Mixed H_2/H_∞ Multi-Channel Linear Parameter-Varying Control in Discrete Time

Pierre Apkarian[*] *Paulo C. Pellanda*[‡] *Hoang Duong Tuan*[¶]

Abstract

This paper develops a new method for the synthesis of Linear Parameter-Varying (LPV) controllers in discrete time. LPV plants under consideration have a Linear Fractional Transformation (LFT) representation. In contrast to earlier results which are restricted to single-objective LPV problems the proposed method can handle a set of H_2/H_∞ specifications that can be defined channel-wise. This practically attractive extension is derived by using specific transformations of both the Lyapunov and scaling/multiplier variables in tandem with appropriate linearizing transformations of the controller data and of the controller scheduling function. It is shown that the controller gain-scheduling function can be constructed as an affine matrix-valued function in the polytopic coordinates of the scheduled parameter, hence is easily implemented on line. Finally, these manipulations give rise to a tractable and practical LMI formulation of the multi-objective LPV control problem.

Key words. LPV synthesis, mixed H_2/H_∞, multi-channel control, LFT, Linear Matrix Inequalities.

1 Introduction

LPV control techniques have received great attention in recent years [17, 2, 4, 14, 19]. The main thrust of these techniques is to provide an elegant and algorithmically attractive setting for addressing the practical needs of gain scheduling or controller interpolation. The most demanding task of these techniques amounts to solving Linear Matrix Inequality (LMI) programs which is relatively easy with currently available Semi-Definite Programming codes. These methods have also been constantly refined and improved in different directions. In [19] generalized classes of scaling are introduced which results in less conservative characterizations. In [24, 1, 3, 23], the

[*]ONERA-CERT, Control System Dept., 2 av. Edouard Belin, 31055 Toulouse, FRANCE - Email : apkarian@cert.fr - Tel : +33 5.62.25.27.84 - Fax : +33 5.62.25.27.64

[‡]Supported by Brazilian Ministry of Defense, at ONERA-CERT, - Email : pellanda@cert.fr - Tel : +33 5.62.25.29.21 - Fax : +33 5.62.25.27.64

[¶]Department of Control and Information, Toyota Technological Institute, Hisakata 2-12-1, Tenpaku, Nagoya 468-8511, JAPAN - Email: tuan@toyota-ti.ac.jp -Tel +81 52.809.1815 - Fax +81 52.809.1721

authors employ parameter-dependent Lyapunov functions to take advantage of the fact that parameter evolutions have a limited range of speed. In [16], a mixed technique which can handle both scheduled and uncertain parameters is discussed. Except from isolated cases [18, 1, 14] which either discuss computationally intensive approaches or propose somewhat conservative schemes, the definition of a genuine mixed H_2/H_∞ and multi-channel LPV methodology is a very challenging issue which remains unsolved in the technical setting of LFT representations and multiplier-based characterizations. Because of the many constraints surrounding most practical designs the development of such a methodology is certainly of crucial importance. All these aspects have motivated the discussion in this paper.

We develop a technique for solving the mixed H_2/H_∞ multi-channel LPV control problem in discrete time which is an extension of previous single-objective results in [17, 2]. The core of the contribution is twofold.

- We show that the Lyapunov variables transformations introduced in [8] can be applied similarly in the context of LFT systems for a specific class of symmetric unstructured scalings. These transformations in return permits to short-circuit the inherent strong interrelations between Lyapunov and scaling variables on one side and LPV controller variables on the other side. An important consequence is that, similarly to the nominal case, different Lyapunov variables and scalings can be used for each channel/specification what reduces conservatism.

- We also establish new linearizing transformations of the LPV controller state-space data and of the controller scheduling function to achieve a full LMI program description of the mixed H_2/H_∞ multi-channel LPV synthesis problem. Note that these transformations are new and are not possible with earlier developed techniques such as those in [20]. The latter are known to be impractical whatever class of scalings is used: generalized full-block scalings or simpler diagonal scalings. Also, the techniques in [17, 2, 21, 13, 22] fundamentally hinge on the use of the Projection Lemma [10], a tool which is inherently restricted to single channel and single objective synthesis problems and not of any help in the problem under consideration. Apart from new linearization transformations, we also show that the controller gain-scheduling function can be searched for as a linear matrix-valued function in the polytopic coordinates of the (plant) scheduled parameter, hence is easily implemented on-line. The proposed characterization offers substantial flexibility to construct the controller gain-scheduling function. A major limitation, however, lies in the fact that more complex functions such as higher-order polynomials can play adversely in terms of LMI solver computational time.

As a byproduct, the proposed derivation provides a different proof of the original single-objective H_∞ LPV synthesis problem in [17, 2].

The paper is structured as follows. Instrumental tools useful in future constructions are developed in Section 2. A comprehensive description of the mixed H_2/H_∞ multi-channel, including the synthesis LMI characterizations up to the LPV controller construction is provided in Section 3. Illustrative examples are discussed in Section 4.

The notation used throughout the paper is fairly standard. M^T is the transpose of the matrix M. The notation $\mathrm{Tr}\, M$ stands for the trace of M. For Hermitian or real symmetric matrices , $M > N$

means that $M-N$ is positive definite and $M \geq N$ means that $M-N$ is positive semi-definite. In symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for terms that are induced by symmetry. $x(k)$ is used to denote the signal x at (discrete) time k.

2 Analysis setup

This section develops analysis tests for robust H_2 and H_∞ performance that will be central in the construction of multi-objective LPV controllers. We are concerned with the robust analysis problem of an uncertain discrete-time plant subject to LFT uncertainty. In other words, the uncertain plant is described as

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ z_\Delta(k) \\ z(k) \end{bmatrix} &= \begin{bmatrix} A & B_\Delta & B_1 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} \\ C_1 & D_{1\Delta} & D_{11} \end{bmatrix} \begin{bmatrix} x(k) \\ w_\Delta(k) \\ w(k) \end{bmatrix} \\ w_\Delta(k) &= \Delta(k)\, z_\Delta(k), \qquad \Delta(k) \in \mathbf{R}^{N\times N}. \end{aligned} \tag{1}$$

where $\Delta(k)$ is a time-varying matrix-valued parameter evolving in a polytopic set $\mathcal{P}_\Delta$, defined as

$$\mathcal{P}_\Delta := \mathrm{co}\,\{\Delta_1, \ldots, \Delta_i, \ldots, \Delta_L\} \ni 0\,, \tag{2}$$

where co stands for the convex hull and the Δ_i's denote the vertices of $\mathcal{P}_\Delta$. It is important to note that the parameter Δ is regarded as an uncertainty throughout this section. The LPV or gain-scheduling problem, that is, the case where Δ is measured in real time is developed in a synthesis context in Section 3.

Closing the uncertainty channel $w_\Delta(k) = \Delta(k)\, z_\Delta(k)$ leads to the alternative state-space representation

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \left\{ \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_\Delta \\ D_{1\Delta} \end{bmatrix} \Delta(k)(I - D_{\Delta\Delta}\Delta(k))^{-1} [C_\Delta \quad D_{\Delta 1}] \right\} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}.$$

From the latter expression, we observe that the plant with inputs w and outputs z has state-space data entries which are fractional functions of the time-varying parameter $\Delta(k)$. As is standard in the robust synthesis literature, we have used the following notation

- x for the state vector,
- w for exogenous inputs,
- z for controlled or performance variables.

2.1 Robust H_2 performance

Guaranteed H_2 performance can be interpreted in different ways: it provides an upper bound on the variance of the output for all admissible parameter trajectories or alternatively, it gives an upper bound on the worst-case (with respect to Δ) output energy in response to impulse inputs. See [15] and references therein for a detailed discussion. A characterization of guaranteed H_2 performance is provided in the following proposition.

Proposition 2.1 (Robust H_2 performance) *The statements (i) and (ii), involving Lyapunov variables X and Z, scaling pairs (Q_1,R_1), (Q_2,R_2) and general slack matrix variables V, H_1, F_1, H_2, F_2 are equivalent and enforce a bound ν on the variance of the output z for all parameter trajectories $\Delta(k) \in \mathcal{P}_\Delta$:*

$$
\begin{aligned}
(i) \quad : \quad & \begin{bmatrix} -X & * & * & * & * \\ 0 & Q_1 & * & * & * \\ 0 & 0 & -\nu I & * & * \\ A & B_\Delta & B_1 & -X^{-1} & 0 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & 0 & -R_1^{-1} \end{bmatrix} < 0, \\
& \begin{bmatrix} -X & * & * & * \\ 0 & Q_2 & * & * \\ C_\Delta & D_{\Delta\Delta} & -R_2^{-1} & * \\ C_1 & D_{1\Delta} & 0 & -Z \end{bmatrix} < 0, \quad \mathrm{Tr}\,(Z) < 1, \\
& \begin{bmatrix} R_1 & \Delta^T \\ \Delta & -Q_1^{-1} \end{bmatrix} > 0, \qquad \begin{bmatrix} R_2 & \Delta^T \\ \Delta & -Q_2^{-1} \end{bmatrix} > 0, \;\; \forall \Delta = \Delta_i.
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
(ii) \quad : \quad & \begin{bmatrix} -X & * & * & * & * \\ 0 & Q_1 & * & * & * \\ 0 & 0 & -\nu I & * & * \\ V^T A & V^T B_\Delta & V^T B_1 & X-(V+V^T) & 0 \\ H_1^T C_\Delta & H_1^T D_{\Delta\Delta} & H_1^T D_{\Delta 1} & 0 & R_1-(H_1+H_1^T) \end{bmatrix} < 0, \\
& \begin{bmatrix} -X & * & * & * \\ 0 & Q_2 & * & * \\ H_2^T C_\Delta & H_2^T D_{\Delta\Delta} & R_2-(H_2+H_2^T) & * \\ C_1 & D_{1\Delta} & 0 & -Z \end{bmatrix} < 0, \quad \mathrm{Tr}\,(Z) < 1, \\
& \begin{bmatrix} R_1 & \Delta^T F_1 \\ F_1^T \Delta & Q_1+F_1+F_1^T \end{bmatrix} > 0, \qquad \begin{bmatrix} R_2 & \Delta^T F_2 \\ F_2^T \Delta & Q_2+F_2+F_2^T \end{bmatrix} > 0, \;\; \forall \Delta = \Delta_i.
\end{aligned} \tag{4}
$$

Proof: The fact that (i) enforces a bound ν on the variance of z for all admissible parameter trajectories $\Delta(k)$ is a standard result [5, 19, 15]. Hence, it suffices to prove the equivalence of (i) and (ii). Necessity of (4) follows from the choice $V := X$, $H_1 := R_1$, $H_2 := R_2$, $F_1 := -Q_1$ and $F_2 := -Q_2$ in conditions (4). Sufficiency is obtained by noting that (4) implies that V, H_1, H_2, F_1, F_2 are non-singular. Thus, one can perform the congruence transformation

$$
\mathrm{diag}\,(I,I,I,W,G_1), \qquad W := V^{-1},\; G_1 := H_1^{-1},
$$

in the first LMI in (4). This yields the equivalent condition

$$
\begin{bmatrix} -X & * & * & * & * \\ 0 & Q_1 & * & * & * \\ 0 & 0 & -\nu I & * & * \\ A & B_\Delta & B_1 & W^T X W-(W+W^T) & 0 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & 0 & G_1^T R_1 G_1-(G_1+G_1^T) \end{bmatrix} < 0. \tag{5}
$$

One then easily infers (3) from the inequalities

$$W^T X W - (W + W^T) \geq -X^{-1}, \qquad G_1^T R_1 G_1 - (G_1 + G_1^T) \geq -R_1^{-1},$$

which hold whenever $X > 0$ and $R_1 > 0$.

The equivalence between the last constraints in (3) and (4) also follows by similar arguments. This completes the proof of the proposition. ■

It is worth mentioning that the conditions in Proposition 2.1 are conservative in two respects. First of all, a fixed Lyapunov function $V(x) := x^T X x$ (not depending on parameters) is employed to assess H_2 performance of the uncertain system. This is a well-recognized source of conservatism [7, 12, 11]. Secondly, we are utilizing a subclass of full-block generalized scalings with zero off-diagonal separators in place of the class of generalized scalings or multipliers introduced in [21]. Therefore, these tests should be refined when used for validation purpose. This subclass is, however, more general than the subclass of structured symmetric scalings used in [17, 2]. More importantly, this new characterizations also offer new potentials for deriving tractable characterizations for discrete-time multi-objective LPV control problems which appears delicate using earlier techniques.

There are a few points to have in mind to understand the conditions (3) and (4) and their usefulness.

- In (4), we get rid of the standard Lyapunov terms XA, XB_1, ... and of the scaling terms $R_1 C_\Delta$, $R_1 D_{\Delta 1}$, ... by means of intermediate (slack) variable V, H_1, H_2, F_1 and F_2. These terms generally impose strong limitations in multi-objective control problems since they preclude the use of multiple Lyapunov functions or scalings. Similar ideas have been presented earlier in [8, 9] for Linear Time-Invariant multi-objective synthesis.

- The LMI condition (4) is significantly more costly than its original form (3) because of the additional general matrix variables V and H_1, H_2, F_1 and F_2 . We shall see however that this extra computational overhead is more than offset by new capabilities in multi-objective LPV synthesis. Firstly, multiple Lyapunov functions X_j and scalings R_j, Q_j can be employed for each channel and specification. Secondly, from a synthesis viewpoint, new linearizing transformations of the LPV controller data can be introduced that lead to a full LMI characterization of the control problem.

- It might appear to the reader that the introduction of slack variables F_i, $i = 1, 2$ is superfluous in the LMIs involving Δ, (4). This is right as long as analysis only is of interest. For the LPV control synthesis considered later, however, the subpart of Δ corresponding to the gain-scheduling block becomes a true variable and slack matrices again are necessary to allow linearization of the problem.

Finally, the conditions in (3) and (4) guarantee well-posedness of the LFT representation in (1). The property of well-posedness is ensured in all results in this paper and will not be discussed further. See for instance [2, 21] for discussions on this property.

2.2 Robust H_∞ performance

The following result for H_∞ performance parallels those for the H_2 performance in Proposition 2.1.

Proposition 2.2 (Robust H_∞ performance) *The following LMIs involving a Lyapunov variable X, a scaling pair (Q,R) and general slack matrix variables V, H and F enforces a bound γ on the L_2-induced gain of the operator mapping w into z. In different words, H_∞ performance for the channel (w,z) is guaranteed for all parameter trajectories $\Delta(k) \in \mathcal{P}_\Delta$.*

$$
\begin{aligned}
&\begin{bmatrix} -X & * & * & * & * & * \\ 0 & Q & * & * & * & * \\ 0 & 0 & -\gamma I & * & * & * \\ V^T A & V^T B_\Delta & V^T B_1 & X-(V+V^T) & * & * \\ H^T C_\Delta & H^T D_{\Delta\Delta} & H^T D_{\Delta 1} & 0 & R-(H+H^T) & * \\ C_1 & D_{1\Delta} & D_{11} & 0 & 0 & -\gamma I \end{bmatrix} < 0, \\
&\begin{bmatrix} R & \Delta^T F \\ F^T \Delta & Q+F+F^T \end{bmatrix} > 0, \ \ \forall \Delta = \Delta_i.
\end{aligned} \tag{6}
$$

Proof: The proof is along the lines of the proof of proposition 2.1 and is omitted for brevity. Additional details can be found in [9, 21]. ■

Again, condition (6) enjoys a separated structure which plays a key role in the synthesis results to be presented below. Finally, we reemphasize the important fact that when multi-channel H_2 and H_∞ performance constraints are specified then different Lyapunov variables X_j and scaling pairs (Q_j, R_j) must be used for each channel/specification.

3 Mixed H_2/H_∞ multi-channel LPV synthesis

Before going further, we reemphasize the fact that, in contrast to the nominal multi-objective case [20, 9], the multi-channel mixed H_2/H_∞ LPV control problem is a very challenging issue that remains unsolved for plants described by LFT representations. The purpose of this section is to derive a tractable and practical characterization of this problem.

3.1 Problem presentation

Hereafter, we first introduce the multi-channel mixed H_2/H_∞ LPV control problem as well as some useful notations. We are given an LPV plant with LFT structure

$$
\begin{aligned}
\begin{bmatrix} x(k+1) \\ z_\Delta(k) \\ z(k) \\ y(k) \end{bmatrix} &= \begin{bmatrix} A & B_\Delta & B_1 & B_2 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ C_1 & D_{1\Delta} & D_{11} & D_{12} \\ C_2 & D_{2\Delta} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w_\Delta(k) \\ w(k) \\ u(k) \end{bmatrix}, \qquad A \in \mathbf{R}^{n\times n} \\
w_\Delta(k) &= \Delta(k) z_\Delta(k), \qquad \Delta(k) \in \mathbf{R}^{N\times N},
\end{aligned} \tag{7}
$$

where Δ is defined in (2). Here x, w, w_Δ, z, and z_Δ have the same meaning as in Section 2, u is the control signal, and y is the measurement signal. The pair (w_Δ, z_Δ) is now regarded as the gain-scheduling channel, i. e., the parameter $\Delta(k)$ is measured on line and hence can be exploited by the controller.

For the LPV plant (7) the control problem consists in seeking an LPV controller with LFT structure

$$\begin{aligned} \begin{bmatrix} x_K(k+1) \\ u(k) \\ z_K(k) \end{bmatrix} &= \begin{bmatrix} A_K & B_{K1} & B_{K\Delta} \\ C_{K1} & D_{K11} & D_{K1\Delta} \\ C_{K\Delta} & D_{K\Delta 1} & D_{K\Delta\Delta} \end{bmatrix} \begin{bmatrix} x_K(k) \\ y(k) \\ w_K(k) \end{bmatrix}, \qquad A_K \in \mathbf{R}^{n\times n} \\ w_K(k) &= \Delta_K(k) z_K(k), \qquad \Delta_K \in \mathbf{R}^{N\times N} \end{aligned} \tag{8}$$

such that H_2 and H_∞ specifications are achieved for a family of channels (w_1, z_1), (w_2, z_2), etc, where the w_j's and z_j's are sub-vectors of w and z, respectively (Figure 1). The notation Δ_K is used for the controller scheduling function which is a function of the plant parameter Δ, that is, $\Delta_K := \Delta_K(\Delta)$. This scheduling function is part of the design procedure and will be determined in the course of the derivation below.

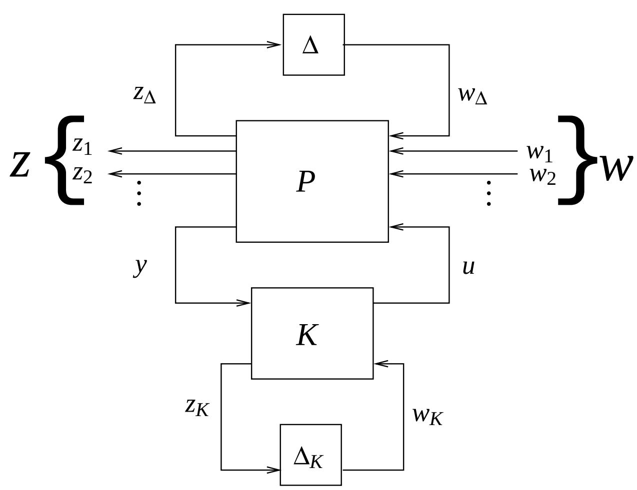

Figure 1: mixed H_2/H_∞ multi-channel LPV interconnection

3.2 LMI characterization

In order to derive closed-loop characterizations of H_2 and H_∞ performance, a standard procedure is to rewrite the LPV plant (7) as an augmented LPV plant with repeated blocks of delay operators $z^{-1}I_n$ and an augmented gain-scheduling block [17, 2]. The resulting closed-loop data are then described as

$$\left[\begin{array}{c|c|c} \mathcal{A} & \mathcal{B}_\Delta & \mathcal{B}_1 \\ \hline \mathcal{C}_\Delta & \mathcal{D}_{\Delta\Delta} & \mathcal{D}_{\Delta 1} \\ \hline \mathcal{C}_1 & \mathcal{D}_{1\Delta} & \mathcal{D}_{11} \end{array}\right] := \left[\begin{array}{cc|cc|c} A & 0 & B_\Delta & 0 & B_1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_\Delta & 0 & D_{\Delta\Delta} & 0 & D_{\Delta 1} \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_1 & 0 & D_{1\Delta} & 0 & D_{11} \end{array}\right] +$$
$$\left[\begin{array}{ccc} 0 & B_2 & 0 \\ I & 0 & 0 \\ \hline 0 & D_{\Delta 2} & 0 \\ 0 & 0 & I \\ \hline 0 & D_{12} & 0 \end{array}\right] \mathcal{K} \left[\begin{array}{cc|cc|c} 0 & I & 0 & 0 & 0 \\ C_2 & 0 & D_{2\Delta} & 0 & D_{21} \\ 0 & 0 & 0 & I & 0 \end{array}\right], \tag{9}$$

with the definition

$$\mathcal{K} := \begin{bmatrix} A_K & B_{K1} & B_{K\Delta} \\ C_{K1} & D_{K11} & D_{K1\Delta} \\ C_{K\Delta} & D_{K\Delta 1} & D_{K\Delta\Delta} \end{bmatrix}.$$

The new uncertainty or parameter structure associated with the closed-loop data (9) is then given by

$$\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_K(\Delta) \end{bmatrix}.$$

With each specification/channel is associated an LMI constraint of the form encountered in Propositions 2.1 and 2.2, LMIs (4) and (6). The desired characterization for LPV output-feedback synthesis with multi-objective/channel specifications can be derived in four steps:

1- introduce different Lyapunov variables and scalings (X_j, Z_j) and (Q_j, R_j) for each specification/channel. Also, an H_2 specification requires two pairs of scaling whereas only one is involved in an H_∞ specification.

2- introduce slack variables V, H and F common to all channels and specifications (this is the conservative step).

3- write down expressions characterizing H_2 and H_∞ performance for each channel using Propositions 2.1 and 2.2 with the closed-loop data $\mathcal{A}$, $\mathcal{B}_\Delta$, ... in (9).

4- perform adequate congruence transformations for each matrix inequality and use specific linearizing changes of variables to end up with LMI synthesis conditions.

The derivation of the final characterizations is rather tedious and lengthy. Hereafter, we clarify the main steps of the proposed procedure. Keeping in mind that all channels (w_1, z_1), (w_2, z_2), etc can be handled in the very same way, we shall only consider the case of an H_2 and H_∞ performance specification for the unique channel (w, z). This greatly simplifies the presentation below. When various channels are under consideration one will simply stack together the corresponding LMI constraints including additional Lyapunov variables and scalings.

In accordance with the partition of $\mathcal{A}$ and $\mathcal{D}_{\Delta\Delta}$ in (9), we introduce a partition of V and of its inverse $W := V^{-1}$, a partition of H and of its inverse $G := H^{-1}$ and a partition of F and of its inverse $E := F^{-1}$ in the form

$$V := \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \quad W := \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \qquad H := \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$
$$F := \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad E := \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

By the strict nature of the LMI constraints involved and a perturbation argument, there is no loss of generality in assuming that V_{21}, W_{21}, H_{21}, G_{21}, F_{21} and E_{21} are invertible. See for instance [6] for a detailed justification. We then introduce the notations

$$\Pi_V := \begin{bmatrix} V_{11} & I \\ V_{21} & 0 \end{bmatrix}, \quad \Pi_W := \begin{bmatrix} I & W_{11} \\ 0 & W_{21} \end{bmatrix}, \qquad \Pi_H := \begin{bmatrix} H_{11} & I \\ H_{21} & 0 \end{bmatrix}, \quad \Pi_G := \begin{bmatrix} I & G_{11} \\ 0 & G_{21} \end{bmatrix},$$
$$\Pi_F := \begin{bmatrix} F_{11} & I \\ F_{21} & 0 \end{bmatrix}, \quad \Pi_E := \begin{bmatrix} I & E_{11} \\ 0 & E_{21} \end{bmatrix}.$$

In turn, these matrices are invertible by the assumptions on V_{21}, W_{21}, H_{21}, G_{21}, F_{21} and E_{21}. One can then readily verify the identities

$$V\Pi_W = \Pi_V, \quad W\Pi_V = \Pi_W, \qquad H\Pi_G = \Pi_H, \quad G\Pi_H = \Pi_G \quad F\Pi_E = \Pi_F, \quad E\Pi_F = \Pi_E.$$

For an H_2 specification, we perform the congruence transformations

$$\operatorname{diag}(\Pi_W, \Pi_E, I, \Pi_W, \Pi_G), \qquad \operatorname{diag}(\Pi_W, \Pi_E, \Pi_G, I),$$

on the first and second inequalities (ii) of Proposition 2.1, respectively. For an H_∞ specification, we perform the congruence transformation

$$\operatorname{diag}(\Pi_W, \Pi_E, I, \Pi_W, \Pi_G, I)$$

in (6) of Proposition 2.2. For inequalities involving uncertainty blocks, last inequalities in (4) and (6), we perform the congruence transformation

$$\operatorname{diag}(\Pi_G, \Pi_E).$$

This yields matrix inequalities which solely involves the terms

$$\begin{bmatrix} \Pi_V^T \mathcal{A} \Pi_W & \Pi_V^T \mathcal{B}_\Delta \Pi_E & \Pi_V^T \mathcal{B}_1 \\ \Pi_H^T \mathcal{C}_\Delta \Pi_W & \Pi_H^T \mathcal{D}_{\Delta\Delta} \Pi_E & \Pi_H^T \mathcal{D}_{\Delta 1} \\ \mathcal{C}_1 \Pi_W & \mathcal{D}_{1\Delta} \Pi_E & \mathcal{D}_{11} \end{bmatrix}, \tag{10}$$

and

$$\begin{array}{ccc} \Pi_W^T X_j \Pi_W, & \Pi_E^T Q_j \Pi_E, & \Pi_G^T R_j \Pi_G, \\ \Pi_W^T V \Pi_W, & \Pi_G^T H \Pi_G, & \Pi_E^T F \Pi_E, \end{array} \tag{11}$$

and

$$\Pi_G^T \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_K \end{bmatrix}^T \Pi_F. \tag{12}$$

The variables (X_j, Z_j, Q_j, R_j) are attached to a given H_2 or H_∞ specification or channel, while (V, W, H, G, F, E) are slack variables common to all specifications and channels.

Explicit computation and inspection of these terms reveal that by invertibility of V_{21}, W_{21}, H_{21}, G_{21}, F_{21} and E_{21}, one can perform the following linearizing changes of variable:

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linearizing changes of variable

$$\begin{aligned}
\mathbf{D}_{K11} &:= D_{K11}, && (13)\\
\mathbf{B}_{K1} &:= V_{21}^T B_{K1} + V_{11}^T B_2 D_{K11}, && (14)\\
\mathbf{C}_{K1} &:= D_{K11} C_2 W_{11} + C_{K1} W_{21}, && (15)\\
\mathbf{A}_K &:= V_{11}^T A W_{11} + V_{21}^T A_K W_{21} + V_{21}^T B_{K1} C_2 W_{11} \\
&\quad + V_{11}^T B_2 C_{K1} W_{21} + V_{11}^T B_2 D_{K11} C_2 W_{11}, && (16)\\
\mathbf{D}_{K1\Delta} &:= D_{K11} D_{2\Delta} E_{11} + D_{K1\Delta} E_{21}, && (17)\\
\mathbf{D}_{K\Delta 1} &:= H_{11}^T D_{\Delta 2} D_{K11} + H_{21}^T D_{K\Delta 1}, && (18)\\
\mathbf{B}_{K\Delta} &:= V_{11}^T B_\Delta E_{11} + V_{21}^T B_{K1} D_{2\Delta} E_{11} + V_{11}^T B_2 D_{K11} D_{2\Delta} E_{11} \\
&\quad + V_{21}^T B_{K\Delta} E_{21} + V_{11}^T B_2 D_{K1\Delta} E_{21}, && (19)\\
\mathbf{C}_{K\Delta} &:= H_{11}^T C_\Delta W_{11} + H_{11}^T D_{\Delta 2} D_{K11} C_2 W_{11} + H_{21}^T D_{K\Delta 1} C_2 W_{11} \\
&\quad + \quad H_{11}^T D_{\Delta 2} C_{K1} W_{21} + H_{21}^T C_{K\Delta} W_{21}, && (20)\\
\mathbf{D}_{K\Delta\Delta} &:= H_{11}^T D_{\Delta\Delta} E_{11} + H_{11}^T D_{\Delta 2} D_{K11} D_{2\Delta} E_{11} + H_{21}^T D_{K\Delta 1} D_{2\Delta} E_{11} \\
&\quad + H_{11}^T D_{\Delta 2} D_{K1\Delta} E_{21} + H_{21}^T D_{K\Delta\Delta} E_{21}, && (21)\\
\mathbf{X}_j &:= \Pi_W^T X_j \Pi_W, && (22)\\
\mathbf{Q}_j &:= \Pi_E^T Q_j \Pi_E, \quad \mathbf{R}_j := \Pi_G^T R_j \Pi_G, && (23)\\
\mathbf{U} &:= V_{11}^T W_{11} + V_{21}^T W_{21}, \mathbf{M} := H_{11}^T G_{11} + H_{21}^T G_{21}, \mathbf{N} := F_{11}^T E_{11} + F_{21}^T E_{21} && (24)\\
\mathbf{\Delta}_K &:= F_{11}^T \Delta G_{11} + F_{21}^T \Delta_K G_{21}. && (25)
\end{aligned}$$

We have adopted a bold notation for the new variables. Note that these transformations are back and forth because of the invertibility of V_{21}, W_{21}, H_{21}, G_{21}, F_{21} and E_{21}. The matrix inequality terms in (10)-(12) then become linear in the new variables:

$$\Pi_V^T \mathcal{A} \Pi_W := \begin{bmatrix} V_{11}^T A + \mathbf{B}_{K1} C_2 & \mathbf{A}_K \\ A + B_2 \mathbf{D}_{K11} C_2 & A W_{11} + B_2 \mathbf{C}_{K1} \end{bmatrix},$$

$$\Pi_V^T \mathcal{B}_\Delta \Pi_E := \begin{bmatrix} V_{11}^T B_\Delta + \mathbf{B}_{K1} D_{2\Delta} & \mathbf{B}_{K\Delta} \\ B_\Delta + B_2 \mathbf{D}_{K11} D_{2\Delta} & B_\Delta E_{11} + B_2 \mathbf{D}_{K1\Delta} \end{bmatrix}, \Pi_V^T \mathcal{B}_1 := \begin{bmatrix} V_{11}^T B_1 + \mathbf{B}_{K1} D_{21} \\ B_1 + B_2 \mathbf{D}_{K11} D_{21} \end{bmatrix}$$

$$\Pi_H^T \mathcal{C}_\Delta \Pi_W := \begin{bmatrix} H_{11}^T C_\Delta + \mathbf{D}_{K\Delta 1} C_2 & \mathbf{C}_{K\Delta} \\ C_\Delta + D_{\Delta 2} \mathbf{D}_{K11} C_2 & C_\Delta W_{11} + D_{\Delta 2} \mathbf{C}_{K1} \end{bmatrix}, \Pi_H^T \mathcal{D}_{\Delta 1} := \begin{bmatrix} H_{11}^T D_{\Delta 1} + \mathbf{D}_{K\Delta 1} D_{21} \\ D_{\Delta 1} + D_{\Delta 2} \mathbf{D}_{K11} D_{21} \end{bmatrix},$$

$$\Pi_H^T \mathcal{D}_{\Delta\Delta} \Pi_E := \begin{bmatrix} H_{11}^T D_{\Delta\Delta} + D_{K\Delta 1} D_{2\Delta} & D_{K\Delta\Delta} \\ D_{\Delta\Delta} + D_{\Delta 2} D_{K11} D_{2\Delta} & D_{\Delta\Delta} E_{11} + D_{\Delta 2} D_{K1\Delta} \end{bmatrix},$$

$$\mathcal{C}_1 \Pi_W := \begin{bmatrix} C_1 + D_{12} D_{K11} C_2 & C_1 W_{11} + D_{12} C_{K1} \end{bmatrix}, \mathcal{D}_{11} := D_{11} + D_{12} D_{K11} D_{21},$$

$$\mathcal{D}_{1\Delta} \Pi_E := \begin{bmatrix} D_{1\Delta} + D_{12} D_{K11} D_{2\Delta} & D_{1\Delta} E_{11} + D_{12} D_{K1\Delta} \end{bmatrix},$$

$$\Pi_W^T \Pi_V := \begin{bmatrix} V_{11} & I \\ \mathbf{U}^T & W_{11}^T \end{bmatrix}, \Pi_G^T \Pi_H := \begin{bmatrix} H_{11} & I \\ M^T & G_{11}^T \end{bmatrix}, \Pi_E^T \Pi_F := \begin{bmatrix} F_{11} & I \\ N^T & E_{11}^T \end{bmatrix}.$$

Thanks to these transformations, the inequalities of Propositions 2.1 and 2.2 which do not involve a parameter block Δ become LMIs as desired.

Inequalities associated with the parameter block are rewritten

$$\begin{bmatrix} R_{\mathbf{j},\mathbf{1}} & R_{\mathbf{j},\mathbf{2}} & \Delta^T F_{11} & \Delta^T \\ R_{\mathbf{j},\mathbf{2}}^T & R_{\mathbf{j},\mathbf{3}} & {\Delta_K}^T & G_{11}^T \Delta^T \\ F_{11}^T \Delta & \Delta_K & Q_{\mathbf{j},\mathbf{1}} + F_{11} + F_{11}^T & Q_{\mathbf{j},\mathbf{2}} + I + N \\ \Delta & \Delta G_{11} & Q_{\mathbf{j},\mathbf{2}}^T + N^T + I & Q_{\mathbf{j},\mathbf{3}} + E_{11}^T + E_{11} \end{bmatrix} > 0, \quad \forall \Delta \in \mathcal{P}_\Delta,\ j = 1, \ldots. \tag{26}$$

They consist of a set indexed by j of parameterized inequalities with respect to Δ. Recalling that $\Delta(k)$ is evolving in a polytopic set $\mathcal{P}_\Delta$, that is,

$$\Delta := \sum_{i=1}^{L} \alpha_i \Delta_i, \quad \sum_{i=1}^{L} \alpha_i = 1, \quad \alpha_i \geq 0,$$

solution candidates can be searched for in the form

$$\Delta_K(\Delta) := \sum_{i=1}^{L} \alpha_i \Delta_{\mathbf{K},\mathbf{i}}, \tag{27}$$

where the $\Delta_{\mathbf{K},\mathbf{i}}$'s are decision variables and the α_i's are the polytopic coordinates of Δ in $\mathcal{P}_\Delta$ (see [3] for other potential methods). Under this restriction, the constraints (26) are converted into a finite set of LMIs

$$\begin{bmatrix} R_{\mathbf{j},\mathbf{1}} & R_{\mathbf{j},\mathbf{2}} & \Delta_i^T & \Delta_i^T F_{11} \\ R_{\mathbf{j},\mathbf{2}}^T & R_{\mathbf{j},\mathbf{3}} & G_{11}^T \Delta_i^T & \Delta_{\mathbf{K},\mathbf{i}}^T \\ \Delta_i & \Delta_i G_{11} & Q_{\mathbf{j},\mathbf{1}} + F_{11} + F_{11}^T & Q_{\mathbf{j},\mathbf{2}} + I + N \\ F_{11}^T \Delta_i & \Delta_{\mathbf{K},\mathbf{i}} & Q_{\mathbf{j},\mathbf{2}}^T + N^T + I & Q_{\mathbf{j},\mathbf{3}} + E_{11}^T + E_{11} \end{bmatrix} > 0, \quad i = 1, \ldots, L,\ j = 1, \ldots \tag{28}$$

where i indexes the vertices of $\mathcal{P}_\Delta$ and j indexes the channels and specifications.

Since for each channel and H_2 and H_∞ specifications, terms are of the form just derived, we conclude that sufficient existence conditions for the multi-objective/channel LPV control problem can be recast as an LMI program in the variables V_{11}, W_{11}, H_{11}, G_{11}, F_{11}, E_{11}, $\Delta_{\mathbf{K},\mathbf{i}}$ and the (bold) variables defined in (13)-(24).

For practical needs, we provide LMI characterizations of H_2 and H_∞ performance in Appendix A.

Remark. When the matrix polytope $\mathcal{P}_\Delta$ is a hypercube centered at 0, what can always be achieved for any hyper-rectangle by translation and scaling if necessary, then it is easily shown from the LMI conditions (28) that solutions $\Delta_{\mathbf{K},\mathbf{i}}$ associated with opposite points of the polytope are also opposite of each other. As a result, the solution family $\{\Delta_{\mathbf{K},\mathbf{i}}\}_{i=1,\dots,L}$ determine a hypercube with a similar arrangement as the original hypercube $\{\Delta_i\}_{i=1,\dots,L}$. With this in mind, the number of LMIs indexed by i in (28) and the number of matrix variables $\Delta_{\mathbf{K},\mathbf{i}}$ can be reduced by a factor of 2.

3.3 LPV controller construction

Once a feasible solution of the LMI constraints (Appendix A) has been computed, the state-space data (8) of the LPV controller are readily obtained as indicated below:

- compute an SVD factorization of $\mathbf{U}-V_{11}^T W_{11}$ and deduce invertible matrices V_{21} and W_{21} according to (24). Analogously, compute an SVD factorization of $M-H_{11}^T G_{11}$ and $N-F_{11}^T E_{11}$ and deduce invertible matrices H_{21}, G_{21}, F_{21} and E_{21} according to (24).
- compute the LPV controller data by sequentially reverting the changes of variable as specified in (13)-(21) .
- deduce the controller gain-scheduling function as

$$\begin{array}{rcl} \Delta_K(\Delta) & := & F_{21}^{-T}\left(\sum_{i=1}^{L}\alpha_i\Delta_{\mathbf{K},\mathbf{i}} - F_{11}^T\sum_{i=1}^{L}\alpha_i\Delta_i G_{11}\right)G_{21}^{-1} \\ & := & \sum_{i=1}^{L}\alpha_i\left(F_{21}^{-T}\Delta_{\mathbf{K},\mathbf{i}}G_{21}^{-1} - F_{21}^{-T}F_{11}^T\Delta_i G_{11}G_{21}^{-1}\right). \end{array} \tag{29}$$

Hence, the scheduling function is affine in polytopic coordinates of the parameter block Δ. In a practical implementation, there is no need to compute the matrix product terms on line in (29). Indeed, $\Delta_K(\Delta)$ can be rewritten as

$$\Delta_K(\Delta) := \sum_{i=1}^{L}\alpha_i\Phi_i, \quad \text{with } \Phi_i := \left(F_{21}^{-T}\Delta_{\mathbf{K},\mathbf{i}}G_{21}^{-1} - F_{21}^{-T}F_{11}^T\Delta_i G_{11}G_{21}^{-1}\right) \tag{30}$$

where the Φ_i's are computed off line.

Remark. An enriched class of scheduling functions $\Delta_K(\Delta)$ can be employed in place of a mere linear expression in (27). This, however, can play adversely in terms of computational time since additional variables are searched for. If structured symmetric scalings were used, and the controller was forced to replicate the parameter block of the plant, i.e., $\Delta_K := \Delta$, then it can be showed that LMIs involving Δ blocks disappear. This simpler characterization is then equivalent to those in [17, 2] for the single-objective H_∞ control problem. It is, however, more conservative than that proposed in this paper.

4 Illustrative examples

In this section, we provide illustrations of the proposed method and comparison results with earlier techniques. We consider an LFT plant (borrowed from [9]) of the form described in (1) and (2).

We shall use the following partitioning notations for parameter and performance channels:

$$\overbrace{\begin{bmatrix} w_{\delta_1} \\ w_{\delta_2} \\ w_{\delta_3} \end{bmatrix}}^{w_\Delta} = \overbrace{\begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix}}^{\Delta} \overbrace{\begin{bmatrix} z_{\delta_1} \\ z_{\delta_2} \\ z_{\delta_3} \end{bmatrix}}^{z_\Delta},$$

$$z := \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}, \; w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

State-space data of the LFT plant are given in Appendix B.

4.1 Single-objective comparisons

It is first assumed that δ_i, $i = 1,2,3$ are gain-scheduling variables with

$$\delta_1 = \delta_2 = \delta_3 \quad |\delta_i| \leq 0.2 .$$

Using different techniques, LPV controllers are synthesized to minimize the H_∞ performance γ of the channel (w,z). Results are recapped in Table 1.

	method [17, 2]	method [13, 22]	proposed method	method [21]
γ	22.07	21.98	21.98	21.98

Table 1: H_∞-performance with various methods

Note that performance levels are nearly the same for all used methods. As theoretically predicted, the proposed method achieves a performance level which is always better than that of method [17, 2] (structured symmetric scalings), and worse than that of method [21] (full-block scalings). One cannot draw definitive conclusions from comparisons with method [13, 22] since there is no inclusion relationships between the scaling sets involved. Hence, in this situation, the advantage of either of these methods is essentially problem dependent.

4.2 A multi-objective example

Hereafter, we consider a multi-objective application of our technique. The gain-scheduling variables are now δ_1, δ_2 with the normalization constraints $|\delta_i| \leq 0.2$, $i = 1,2$. The input-output pair $(w_{\delta_3}, z_{\delta_3})$ is now regarded as the H_∞-performance channel. The H_2 norm of the channel (w,z) is then minimized subject to various constraints on the H_∞ performance. This yields the tradeoff curve in Figure 2.

It is interesting to note that the problem becomes infeasible when the parameters δ_1 and δ_2 are handled as uncertain parameters as opposed to scheduling variables except for very large values of the H_∞ index associated with δ_3. This enlightens the positive feature of the gain-scheduling nature of the controller. A tradeoff point which provides a fairly good balance between H_2 and H_∞ criteria is pointed in Figure 2.

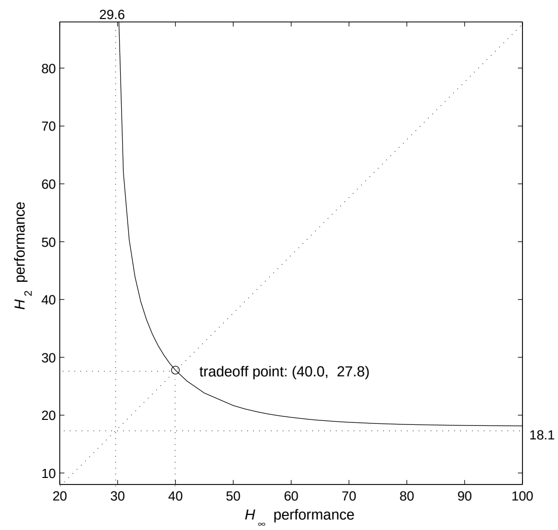

Figure 2: H_2/H_∞ tradeoff curve in LPV synthesis

5 Conclusion

In this paper, we have developed a new method for addressing the mixed H_2/H_∞ multi-channel LPV control problem in discrete-time. We have introduced new conditions for H_2 and H_∞ performances of LPV systems. These conditions are then combined with appropriate transformations on the controller data and on the controller scheduling function to end up with an LMI program description of the solutions. A very favorable feature is that different Lyapunov/scaling pairs can be used for each specification and channel what immediately reduces conservatism as compared to earlier methods. Extensions of this method to the continuous-time case and to more general classes of Lyapunov functions or scalings still remain challenging and will be considered in a future research.

Appendix A - LMIs for mixed H_2/H_∞ multi-channel LPV synthesis

The formulas below describe the structure of the LMI constraints that must be encoded. One must have in mind that for multiple channel specifications, the matrices

$$\begin{bmatrix} B_1 \\ D_{\Delta 1} \\ D_{11} \\ D_{21} \end{bmatrix}, \qquad \begin{bmatrix} C_1 & D_{1\Delta} & D_{11} & D_{12} \end{bmatrix}$$

appearing in the descriptions below should be modified in the appropriate way. Also, variables X_j, Z_j, Q_j, R_j, ν_j and γ_j should be introduced for each channel and specification indexed by j what reduces conservatism. All other variables are common to all channels and specifications.

H_2 performance

$$
\begin{bmatrix} \mathbf{LMI}_{11}^{1} & * \\ \mathbf{LMI}_{21}^{1} & \mathbf{LMI}_{22}^{1} \end{bmatrix} < 0, \quad \begin{bmatrix} \mathbf{LMI}_{11}^{2} & * \\ \mathbf{LMI}_{21}^{2} & \mathbf{LMI}_{22}^{2} \end{bmatrix} < 0, \quad \mathrm{Tr}\,(Z) < 1\,,
$$

$$
\begin{bmatrix} R_1 & * \\ \begin{bmatrix} F_{11}^T\Delta_i & \Delta_{\mathbf{K},\mathbf{i}} \\ \Delta_i & \Delta_i G_{11} \end{bmatrix} & Q_1 + \begin{bmatrix} F_{11}+F_{11}^T & * \\ (N+I)^T & E_{11}+E_{11}^T \end{bmatrix} \end{bmatrix} > 0, \quad \begin{bmatrix} R_2 & * \\ \begin{bmatrix} F_{11}^T\Delta_i & \Delta_{\mathbf{K},\mathbf{i}} \\ \Delta_i & \Delta_i G_{11} \end{bmatrix} & Q_2 + \begin{bmatrix} F_{11}+F_{11}^T & * \\ (N+I)^T & E_{11}+E_{11}^T \end{bmatrix} \end{bmatrix} > 0,
$$

$$
i = 1,\dots,L
$$

where

$$
\mathbf{LMI}_{11}^{1} := \begin{bmatrix} -\mathbf{X} & * & * \\ 0 & Q_1 & * \\ 0 & 0 & -\nu I \end{bmatrix},
$$

$$
\mathbf{LMI}_{21}^{1} := \begin{bmatrix} V_{11}^T A + B_{K1}C_2 & A_K & V_{11}^T B_\Delta + B_{K1}D_{2\Delta} & B_{K\Delta} & V_{11}^T B_1 + B_{K1}D_{21} \\ A + B_2 D_{K11} C_2 & AW_{11} + B_2 C_{K1} & B_\Delta + B_2 D_{K11} D_{2\Delta} & B_\Delta E_{11} + B_2 D_{K1\Delta} & B_1 + B_2 D_{K11} D_{21} \\ H_{11}^T C_\Delta + D_{K\Delta 1} C_2 & C_{K\Delta} & H_{11}^T D_{\Delta\Delta} + D_{K\Delta 1} D_{2\Delta} & D_{K\Delta\Delta} & H_{11}^T D_{\Delta 1} + D_{K\Delta 1} D_{21} \\ C_\Delta + D_{\Delta 2} D_{K11} C_2 & C_\Delta W_{11} + D_{\Delta 2} C_{K1} & D_{\Delta\Delta} + D_{\Delta 2} D_{K11} D_{2\Delta} & D_{\Delta\Delta} E_{11} + D_{\Delta 2} D_{K1\Delta} & D_{\Delta 1} + D_{\Delta 2} D_{K11} D_{21} \end{bmatrix},
$$

$$
\mathbf{LMI}_{22}^{1} := \begin{bmatrix} \mathbf{X} - \begin{bmatrix} V_{11}+V_{11}^T & * \\ (\mathbf{U}+I)^T & W_{11}+W_{11}^T \end{bmatrix} & * \\ 0 & R_1 - \begin{bmatrix} H_{11}+H_{11}^T & * \\ (M+I)^T & G_{11}+G_{11}^T \end{bmatrix} \end{bmatrix},
$$

$$
\mathbf{LMI}_{11}^{2} := \begin{bmatrix} -\mathbf{X} & 0 \\ 0 & Q_2 \end{bmatrix},
$$

$$
\mathbf{LMI}_{21}^{2} := \begin{bmatrix} H_{11}^T C_\Delta + D_{K\Delta 1} C_2 & C_{K\Delta} & H_{11}^T D_{\Delta\Delta} + D_{K\Delta 1} D_{2\Delta} & D_{K\Delta\Delta} \\ C_\Delta + D_{\Delta 2} D_{K11} C_2 & C_\Delta W_{11} + D_{\Delta 2} C_{K1} & D_{\Delta\Delta} + D_{\Delta 2} D_{K11} D_{2\Delta} & D_{\Delta\Delta} E_{11} + D_{\Delta 2} D_{K1\Delta} \\ C_1 + D_{12} D_{K11} C_2 & C_1 W_{11} + D_{12} C_{K1} & D_{1\Delta} + D_{12} D_{K11} D_{2\Delta} & D_{1\Delta} E_{11} + D_{12} D_{K1\Delta} \end{bmatrix},
$$

$$
\mathbf{LMI}_{22}^{2} := \begin{bmatrix} R_2 - \begin{bmatrix} H_{11}+H_{11}^T & * \\ (M+I)^T & G_{11}+G_{11}^T \end{bmatrix} & * \\ 0 & -Z \end{bmatrix}.
$$

H_∞ performance

$$\begin{bmatrix} \mathbf{LMI}_{11} & * \\ \mathbf{LMI}_{21} & \mathbf{LMI}_{22} \end{bmatrix} < 0,$$

$$\begin{bmatrix} R & * \\ \begin{bmatrix} F_{11}^T\Delta_i & \Delta_{\mathbf{K,i}} \\ \Delta_i & \Delta_i G_{11} \end{bmatrix} & Q + \begin{bmatrix} F_{11}+F_{11}^T & * \\ (N+I)^T & E_{11}+E_{11}^T \end{bmatrix} \end{bmatrix} > 0, \quad i = 1,\ldots,L$$

where

$$\mathbf{LMI}_{11} := \begin{bmatrix} -\mathbf{X} & * & * \\ 0 & Q & * \\ 0 & 0 & -\gamma I \end{bmatrix},$$

$$\mathbf{LMI}_{21} := \begin{bmatrix} V_{11}^T A + B_{K1} C_2 & A_K & V_{11}^T B_\Delta + B_{K1} D_{2\Delta} & B_{K\Delta} & V_{11}^T B_1 + B_{K1} D_{21} \\ A + B_2 D_{K11} C_2 & AW_{11} + B_2 C_{K1} & B_\Delta + B_2 D_{K11} D_{2\Delta} & B_\Delta E_{11} + B_2 D_{K1\Delta} & B_1 + B_2 D_{K11} D_{21} \\ H_{11}^T C_\Delta + D_{K\Delta 1} C_2 & C_{K\Delta} & H_{11}^T D_{\Delta\Delta} + D_{K\Delta 1} D_{2\Delta} & D_{K\Delta\Delta} & H_{11}^T D_{\Delta 1} + D_{K\Delta 1} D_{21} \\ C_\Delta + D_{\Delta 2} D_{K11} C_2 & C_\Delta W_{11} + D_{\Delta 2} C_{K1} & D_{\Delta\Delta} + D_{\Delta 2} D_{K11} D_{2\Delta} & D_{\Delta\Delta} E_{11} + D_{\Delta 2} D_{K1\Delta} & D_{\Delta 1} + D_{\Delta 2} D_{K11} D_{21} \\ C_1 + D_{12} D_{K11} C_2 & C_1 W_{11} + D_{12} C_{K1} & D_{1\Delta} + D_{12} D_{K11} D_{2\Delta} & D_{1\Delta} E_{11} + D_{12} D_{K1\Delta} & D_{11} + D_{12} D_{K11} D_{21} \end{bmatrix},$$

$$\mathbf{LMI}_{22} := \begin{bmatrix} \mathbf{X} - \begin{bmatrix} V_{11}+V_{11}^T & * \\ (\mathbf{U}+I)^T & W_{11}+W_{11}^T \end{bmatrix} & * & * \\ 0 & R - \begin{bmatrix} H_{11}+H_{11}^T & * \\ (M+I)^T & G_{11}+G_{11}^T \end{bmatrix} & * \\ 0 & 0 & -\gamma I \end{bmatrix}.$$

Appendix B - State-space data

$$
\left[\begin{array}{c|c|c|c} A & B_\Delta & B_1 & B_2 \\ \hline C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ \hline C_1 & D_{1\Delta} & D_{11} & D_{12} \\ \hline C_2 & D_{2\Delta} & D_{21} & 0 \end{array}\right] := \left[\begin{array}{ccc|ccc|cc|c} 2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0.5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]
$$

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