i = 1, 2, ..., L \\ Q_{ij} + Q_{ij}^M \leq 0, \qquad 1 \leq i \leq j \leq L \end{array} \right. \tag{18}$$

then $\bar{f}(\alpha) < 0\ \forall\ \alpha \geq a$ and thus $f(a) < 0\ \forall\ \alpha \geq a$. □

2.1 Extension to the semidefinite matrix cone

In our robust control problems, we are dealing with PLMIs of the form

$$\mathcal{Q}(\alpha) := Q_0 + \sum_{i=1}^{L} \alpha_i Q_i + \sum_{i,j=1}^{L} \alpha_i \alpha_j Q_{i,j} < 0\,,$$

where Q_0, Q_i and $Q_{i,j}$ are symmetric matrices and the inequality sign must be regarded as an inequality in the cone of positive-definite matrices, namely $x^T \mathcal{Q} x < 0,\ \forall x \neq 0$. As a result, for a given x a PLMI constraint can be viewed as a standard quadratic inequality as in (1) with the obvious definitions

$$\begin{aligned} Q \;&:=\; \begin{bmatrix} x^T Q_{11} x & x^T Q_{12} x & \ldots. & x^T Q_{1L} x \\ x^T Q_{12} x & x^T Q_{22} x & \ldots. & x^T Q_{2L} x \\ \ldots & \ldots & \ldots. & \ldots. \\ x^T Q_{1L} x & x^T Q_{2L} x & \ldots & x^T Q_{LL} x \end{bmatrix}, \\ q \;&:=\; [x^T Q_1 x \quad x^T Q_2 x \quad \ldots. \quad x^T Q_L x], \qquad p := x^T Q_0 x \end{aligned} \tag{19}$$

Thus we have to check (1) for all $x \neq 0 \in R^n$ and therefore all inequalities (1)-(18) must be understood in the matrix sense. It follows that Lemma 2.1 and Theorem 2.2 trivially extend to the positive-definite matrix cone by simply regarding the associated conditions of scalar type as matrix inequalities.

For Theorem (2.2) a particular class of Q^M, q^M, p^M in this case is

$$Q^M_{ij} = q^M_{ij} I,\ q^M_i = \tilde{q}_i I,\ p^M = \tilde{p} I,\ q^M_{ij} \in R,\ \tilde{q}_i \in R,\ \tilde{p} \in R \tag{20}$$

where I is the identity matrix of appropriate size. Clearly, for such choice, (13) is guaranteed by

$$\begin{aligned} &\begin{bmatrix} q^M_{11} & q^M_{12} & \ldots & q^M_{1L} \\ q^M_{12} & q^M_{22} & \ldots & q^M_{2L} \\ \ldots & \ldots & \ldots & \ldots \\ q^M_{1L} & q^M_{2L} & \ldots & q^M_{LL} \end{bmatrix} + \epsilon I \geq 0, \\ &\tilde{q}_i - \epsilon(a_i + b_i) \geq 0,\ i = 1, 2, ..., L;\ \tilde{p} + \epsilon \sum_{i=1}^{L} a_i b_i \geq 0,\ \epsilon \geq 0. \end{aligned} \tag{21}$$

From (2) and (3), we see that the convexity based approach requires checking a system of $2^L + L$ LMIs (of *exponential order* with respect to L) for (1), while the result of Theorem 2.2 requires checking a system of $1+L+L(L+1)/2$ LMIs (of *polynomial order* with respect to L). At first glance, it seems that the result of Theorem 2.2 is more conservative than (2) and (3). It is not a case, however, and our computational examples in Section 4 will actually show that even the result of Theorem 2.2 is less conservative than (2) and (3).

3 Application to robust control problems

3.1 $\mu-$analysis problem

We begin this subsection by stating the following intermediate result.

Lemma 3.1 *The following conditions are equivalent for a matrix A of size $n \times n$.*

(i) A is nonsingular.

(ii) $A^T A > 0$.

Proof. Let A be nonsingular. Then $Ax \neq 0 \ \forall x \neq 0$ and thus $x^T A^T A x > 0 \ \forall x \neq 0$ proving the positive definiteness of $A^T A$. Now, suppose that $A^T A > 0$, i.e. $x^T A^T A x > 0 \ \forall x \neq 0$ so $Ax \neq 0 \ \forall x \neq 0$ showing the nonsingularity of A. □

A general class of robust stability analysis problems can be expressed as to check whether

$$\mathcal{A}\mathcal{B}(\alpha) \text{ is nonsingular for all, } \alpha \in \Gamma, \ \mathcal{B}(\alpha) = \mathcal{B}_0 + \sum_{i=1}^{L} \alpha_i \mathcal{B}_i. \tag{22}$$

Then using Lemma 3.1 it can be seen that

$$\begin{aligned}
(22) \quad &\Leftrightarrow \quad \mathcal{B}^T(\alpha)\mathcal{A}^T\mathcal{A}\mathcal{B}(\alpha) > 0, \ \forall \alpha \in \Gamma \\
&\Leftrightarrow \quad \exists \Theta : \ \mathcal{A}^T\mathcal{A} \geq \Theta, \mathcal{B}(\alpha)^T \Theta \mathcal{B}(\alpha) > 0, \ \forall \ \alpha \in \Gamma \\
&\Leftrightarrow \quad \exists \Theta : \ \mathcal{N}_{\mathcal{A}}^T \Theta \mathcal{N}_{\mathcal{A}} \leq 0 \ \ \text{(by Finsler lemma)} \qquad (23) \\
&\qquad\quad \text{and } \mathcal{B}(\alpha)^T \Theta \mathcal{B}(\alpha) > 0, \ \forall \ \alpha \in \Gamma \qquad (24)
\end{aligned}$$

where $\mathcal{N}_{\mathcal{A}}$ is any base of the nullspace of $\mathcal{A}$.

A particular case of (22) is the $\mu-$analysis problem of checking whether

$$(\mu I - G\Delta(\alpha)) \text{ is not singular for all } \Delta(\alpha) = \sum_{i=1}^{L} \alpha_i \Delta_i, \ \alpha \in [a, b] \subset R_+^L \tag{25}$$

$$\Leftrightarrow \quad (22) \text{ with } \mathcal{A} = [\,\mu I \quad -G\,], \ \mathcal{B}(\alpha) = \begin{bmatrix} I \\ \Delta(\alpha) \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} + \sum_{i=1}^{L} \alpha_i \begin{bmatrix} 0 \\ \Delta_i \end{bmatrix} \tag{26}$$

and the corresponding $\mathcal{N}_{\mathcal{A}}$ in (23) is given as

$$\mathcal{N}_{\mathcal{A}} = \begin{bmatrix} G \\ \mu I \end{bmatrix}.$$

A particular case $\Theta = \mathcal{A}^T\mathcal{A}$ satisfying (23) implies that it is not always natural to restrict Θ with $\Theta_{11}(i, i) < 0$ as done in the directional convexity approach [5]. Now we shall use the concept of monotonic optimization to avoid such restriction. For this note that

$$(24) \quad \Leftrightarrow \quad (1), (19) \text{ with} \begin{cases} Q_{ij} = -\frac{1}{2}[\mathcal{B}_i^T \Theta \mathcal{B}_j + \mathcal{B}_j^T \Theta \mathcal{B}_i], \\ q_i = -[\mathcal{B}_0^T \Theta \mathcal{B}_i + \mathcal{B}_i^T \Theta \mathcal{B}_0], \ p = -\mathcal{B}_0^T \Theta \mathcal{B}_0. \end{cases} \tag{27}$$

The result of Theorem 2.2 suitably adapted to (24) is thus.

Theorem 3.2 *(22) holds true if there exist a matrix Θ satisfying (23) and matrices Q_{ij}^M, q_i^M, p^M satisfying either (12) or (13) (in the matrix sense) such that one of the following conditions is fulfilled.*

(i)

$$\mathcal{B}^T(b)\Theta\mathcal{B}(b) - \sum_{i,j=1}^{L} Q_{ij}^M b_i b_j - \sum_{i=1}^{L} q_i^M b_i - p^M > 0 \tag{28}$$

and additionally either

$$\begin{cases} [\sum_{j=1}^{L} b_j(\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + \mathcal{B}_0^T\Theta\mathcal{B}_i] + (*) - q_i^M \leq 0, \ i = 1, 2, ..., L, \\ (\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + (*) \geq 0, \ 1 \leq i \leq j \leq L. \end{cases} \tag{29}$$

or

$$\begin{cases} [\sum_{j=1}^{L} a_j(\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + \mathcal{B}_0^T\Theta\mathcal{B}_i] + (*) - q_i^M \leq 0, \ i = 1, 2, ..., L, \\ [\sum_{j=1}^{L} a_j(\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + \mathcal{B}_0^T\Theta\mathcal{B}_i + \tau(\mathcal{B}_i^T\Theta\mathcal{B}_\ell - Q_{i\ell}^M)] + (*) - q_i^M \leq 0, \\ i, j = 1, 2, ..., L \end{cases} \tag{30}$$

(ii)

$$\mathcal{B}(a)^T\Theta\mathcal{B}(a) - \sum_{i,j=1}^{L} Q_{ij}^M a_i a_j - \sum_{i=1}^{L} q_i^M a_i - p^M > 0 \tag{31}$$

and additionally either

$$\begin{cases} [\sum_{j=1}^{L} b_j(\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + \mathcal{B}_0^T\Theta\mathcal{B}_i] + (*) - q_i^M \geq 0, \ i = 1, 2, ..., L, \\ (\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + (*) \leq 0, 1 \leq i \leq j \leq L, \end{cases} \tag{32}$$

or

$$\begin{cases} [\sum_{j=1}^{L} a_j(\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + \mathcal{B}_0^T\Theta\mathcal{B}_i] + (*) - q_i^M \geq 0, \ i = 1, 2, ..., L, \\ [\sum_{j=1}^{L} a_j(\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + \mathcal{B}_0^T\Theta\mathcal{B}_i + \tau(\mathcal{B}_i^T\Theta\mathcal{B}_\ell - Q_{i\ell}^M)] + (*) - q_i^M \geq 0, \\ i, \ell = 1, 2, ..., L. \end{cases} \tag{33}$$

If (31), (12) and

$$\begin{cases} [\sum_{j=1}^{L} a_j(\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + \mathcal{B}_0^T\Theta\mathcal{B}_i] + (*) - q_i^M \geq 0, \ i = 1, 2, ..., L, \\ (\mathcal{B}_i^T\Theta\mathcal{B}_j - Q_{ij}^M) + (*) \geq 0, \ 1 \leq i \leq j \leq L, \end{cases} \tag{34}$$

are fulfilled then (27) holds true for all $\alpha \geq a$.

Remark. It can be easily seen that for $\Gamma = [-a, b] \subset R^L, a > 0, b > 0$ as often in the $\mu-$analysis problem, without translating Γ to R_+^L, the result of Theorems 3.2 and 3.3 are still valid with only condition (12), (13) involving Q^M, q^M, p^M slightly modified to the following

$$Q^M + \epsilon I \geq 0, \ q^M = \epsilon(b - a), \ p^M - \epsilon\sum_{i=1}^{L} a_i b_i \geq 0, \ \epsilon \geq 0. \tag{35}$$

It should be mentioned that in such case (31), (34) yield (27) only whenever $\alpha \in \Gamma$ but not for all $\alpha \geq -a$.

3.2 Robust stability

It is well known that the linear uncertain system

$$\dot{x} = [A_0 + \sum_{i=1}^{L} \alpha_i A_i]x, \;\; \alpha \in \Gamma = [0,1]^L \tag{36}$$

is robustly stable if there is Lyapunov function

$$P = P_0 + \sum_{i=1}^{L} \alpha_i P_i > 0 \tag{37}$$

such that

$$(A_0 + \sum_{i=1}^{L} \alpha_i A_i)(P_0 + \sum_{i=1}^{L} \alpha_i P_i) + (*) < 0 \;\forall\; \alpha \in \Gamma \tag{38}$$

$$\Leftrightarrow \quad (1),(19) \text{ with } \begin{cases} Q_{ij} = \frac{1}{2}[(A_i P_j + A_j P_i) + (*)], \\ q_i = (A_0 P_i + A_i P_0) + (*), \\ p = A_0 P_0 + (*), \;\; a = (0,0,...,0)^T, \;\; b = (1,1,...,1)^T \end{cases} \tag{39}$$

Applying Theorem 2.2 to system (1), (39) we have the following result

Theorem 3.3 *System (36) is robustly stable if there are matrices Q_{ij}^M, q_i^M, p^M satisfying either (12) or (13) such that one of the following conditions is satisfied*

(i)

$$[(A_0 + \sum_{i=1}^{L} A_i)(P_0 + \sum_{i=1}^{L} P_i) + (*)] + \sum_{i,j=1}^{L} Q_{ij}^M + \sum_{i=1}^{L} q_i^M + p^M < 0 \tag{40}$$

and either

$$\begin{cases} [\sum_{j=1}^{L}(A_i P_j + A_j P_i + Q_{ij}^M) + A_i P_0 + A_0 P_i] + (*) + q_i^M \geq 0, \;\; i = 1,2,...,L \\ (A_i P_j + A_j P_i + Q_{ij}^M) + (*) \leq 0, \;\; 1 \leq i \leq j \leq L, \end{cases} \tag{41}$$

or

$$\begin{cases} (A_i P_0 + A_0 P_i) + (*) + q_i^M \geq 0, \;\; i = 1,2,...,L \\ [A_i P_0 + A_0 P_i + L(A_i P_\ell + A_\ell P_i + Q_{ij}^M)] + (*) + q_i^M \geq 0, \;\; i,\ell = 1,2,...,L. \end{cases} \tag{42}$$

(ii)

$$(A_0 P_0 + (*)) + p^M < 0 \tag{43}$$

and either

$$\begin{cases} [\sum_{j=1}^{L}(A_i P_j + A_j P_i + Q_{ij}^M) + A_i P_0 + A_0 P_i] + (*) + q_i^M \leq 0, \;\; i = 1,2,....,L, \\ (A_i P_j + A_j P_i + Q_{ij}^M) + (*) \geq 0, \;\; 1 \leq i \leq j \leq L, \end{cases} \tag{44}$$

$$\begin{cases} (A_i P_0 + A_0 P_i) + (*) + q_i^M \leq 0, \;\; i = 1,2,...,L, \\ [A_i P_0 + A_0 P_i + L(A_i P_\ell + A_\ell P_i + Q_{i\ell}^M)] + (*) + q_i^M \leq 0, \;\; i,\ell = 1,2...,L. \end{cases} \tag{45}$$

If (43), (12) and

$$\left\{ \begin{array}{l} (A_iP_0 + A_0P_i) + (*) + q_i^M \le 0, \ i = 1, 2, ..., L, \\ (A_iP_j + A_jP_i + Q_{ij}^M) + (*) \le 0, \ 1 \le i \le j \le L, \end{array} \right. \tag{46}$$

are fulfilled, then then (37 is stable for all $\alpha \in R_+^L$.

4 Computational examples

4.1 $\mu-$computation

The following data for the μ analysis problem in (25) is taken from [9]

$$G = \begin{bmatrix} 0 & 1 & 0 & 1 \\ b & 0 & c & 0 \\ 2a & 0 & a & 0 \\ 0 & -2a & 0 & -a \end{bmatrix}, \Delta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \Delta_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It known that for $a = 0.5$, $c = 0.25$ and $b \in [-5, 0]$ the minimum μ_{opt} of such $\mu > 0$ satisfying (25) is 0.5. The relaxation result of [7] for computing an upper bound of μ_{opt} yields a conservative result (large than μ_{opt} by 10% [7, Example 1, case 2]). In contrast, the characterizations given in Theorem 3.2 all yield the exact value 0.5 of μ_{opt}. Note that the method of [3] also hits this value 0.5 of μ_{opt}.

4.2 Robust stability analysis

We consider the following example borrowed from [11] where the matrix A of system (36) is given as

$$A(\alpha_1, \alpha_2) = \begin{bmatrix} -2 + \alpha_1 & 0 & -1 + \alpha_1 \\ 0 & -3 + \alpha_2 & 0 \\ -1 + \alpha_1 & -1 + \alpha_2 & -4 + \alpha_1 \end{bmatrix} = \bar{A}_0 - \alpha_1 A_1 - \alpha_2 A_2 \,,$$

with

$$\bar{A}_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}, \ A_1 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

The actual domain of stability for $A(\alpha_1, \alpha_2)$ is

$$\alpha_1 < 1.7499, \quad \alpha_2 < 3. \tag{47}$$

To make the result of Theorem 3.3 applicable for this case, first use the changes of variable $\alpha_1 \leftarrow -\alpha_1$, $\alpha_2 \leftarrow -\alpha_2$ and re-express $A(\alpha_1, \alpha_2)$ as

$$A(\alpha_1, \alpha_2) = A_0 + \alpha_1 A_1 + \alpha_2 A_2, \ A_0 = \bar{A}_0 - 1.7499 A_1 - 2.9999 A_2 \tag{48}$$

Then, using the LMI Control Toolbox [6] it is immediate to check that LMI system (43), (46) is feasible with all Q_{ij}^M, q_i^M, p^M set equal to zero. Therefore the matrix $A(\alpha_1, \alpha_2)$ defined by (48) is stable for all $\alpha_1 \ge 0, \alpha_2 \ge 0$, confirming the true stability domain (47).

The result of Theorem 3.3 clearly outperforms existing results in [11, 1, 12] which require a trial and error process to arrive at the final result.

Next, take the following example of [8]

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} + \Delta A, |\Delta A_{ij}| \leq rS(i,j),\ \ i,j = 1,2,3;$$
$$S = \begin{bmatrix} 0.1651 & 0.9394 & 0.5691 \\ 0.2451 & 0.4727 & 0.1457 \\ 0.7004 & 0.4014 & 0.3141 \end{bmatrix}.$$

It was guessed in [2] that A is robustly stable for $r = 0.5$.

Now, we can transform A to the form (36) with $L = 9$,

$$A_0 = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} - rS$$

and all entries of matrices $A_i, i = 1,2,...,9$ are zeros exept

$$\begin{array}{l} A_1(1,1) = 2rS(1,1),\ \ A_2(1,2) = 2rS(1,2),\ \ A_3(1,3) = 2rS(1,3), \\ A_4(2,1) = 2rS(2,1),\ \ A_5(2,2) = 2rS(2,2),\ \ A_6(2,3) = 2rS(2,3), \\ A_7(3,1) = 2rS(3,1),\ \ A_8(3,2) = 2rS(3,2),\ \ A_9(3,3) = 2rS(3,3). \end{array}$$

Clearly, the previously developed convexity-based methods [5, 1, 12] require solving a systems of $2^L + L = 2^9 + 9 = 521$ LMIs constraints which are not readily solved by the existing LMI softwares such as MATLAB LMI control toolbox [6]. In contrast, for $r = 0.5$ applying the result (40), (41) with $Q_{ij}^M = 0, q_i^M = 0, p^M = 0$ involving total 56 LMIs constraints gives the following feasible solutions confirming the robust stability of A,

$$P_0 = 10^{-4}\begin{bmatrix} 0.090307 & -0.023254 & 0.043219 \\ -0.023254 & 0.061041 & -0.010110 \\ 0.043219 & -0.010111 & 0.144195 \end{bmatrix},\ P_1 = 10^{-5}\begin{bmatrix} -0.046081 & 0.021889 & -0.025640 \\ 0.021889 & -0.033281 & 0.003796 \\ -0.025640 & 0.003796 & -0.100189 \end{bmatrix},$$
$$P_2 = 10^{-5}\begin{bmatrix} -0.206172 & 0.007955 & -0.075246 \\ 0.007955 & 0.006530 & -0.005840 \\ -0.075246 & -0.005840 & -0.158144 \end{bmatrix},\ P_3 = 10^{-6}\begin{bmatrix} -0.624499 & 0.165445 & -0.243885 \\ 0.165445 & -0.165275 & 0.232036 \\ -0.243885 & 0.232036 & -0.636828 \end{bmatrix},$$
$$P_4 = 10^{-6}\begin{bmatrix} -0.332545 & 0.112977 & -0.181188 \\ 0.112977 & 0.331373 & 0.133355 \\ -0.181188 & 0.133355 & -0.855959 \end{bmatrix},\ P_5 = 10^{-5}\begin{bmatrix} -0.080029 & -0.037127 & -0.062369 \\ -0.037127 & -0.051059 & -0.022919 \\ -0.062369 & -0.022919 & -0.134420 \end{bmatrix},$$
$$P_6 = 10^{-6}\begin{bmatrix} -0.524775 & 0.029002 & -0.314483 \\ 0.029002 & -0.087495 & 0.028610 \\ -0.314483 & 0.028610 & -0.9428149 \end{bmatrix},\ P_7 = 10^{-6}\begin{bmatrix} -0.301095 & 0.110264 & -0.059123 \\ 0.110265 & 0.151310 & 0.682579 \\ -0.059123 & 0.682579 & -0.519952 \end{bmatrix},$$
$$P_8 = 10^{-5}\begin{bmatrix} -0.054156 & -0.000801 & -0.059787 \\ -0.000810 & -0.019287 & 0.012637 \\ -0.059787 & 0.012637 & -0.117466 \end{bmatrix},\ P_9 = 10^{-5}\begin{bmatrix} -0.097658 & -0.013450 & -0.071830 \\ -0.013450 & -0.065852 & -0.006614 \\ -0.071830 & -0.006614 & -0.161458 \end{bmatrix}.$$

4.3 LPV synthesis example

The LPV model of the longitudinal dynamics of the missile are given as [10, 1, 12]

$$
\begin{aligned}
\begin{bmatrix} \dot{\hat{x}} \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} -0.89 & 1 \\ -142.6 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ q \end{bmatrix} + \begin{bmatrix} 0 & -0.89 \\ 178.25 & 0 \end{bmatrix} \begin{bmatrix} w_{\alpha_1} \\ w_{\alpha_2} \end{bmatrix} + \begin{bmatrix} -0.119 \\ -130.8 \end{bmatrix} \delta_{\text{fin}} \\
\begin{bmatrix} w_{\alpha_1} \\ w_{\alpha_2} \end{bmatrix} &= \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ q \end{bmatrix} \\
\begin{bmatrix} \eta_z \\ q \end{bmatrix} &= \begin{bmatrix} -1.52 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ q \end{bmatrix}
\end{aligned}
$$

where $\hat{x}$, q, η_z and δ_{fin} denote the angle of attack, the pitch rate, the vertical accelerometer measurement, the fin deflection, respectively; and α_1, α_2 are two time-varying parameters, measured in real time, resulting from changes in missile aerodynamic conditions (angle of attack from 0 up to 20 degrees).

The problem specifications are as follows:

- A settling time of 0.2 second with minimal overshoot and zero steady-sta te error for the vertical acceleration η_z in response to a step command η_c.
- The controller must achieve an adequate high-frequency roll-off for noise attenuation and withstand neglected dynamics and flexible modes. Magnitude constraints of 2 are also imposed to the control signal δ_{fin}.

Moreover, those specifications must be met for all parameter values:

$$|\alpha_1| \leq 1, \qquad |\alpha_2| \leq 1.$$

An integrator has been introduced on the acceleration channel to ensure zero steady-state error. It turns out that the resulting LPV controller K is obtained as the composition of the operators K_0 and

$$\begin{bmatrix} \frac{2+0.06\,s}{s} & 0 \\ 0 & 1 \end{bmatrix}.$$

The weighting functions W_e and W_u were chosen to be

$$W_e = 0.8, \qquad W_u = \frac{0.001s^3 + 0.03s^2 + 0.3s + 1}{1e-5s^3 + 3e-2s^2 + 30s + 10000}.$$

It is not difficult to rewrite this example in the form

$$
\begin{aligned}
\dot{x} &= A(\alpha_1, \alpha_2)x + B_1 w + B_2 u \\
z &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w,
\end{aligned}
\tag{49}
$$

with $A(\alpha_1, \alpha_2) = A_0 + \alpha_1 A_1 + \alpha_2 A_2$. Then this problem can be reduced to the general LPV control problem with guaranteed L_2-gain performance. It consists in finding a dynamic LPV controller with state-space equations

$$
\begin{aligned}
\dot{x}_K &= A_K(\alpha, \dot{\alpha}) x_K + B_K(\alpha, \dot{\alpha}) y \\
u &= C_K(\alpha, \dot{\alpha}) x_K + D_K(\alpha, \dot{\alpha}) y
\end{aligned}
\tag{50}
$$

method	achieved perf. level γ	cpu time (sec.)
DCC in [1]	0.1290	170.790
separated convexification in [12]	0.1284	256.00
d.c. convexification in [12]	0.1290	273.300
monotonic relaxation (Theorem 2.2,(ii), (16))	0.1154	109.470

Table 1: Numerical results of LPV synthesis techniques

which ensures internal stability and a guaranteed L_2-gain bound γ for the closed-loop operator (49)-(50) from the disturbance signal w to the error signal z, that is,

$$\int_0^T z'z\,d\tau \le \gamma^2 \int_0^T w'w\,d\tau, \quad \forall T \ge 0,$$

for all admissible parameter trajectories $\alpha(t)$.

By [12, 1] this problem can be reduced to form (1) and the results of Theorem 2.2 are readily applicable. Table 1 displays the achieved performance level γ for different existing techniques. It appears that the result of monotonic relaxation (Theorem 2.2) provide the best performance level, which is much better than those provided by other methods. Furthermore, due to a lower number of LMI constraints, the computational time (in Pentium 330Mhz for this example) of the former is much smaller than that required for competing techniques.

5 Conclusions

We have demonstrated how to solve PLMIs with parameters restricted to a hyper-rectangle by solving a finite number of LMIs which is of polynomial order with respect to parameter dimension. From the included numerical examples, it appears that our method is not only more computationally tractable than earlier convexification techniques requiring the solution to an exponentional number of LMIs but also provides less conservative results.

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