New Fuzzy Control Model and Dynamic Output Feedback Parallel Distributed Compensation

Hoang Duong Tuan, Pierre Apkarian, Tatsuo Narikiyo, and Masaaki Kanota

***Abstract*—A new fuzzy modeling based on fuzzy linear fractional transformations model is introduced. This new representation is shown to be a flexible tool for handling complicated nonlinear models. Particularly, the new fuzzy model provides an efficient and tractable way to handle the output feedback parallel distributed compensation problem. We demonstrate that this problem can be given a linear matrix inequality characterization and hence is immediately solvable through available semidefinite programming codes. The capabilities of the new fuzzy modeling is illustrated through numerical examples.**

***Index Terms*—Fuzzy control, linear matrix inequality (LMI), output feedback.**

I. Introduction

A CONVENIENT and flexible tool for handling complex nonlinear systems is the Tagaki–Sugeno (T–S) fuzzy model [12], where the consequent parts are linear systems connected by IF–THEN rules. Suppose that x is the state vector with dimension n_x, u is the control input with dimension n_u, y is the measurement output with dimension n_y, w and z are the disturbance and controlled output of the system with the same dimension n_{wz}, and L denotes the number of IF–THEN rules. Then, each ith plant rule has the form

$$\begin{aligned}&\text{IF } z_1(t) \text{ is } N_{i1} \text{ and } \dots z_p1(t) \text{ is } N_{ip}\\ &\text{THEN} \begin{bmatrix}\dot{x}\\ z\\ y\end{bmatrix} = \begin{bmatrix}A_i & B_{1i} & B_{2i}\\ C_{1i} & D_{11i} & D_{12i}\\ C_{2i} & D_{21i} & D_{22i}\end{bmatrix}\begin{bmatrix}x\\ w\\ u\end{bmatrix}.\end{aligned} \tag{1}$$

Here z_i are premise variables assumed independent of the control u and N_{ij} are fuzzy sets. One of the main advantages of the previous IF–THEN rule is the ease of its on-line implementation. Conformably to this description, it is quite natural to seek a dynamic output feedback for (1) in the form

$$\begin{aligned}&\text{IF } z_p(t) \text{ is } N_{i1} \text{ and } \dots z_1(t) \text{ is } N_{ip}\\ &\text{THEN} \begin{bmatrix}\dot{x}_K\\ u\end{bmatrix} = \begin{bmatrix}A_{Ki} & B_{Ki}\\ C_{Ki} & D_{Ki}\end{bmatrix}\begin{bmatrix}x_K\\ y\end{bmatrix}.\end{aligned} \tag{2}$$

This specific form will be referred to as a parallel distributed compensation (PDC) hereafter.

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However, in contrast to the case of the state feedback PDC [13], [14], [17], there is no known satisfactory method for designing the dynamic output feedback PDC (2). To see the difficulty of this design problem, let us first rewrite (1) and (2) in the gain-scheduling form.

Denoting $N_{ij}(z_i(t))$ the grade of membership of $z_i(t)$ in N_{ij} and normalizing the weight $w_i(t) = \prod_{j=1}^{p} N_{ij}(z_i(t))$ of each ith IF–THEN rule by

$$\alpha_i(t) = \frac{w_i(t)}{\sum_{j=1}^{L} w_j(t)} \geq 0, \qquad i = 1, 2, \dots, L \tag{3}$$

$$\begin{aligned}\Rightarrow \alpha(t) &= (\alpha_1(t), \dots, \alpha_L(t)) \in \Gamma\\ &:= \left\{\alpha \in R^L : \sum_{i=1}^{L} \alpha_i = 1,\ \alpha_i \geq 0\right\}\end{aligned} \tag{4}$$

the state-space representation of the T–S model is

$$\begin{bmatrix}\dot{x}\\ z\\ y\end{bmatrix} = \begin{bmatrix}A(\alpha(t)) & B_1(\alpha(t)) & B_2(\alpha(t))\\ C_1(\alpha(t)) & D_{11}(\alpha(t)) & D_{12}(\alpha(t))\\ C_2(\alpha(t)) & D_{21}(\alpha(t)) & D_{22}(\alpha(t))\end{bmatrix}\begin{bmatrix}x\\ w\\ u\end{bmatrix} \tag{5}$$

where

$$\begin{bmatrix}A(\alpha(t)) & B_1(\alpha(t)) & B_2(\alpha(t))\\ C_1(\alpha(t)) & D_{11}(\alpha(t)) & D_{12}(\alpha(t))\\ C_2(\alpha(t)) & D_{21}(\alpha(t)) & D_{22}(\alpha(t))\end{bmatrix} = \sum_{i=1}^{L} \alpha_i(t) \begin{bmatrix}A_i & B_{1i} & B_{2i}\\ C_{1i} & D_{11i} & D_{12i}\\ C_{2i} & D_{21i} & D_{22i}\end{bmatrix}. \tag{6}$$

Analogously, the PDC (2) can be rewritten as

$$\begin{bmatrix}\dot{x}_K\\ u\end{bmatrix} = \begin{bmatrix}A_K(\alpha(t)) & B_K(\alpha(t))\\ C_K(\alpha(t)) & D_K(\alpha(t))\end{bmatrix}\begin{bmatrix}x_K\\ y\end{bmatrix} \tag{7}$$

with

$$\begin{bmatrix}A_K(\alpha(t)) & B_K(\alpha(t))\\ C_K(\alpha(t)) & D_K(\alpha(t))\end{bmatrix} = \sum_{i=1}^{L} \alpha_i(t) \begin{bmatrix}A_{Ki} & B_{Ki}\\ C_{Ki} & D_{Ki}\end{bmatrix}. \tag{8}$$

Since $\alpha(t)$ in (5) is available on-line, the control problem described by (5) and (7) belongs to the more general class of gain-scheduling problems, an intensively studied subject in the past decade (see, e.g., [1], [2], and the references therein). Specifically, gain-scheduling is a widely used method for the control of nonlinear plants or a family of linear models. From (6) and (8), the special feature of system (5) and control (7) is that both of them are linear in the "gain scheduling" parameter $\alpha(t)$. While the system structure (6) is quite natural and widely studied in gain-scheduling control theory, the control structure (8) is not properly considered. Indeed, most of the results for gain scheduling control do not impose any special structure like (8) [2], [3],

[16]. With the structure assumption (8), the problem becomes very complex and to our knowledge, there is no effective solution method so far. This is in contrast with the state feedback control where it has been shown in [17] that the linear structure in the scheduling parameter $\alpha(t)$ is quite natural and can be assumed without loss of generality. Not surprisingly, a general structure of fuzzy control has been assumed in [9] to make it directly applicable from the general results of gain-scheduling control. Meanwhile, it is also well known in gain scheduling control as well as in fuzzy control that the simple linear structure (6), (8) is crucial to ensure efficiency and ease of on-line implementation. This motivates us to introduce a new class of fuzzy control models which not only includes model (1) as a particular case but also better approximates the nonlinear models while retaining favorable implementation properties.

Another feature of the proposed fuzzy model is its flexibility. As it is known, the T–S modeling works well for nonlinear systems with polynomial or other simple nonlinearities. When a system involves also fractional terms, one has first to try to transform it to a new simplified system involving only simpler nonlinear terms in order to successfully apply the T–S modeling [14]. Even when such transformations are possible, they do not allow to address the performance problem of the original system neither they facilitate the solution of the output feedback problem. In contrast, the proposed fuzzy modeling is free of such difficulties.

The paper is organized as follows. The new fuzzy modeling is introduced in Section II with some demonstration on its advantages over the T–S modeling. Section III gives a new LMI formulation for sub-optimal $\mathcal{H}_\infty$ dynamic output feedback PDC. The theoretical developments of the previous sections are illustrated by numerical examples in Section IV.

The notation of this paper is fairly standard. M^T is the transpose of the matrix M. For symmetric matrices, $M - N < 0$ ($M - N > 0$, resp.) means $M - N$ is negative definite (positive–definite, respectively). In symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for terms that are induced by symmetry, e.g.,

$$K\begin{bmatrix} S+(*) & * \\ M & Q \end{bmatrix} * \equiv K \begin{bmatrix} S+S^T & M^T \\ M & Q \end{bmatrix} K^T.$$

II. Fuzzy LFT Model

The new fuzzy LFT model discussed in this paper is described as

$$\begin{bmatrix} \dot{x}(t) \\ z_\Delta(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_\Delta & B_1 & B_2 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ C_1 & D_{1\Delta} & D_{11} & D_{12} \\ C_2 & D_{2\Delta} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w_\Delta(t) \\ w(t) \\ u(t) \end{bmatrix} \quad (9)$$

IF $z_1(t)$ is N_{i1} and $\ldots$ $z_p(t)$ is N_{ip}
THEN $w_\Delta(t) = \Delta_i z_\Delta(t)$. (10)

Correspondingly, the dynamic output feedback PDC has the same structure as that of (9) and (10)

$$\begin{bmatrix} \dot{x}_K(t) \\ u(t) \\ z_K(t) \end{bmatrix} = \begin{bmatrix} A_K & B_{K1} & B_{K\Delta} \\ C_{K1} & D_{K11} & D_{K1\Delta} \\ C_{K\Delta} & D_{K\Delta 1} & D_{K\Delta\Delta} \end{bmatrix} \begin{bmatrix} x_K(t) \\ y(t) \\ w_K(t) \end{bmatrix} \quad (11)$$

IF $z_1(t)$ is N_{i1} and $\ldots$ $z_p(t)$ is N_{ip}
THEN $w_K(t) = \Delta_{Ki} z_K(t)$. (12)

Here, in (9)–(12), the variables x, u, z, y have the same meaning and the same dimension as those in (1). The new variables z_Δ, w_Δ have the same dimension n_Δ. The controller variables x_K, z_K, w_K have the same dimensions as x, z_Δ, w_Δ, respectively. Therefore, both feedback connection Δ_i and control feedback connection Δ_{Ki} in the IF-THEN rule must be of dimension $n_\Delta \times n_\Delta$.

As it is well known [20, p. 255], any nonlinear system $[\dot{x}^T \quad z^T \quad y]^T = f(x) + g_1(x)w + g_2(x)u$ with rational functions $f(x), g_1(x), g_2(x)$ can be represented by (9) with

$$w_\Delta = \Delta(x) z_\Delta \quad (13)$$

where the function $\Delta(x)$ has a very simple structure. System (9), (13) is just

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \left(\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} + \begin{bmatrix} B_\Delta \\ D_{1\Delta} \\ D_{2\Delta} \end{bmatrix} \times \Delta(x)(I - D_{\Delta\Delta}\Delta(x))^{-1} \times [C_\Delta \quad D_{\Delta 1} \quad D_{\Delta 2}] \right) \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix}. \quad (14)$$

However, the representation (9), (13) is much more powerful for studying nonlinear systems than the LFT representation (14). Thus, our fuzzy LFT (9) and (10) can be interpreted as a fuzzification applied to the feedback part with simple structure (13) instead of applying fuzzification to the whole system in (14) in the T–S modeling. In fact, it is not so easy to apply fuzzy rules to system (14) because of complicated fractional terms. In such cases, with the T–S modeling, one has to try to use nonlinear transformations and state feedback linearization to simplify the system to a form that is more convenient for fuzzification. Unfortunately, such transformations do not exist in general and do not bring any positive feature for the output feedback control problem. The fuzzy LFT model (9) and (10) is free of these drawbacks as mentioned previously. The fuzzy rules are applied to the simpler object (13). Of course, the T–S model (1) can be regarded as a particular case of the fuzzy LFT (9) and (10) since the former can be easily rewritten in the form of the later.

We now demonstrate clear advantage of the fuzzy LFT modeling in comparison with the T–S modeling of the nonlinear benchmark model [5] of rotational-translational actuator (RTAC). The normalized RTAC model is

$$\begin{aligned} \ddot{\xi} + \xi &= \epsilon(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) + w \\ \ddot{\theta} &= -\epsilon\ddot{\xi}\cos\theta + u \end{aligned} \quad (15)$$

where

$$\xi = \sqrt{\frac{M+m}{k(I+me^2)}}q \quad \epsilon = \sqrt{\frac{k}{M+m}}t$$
$$u = \frac{M+m}{k(I+me^2)}N \quad w = \frac{1}{k}\sqrt{\frac{M+m}{I+me^2}}F.$$

The state-space representation of (15) is thus

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{-x_1+\epsilon x_4^2 \sin x_3}{1-\epsilon^2 \cos^2 x_3} \\ x_4 \\ \frac{\epsilon \cos x_3 \left(x_1 - \epsilon x_4^2 \sin x_3\right)}{1-\epsilon^2 \cos^2 x_3} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-\epsilon \cos x_3}{1-\epsilon^2 \cos^2 x_3} \\ 0 \\ \frac{1}{1-\epsilon^2 \cos^2 x_3} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{1}{1-\epsilon^2 \cos^2 x_3} \\ 0 \\ \frac{-\epsilon \cos x_3}{1-\epsilon^2 \cos^2 x_3} \end{bmatrix} w \quad (16)$$

with the redefinition $x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T = [\xi \quad \dot{\xi} \quad \theta \quad \dot{\theta}]^T$.

Clearly, because of many fractional terms involved in (16), the corresponding T–S modeling may lead to a very loose approximation. To overcome this difficulty, a known nonlinear transformations of variables [19] with the state feedback control linearization has been applied in [14] to transform (16) to a new simplified system involving only one nonlinear term $\sin x_3$ which is thus convenient for the T–S modeling. Clearly, with such an approach one can hardly handle the performance problem as well as the output feedback problem.

Now, it is immediate to verify (see the Appendix) that (16) can be rewritten as

$$\begin{bmatrix} \dot{x} \\ z_\Delta \end{bmatrix} = \begin{bmatrix} A & B_\Delta & B_1 & B_2 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \end{bmatrix} \begin{bmatrix} x \\ w_\Delta \\ w \\ u \end{bmatrix} \quad (17)$$

$$w_\Delta(t) = [(\epsilon \cos x_3(t) - a_1)\tilde{\Delta}_1 + \epsilon x_4 \sin x_3(t) \tilde{\Delta}_2] z_\Delta \quad (18)$$

with $A, B_\Delta, B_1, B_2, C_\Delta, D_{\Delta\Delta}, D_{\Delta 1}, D_{\Delta 2}$, and a_1 defined by (52) and (56) and

$$\tilde{\Delta}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \tilde{\Delta}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (19)$$

Thus it remains a simple task to apply fuzzy rules to the quite simple nonlinear terms in (18). The performance and regulator problems can then be directly addressed (see Section IV for more details). Also, noticing that the term Δ connecting w_Δ to z_Δ in the LFT representation of a nonlinear system has a simple structure, one needs much fewer IF–THEN rules to handle their connection in comparison with the T–S modeling.

III. $\mathcal{H}_\infty$ Control

The optimal $\mathcal{H}_\infty$ control problem consists in finding a controller (11) and (12) for (9) and (10) such that

$$\begin{aligned} &\gamma \to \min : \gamma > 0 \\ &\int_0^T \|z(t)\|^2 dt < \gamma^2 \int_0^T \|w(t)\|^2 dt \qquad \forall\, w \qquad \forall\, T > 0 \\ &x(0) = 0. \end{aligned} \quad (20)$$

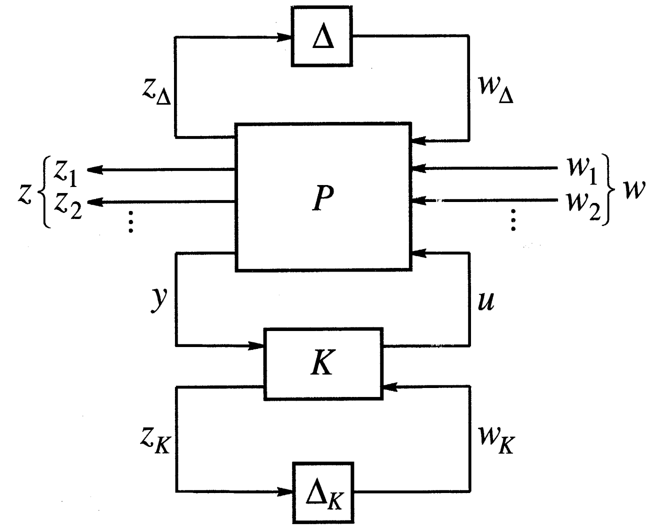

Fig. 1. Online implementation of fuzzy LFT system and PDC controller.

By using the normalized membership functions $\alpha_i(t)$ defined by (3) and (4), we can write our model (9) and (10) and output feedback PDC (11) and (12) as (9), (11) with

$$w_\Delta(t) = \Delta(\alpha(t)) z_\Delta(t) = \sum_{i=1}^{L} \alpha_i(t) \Delta_i z_\Delta(t) \quad (21)$$

$$w_K(t) = \Delta_K(\alpha(t)) z_K(t) = \sum_{i=1}^{L} \alpha_i(t) \Delta_{Ki} z_K(t) \quad (22)$$

see Fig. 1.

Before going further, let us mention that with the expressions (21), (22) in mind, the fuzzy system (9) and (10) and the control (11) and (12) are nothing else than an LFT gain-scheduling system and an LFT gain-scheduling control with the membership functions $\alpha(t)$ playing the role of the gain-scheduling parameters. However, it should be emphasized again that most existing techniques handling such LFT gain scheduling models (see, e.g., [1], [6], and the references therein) such as the linearization method in [11] or the Projection Lemma based technique in [1] and [6], are restricted to the norm constraint $\|\Delta(\alpha)\| \le 1$ for the matrix $\Delta(\alpha(t))$ in (21) but are unable to handle the polytopic constraints such as (21) and (22). Also, as mentioned earlier, the PDC structure of control (22) cannot be properly addressed. The approach below is based on the new linearizing technique recently developed in [4] for the discrete-time systems.

Now, denoting $x_{cl} = \begin{bmatrix} x \\ x_K \end{bmatrix}$, $z_{\Delta cl} = \begin{bmatrix} z_\Delta \\ z_K \end{bmatrix}$, $w_{\Delta cl} = \begin{bmatrix} w_\Delta \\ w_K \end{bmatrix}$, we can write down the closed-loop system as

$$\begin{bmatrix} \dot{x}_{cl} \\ z_{\Delta cl} \\ z_{cl} \end{bmatrix} = \left[\begin{array}{c|c|c} \mathcal{A} & \mathcal{B}_\Delta & \mathcal{B}_1 \\ \hline \mathcal{C}_\Delta & \mathcal{D}_{\Delta\Delta} & \mathcal{D}_{\Delta 1} \\ \hline \mathcal{C}_1 & \mathcal{D}_{1\Delta} & \mathcal{D}_{11} \end{array}\right] \begin{bmatrix} x_{cl} \\ w_{\Delta cl} \\ w \end{bmatrix}$$

$$w_{\Delta cl} = \Delta(\alpha) z_{\Delta cl} \quad (23)$$

where

$$\left[\begin{array}{c|c|c} \mathcal{A} & \mathcal{B}_\Delta & \mathcal{B}_1 \\ \hline \mathcal{C}_\Delta & \mathcal{D}_{\Delta\Delta} & \mathcal{D}_{\Delta 1} \\ \hline \mathcal{C}_1 & \mathcal{D}_{1\Delta} & \mathcal{D}_{11} \end{array}\right] := \left[\begin{array}{cc|cc|c} A & 0 & B_\Delta & 0 & B_1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_\Delta & 0 & D_{\Delta\Delta} & 0 & D_\Delta \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_1 & 0 & D_{1\Delta} & 0 & D_{11} \end{array}\right] + \left[\begin{array}{ccc} 0 & B_2 & 0 \\ I & 0 & 0 \\ \hline 0 & D_{\Delta 2} & 0 \\ 0 & 0 & I \\ \hline 0 & D_{12} & 0 \end{array}\right] \times \mathcal{K} \left[\begin{array}{cc|cc|c} 0 & I & 0 & 0 & 0 \\ C_2 & 0 & D_{2\Delta} & 0 & D_{21} \\ 0 & 0 & 0 & I & 0 \end{array}\right]$$

$$\bar{\Delta}(\alpha) = \begin{bmatrix} \Delta(\alpha) & 0 \\ 0 & \Delta_K(\alpha) \end{bmatrix} = \sum_{i=1}^{L} \alpha_i \bar{\Delta}_i \tag{24}$$

with the definition

$$\mathcal{K} := \begin{bmatrix} A_K & B_{K1} & B_{K\Delta} \\ C_{K1} & D_{K11} & D_{K1\Delta} \\ C_{K\Delta} & D_{K\Delta 1} & D_{K\Delta\Delta} \end{bmatrix}$$

$$\bar{\Delta}_i = \begin{bmatrix} \Delta_i & 0 \\ 0 & \Delta_{Ki} \end{bmatrix}, \qquad i = 1, 2, \ldots, L.$$

To handle the nonlinear relationship between $w_{\Delta cl}$ and $z_{\Delta cl}$, we use the following symmetric scalings R, Q:

$$\begin{bmatrix} R & \bar{\Delta}^T(\alpha) \\ \bar{\Delta}(\alpha) & -Q^{-1} \end{bmatrix} > 0 \qquad \forall\, \alpha \in \Gamma. \tag{25}$$

Clearly, for such scalings, one has

$$R + \bar{\Delta}^T(\alpha) Q \bar{\Delta}(\alpha) > 0 \text{ (by Schur complement)}$$
$$\Rightarrow z_{\Delta cl}^T \left(R + \bar{\Delta}^T(\alpha) Q \bar{\Delta}(\alpha)\right) z_{\Delta cl} \geq 0 \qquad \forall\, z_{\Delta cl}$$

i.e.,

$$z_{\Delta cl}^T R z_{\Delta cl} + w_{\Delta cl}^T Q w_{\Delta cl} \geq 0 \tag{26}$$

for all $w_{\Delta cl}, z_{\Delta cl}$ satisfying (23).

By virtue of (26), if there is a matrix

$$\mathcal{X} > 0 \tag{27}$$

such that

$$\frac{d}{dt}\left[x_{cl}^T(t)\mathcal{X}x_{cl}(t)\right] + z_{\Delta cl}^T(t) R z_{\Delta cl}(t) + w_{\Delta cl}^T(t) Q w_{\Delta cl}(t) + \gamma^{-1}\|z_{cl}(t)\|^2 - \gamma\|w(t)\|^2 < 0 \tag{28}$$

then

$$\frac{d}{dt}\left[x_{cl}^T(t)\mathcal{X}x_{cl}(t)\right] + \gamma^{-1}\|z_{cl}(t)\|^2 - \gamma\|w(t)\|^2 < 0 \tag{29}$$

so (20) holds true for all $z_{\Delta cl}, w_{\Delta cl}$ satisfying (23).

Rewriting the left-hand side of (28) as a quadratic functional in $(x_{cl}, w_{\Delta cl}, w)$ and by a Schur complement argument the following equivalent inequality is obtained:

$$\begin{bmatrix} \mathcal{X}\mathcal{A} + \mathcal{A}^T\mathcal{X} & * & * \\ \begin{bmatrix} \mathcal{B}_\Delta^T \\ \mathcal{B}_1^T \end{bmatrix}\mathcal{X} & \begin{bmatrix} Q & * \\ 0 & -\gamma I \end{bmatrix} & * \\ \begin{bmatrix} R\mathcal{C}_\Delta \\ \mathcal{C}_1 \end{bmatrix} & \begin{bmatrix} R\mathcal{D}_{\Delta\Delta} & R\mathcal{D}_{\Delta 1} \\ \mathcal{D}_{1\Delta} & \mathcal{D}_{11} \end{bmatrix} & \begin{bmatrix} -R & * \\ 0 & -\gamma I \end{bmatrix} \end{bmatrix} < 0. \tag{30}$$

Meanwhile, it is easily seen that (25) is equivalent to

$$\begin{bmatrix} R & \bar{\Delta}_i^T Q \\ Q\bar{\Delta}_i & -Q \end{bmatrix} > 0, \qquad i = 1, 2, \ldots, L. \tag{31}$$

In accordance with the partition of $\mathcal{A}$ and $\mathcal{D}_{\Delta\Delta}$ in (24), we introduce the following partitions of $\mathcal{X}$, R, Q and their inverses $\mathcal{Y} := \mathcal{X}^{-1}$, $H := R^{-1}$, $E := Q^{-1}$ in the form

$$\mathcal{X} := \begin{bmatrix} \mathbf{X} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \quad \mathcal{X}^{-1} = \mathcal{Y} := \begin{bmatrix} \mathbf{Y} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$$
$$R := \begin{bmatrix} \mathbf{R} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} \quad R^{-1} = H := \begin{bmatrix} \mathbf{H} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}$$
$$Q := \begin{bmatrix} \mathbf{Q} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \quad Q^{-1} = E := \begin{bmatrix} \mathbf{E} & E_{12} \\ E_{12}^T & E_{22} \end{bmatrix}.$$

where by the strict nature of the LMI constraints involved and a perturbation argument, there is no loss of generality in assuming that $X_{12}, Y_{12}, H_{12}, R_{12}, Q_{12}$ and E_{12} are invertible.

Because of the trivial indentities $\mathcal{X}\mathcal{X}^{-1} = I$, $RR^{-1} = I$, $QQ^{-1} = I$, one may note that

$$\mathbf{XY} + X_{12}Y_{12}^T = I$$
$$\mathbf{RH} + R_{12}H_{12}^T = I$$
$$\mathbf{QE} + Q_{12}E_{12}^T = I. \tag{32}$$

Therefore, with the notations

$$\Pi_{\mathcal{X}} := \begin{bmatrix} \mathbf{X} & I \\ X_{12}^T & 0 \end{bmatrix} \quad \Pi_{\mathcal{Y}} := \begin{bmatrix} I & \mathbf{Y} \\ 0 & Y_{12}^T \end{bmatrix}$$
$$\Pi_R := \begin{bmatrix} \mathbf{R} & I \\ R_{12}^T & 0 \end{bmatrix} \quad \Pi_H := \begin{bmatrix} I & \mathbf{H} \\ 0 & H_{12}^T \end{bmatrix}$$
$$\Pi_Q := \begin{bmatrix} \mathbf{Q} & I \\ Q_{12}^T & 0 \end{bmatrix} \quad \Pi_E := \begin{bmatrix} I & \mathbf{E} \\ 0 & E_{12}^T \end{bmatrix}$$

it is immediate to check that

$$\Pi_{\mathcal{Y}}^T \mathcal{X} \Pi_{\mathcal{Y}} = \begin{bmatrix} \mathbf{X} & I \\ I & \mathbf{Y} \end{bmatrix}$$
$$\Pi_H^T R \Pi_H = \begin{bmatrix} \mathbf{R} & I \\ I & \mathbf{H} \end{bmatrix}$$
$$\Pi_E^T Q \Pi_E = \begin{bmatrix} \mathbf{Q} & I \\ I & \mathbf{E} \end{bmatrix}. \tag{33}$$

These last equalities enable us to see the following.

- Equation (27), i.e., $\mathcal{X} > 0$ if and only $\Pi_{\mathcal{Y}}^T \mathcal{X} \Pi_{\mathcal{Y}} > 0$ (the congruence trasformation of (27) by $\Pi_{\mathcal{Y}}$):

$$\begin{bmatrix} \mathbf{X} & I \\ I & \mathbf{Y} \end{bmatrix} > 0. \tag{34}$$

- The congruence transformation

$$\mathrm{diag}(\Pi_H, \Pi_E)$$

applied to (31) in conjunction with the linearizing changes of variable

$$\mathbf{\Delta_{Ki}} := \mathbf{Q}\Delta_i \mathbf{H} + Q_{12}\Delta_{Ki} H_{12}^T, \qquad i = 1, 2, \ldots, L \tag{35}$$

makes (31) equivalent to the LMI

$$\begin{bmatrix} \mathbf{R} & I & \Delta_i^T \mathbf{Q} & \Delta_i^T \\ I & \mathbf{H} & \mathbf{\Delta}_{\mathbf{Ki}}^T & \mathbf{H}\Delta_i^T \\ \mathbf{Q}\Delta_i & \mathbf{\Delta_{Ki}} & -\mathbf{Q} & -I \\ \Delta_i & \Delta_i \mathbf{H} & -I & -\mathbf{E} \end{bmatrix} > 0, \qquad i = 1, 2, \ldots, L. \tag{36}$$

Here, the following consequence of (33) is also used:

$$\Pi_H^T \bar{\Delta}_i^T Q \Pi_E = \Pi_H^T \bar{\Delta}_i^T \Pi_Q = \begin{bmatrix} \Delta_i^T \mathbf{Q} & \Delta_i^T \\ \mathbf{\Delta}_{\mathbf{Ki}}^T & \mathbf{H}\Delta_i^T \end{bmatrix}.$$

Furthermore, for the following linearizing changes of variable [4]

$$\mathbf{D_{K11}} := D_{K11} \tag{37}$$
$$\mathbf{B_{K1}} := X_{12} B_{K1} + \mathbf{X} B_2 D_{K11} \tag{38}$$
$$\mathbf{C_{K1}} := D_{K11} C_2 \mathbf{Y} + C_{K1} Y_{12}^T \tag{39}$$
$$\begin{aligned} \mathbf{A_K} :=\ & \mathbf{X}A\mathbf{Y} + X_{12} A_K Y_{12}^T + X_{12} B_{K1} C_2 \mathbf{Y} \\ & + \mathbf{X} B_2 C_{K1} Y_{12}^T + \mathbf{X} B_2 D_{K11} C_2 \mathbf{Y} \end{aligned} \tag{40}$$
$$\mathbf{D_{K1\Delta}} := D_{K11} D_{2\Delta} \mathbf{E} + D_{K1\Delta} E_{12}^T \tag{41}$$
$$\mathbf{D_{K\Delta 1}} := \mathbf{R} D_{\Delta 2} D_{K11} + R_{12} D_{K\Delta 1} \tag{42}$$
$$\begin{aligned} \mathbf{B_{K\Delta}} :=\ & \mathbf{X} B_\Delta \mathbf{E} + X_{12} B_{K1} D_{2\Delta} \mathbf{E} + \mathbf{X} B_2 D_{K11} D_{2\Delta} \mathbf{E} \\ & + X_{12} B_{K\Delta} E_{12}^T + \mathbf{X} B_2 D_{K1\Delta} E_{12}^T \end{aligned} \tag{43}$$
$$\begin{aligned} \mathbf{C_{K\Delta}} :=\ & \mathbf{R} C_\Delta \mathbf{Y} + \mathbf{R} D_{\Delta 2} D_{K11} C_2 \mathbf{Y} + R_{12} D_{K\Delta 1} C_2 \mathbf{Y} \\ & + \mathbf{R} D_{\Delta 2} C_{K1} Y_{12}^T + R_{12} C_{K\Delta} Y_{12}^T \end{aligned} \tag{44}$$
$$\begin{aligned} \mathbf{D_{K\Delta\Delta}} :=\ & \mathbf{R} D_{\Delta\Delta} \mathbf{E} + \mathbf{R} D_{\Delta 2} D_{K11} D_{2\Delta} \mathbf{E} \\ & + R_{12} D_{K\Delta 1} D_{2\Delta} \mathbf{E} + \mathbf{R} D_{\Delta 2} D_{K1\Delta} E_{12}^T \\ & + R_{12} D_{K\Delta\Delta} E_{12}^T \end{aligned} \tag{45}$$

by lengthy but straightforward manipulations, it can be checked that

$$\begin{aligned}
&\Pi_{\mathcal{Y}}^T \mathcal{X} \mathcal{A} \Pi_{\mathcal{Y}} = \Pi_{\mathcal{X}}^T \mathcal{A} \Pi_{\mathcal{Y}} \\
&\quad := \begin{bmatrix} \mathbf{X}A + \mathbf{B_{K1}} C_2 & \mathbf{A_K} \\ A + B_2 \mathbf{D}_{K11} C_2 & A\mathbf{Y} + B_2 \mathbf{C_{K1}} \end{bmatrix} \\
&\Pi_E^T \mathcal{B}_\Delta^T \mathcal{X} \Pi_{\mathcal{Y}} = \Pi_E^T \mathcal{B}_\Delta^T \Pi_{\mathcal{X}} \\
&\quad := \begin{bmatrix} B_\Delta^T \mathbf{X} + D_{2\Delta}^T \mathbf{B_{K1}}^T & B_\Delta^T + D_{2\Delta}^T \mathbf{D_{K11}}^T B_2^T \\ \mathbf{B}_{\mathbf{K\Delta}}^T & \mathbf{E} B_\Delta^T + \mathbf{D}_{\mathbf{K1\Delta}}^T B_2^T \end{bmatrix} \\
&\mathcal{B}_1^T \mathcal{X} \Pi_{\mathcal{Y}} = \mathcal{B}_1^T \Pi_{\mathcal{X}} \\
&\quad := [\, B_1^T \mathbf{X} + D_{21}^T \mathbf{B_{K1}}^T \quad B_1^T + D_{21}^T \mathbf{D_{K11}}^T B_2^T \,] \\
&\Pi_H^T R \mathcal{C}_\Delta \Pi_{\mathcal{Y}} = \Pi_R^T \mathcal{C}_\Delta \Pi_{\mathcal{Y}} \\
&\quad := \begin{bmatrix} \mathbf{R} C_\Delta + \mathbf{D_{K\Delta 1}} C_2 & \mathbf{C_{K\Delta}} \\ C_\Delta + D_{\Delta 2} \mathbf{D_{K11}} C_2 & C_\Delta \mathbf{Y} + D_{\Delta 2} \mathbf{C_{K1}} \end{bmatrix} \\
&\mathcal{C}_1 \Pi_{\mathcal{Y}} := [\, C_1 + D_{12} \mathbf{D_{K11}} C_2 \quad C_1 \mathbf{Y} + D_{12} \mathbf{C_{K1}} \,] \\
&\Pi_H^T R \mathcal{D}_{\Delta\Delta} \Pi_E = \Pi_R^T \mathcal{D}_{\Delta\Delta} \Pi_E \\
&\quad := \begin{bmatrix} \mathbf{R} D_{\Delta\Delta} + \mathbf{D_{K\Delta 1}} D_{2\Delta} & \mathbf{D_{K\Delta\Delta}} \\ D_{\Delta\Delta} + D_{\Delta 2} \mathbf{D_{K11}} D_{2\Delta} & D_{\Delta\Delta} \mathbf{E} + D_{\Delta 2} \mathbf{D_{K1\Delta}} \end{bmatrix} \\
&\Pi_H^T R \mathcal{D}_{\Delta 1} = \Pi_R^T \mathcal{D}_{\Delta 1} \\
&\quad := \begin{bmatrix} \mathbf{R} D_{\Delta 1} + \mathbf{D_{K\Delta 1}} D_{21} \\ D_{\Delta 1} + D_{\Delta 2} \mathbf{D_{K11}} D_{21} \end{bmatrix} \\
&\mathcal{D}_{1\Delta} \Pi_E \\
&\quad := [\, D_{1\Delta} + D_{12} \mathbf{D_{K11}} D_{2\Delta} \quad D_{1\Delta} \mathbf{E} + D_{12} \mathbf{D_{K1\Delta}} \,] \\
&\mathcal{D}_{11} := D_{11} + D_{12} \mathbf{D_{K11}} D_{21}.
\end{aligned} \tag{46}$$

So, perform the congruence transformation

$$\mathrm{diag}\left(\Pi_{\mathcal{Y}}, \begin{bmatrix} \Pi_E & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} \Pi_H & 0 \\ 0 & I \end{bmatrix} \right)$$

in (30) to get the equivalent inequality shown in (47) at the bottom of the page, which, by (46), is nothing else as the following LMI:

$$\begin{bmatrix} \mathbf{LMI_1} & * \\ \mathbf{LMI_2} & \mathbf{LMI_3} \end{bmatrix} < 0 \tag{48}$$

with (49), as shown at the bottom of the next page, holding true.

In summary, with the new definition

$$\hat{\mathbf{K}} := \begin{bmatrix} \mathbf{A_K} & \mathbf{B_{K1}} & \mathbf{B_{K\Delta}} \\ \mathbf{C_{K1}} & \mathbf{D_{K11}} & \mathbf{D_{K1\Delta}} \\ \mathbf{C_{K\Delta}} & \mathbf{D_{K\Delta 1}} & \mathbf{D_{K\Delta\Delta}} \end{bmatrix}$$

we can reformulate our $\mathcal{H}_\infty$ control problem as

$$\min_{\mathbf{X}, \mathbf{Y}, \hat{\mathbf{K}}, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \mathbf{\Delta_{Ki}}, \gamma} \gamma : (34), (48), (36). \tag{50}$$

Theorem 3.1: The suboptimal dynamic output feedback PDC (11) and (12) for the $\mathcal{H}_\infty$-control problem (20) subject

$$\begin{bmatrix} \Pi_{\mathcal{Y}}^T \mathcal{X} \mathcal{A} \Pi_{\mathcal{Y}} + (*) & * & * & * & * \\ \Pi_E^T \mathcal{B}_\Delta^T \mathcal{X} \Pi_{\mathcal{Y}} & \Pi_E^T Q \Pi_E & * & * & * \\ \mathcal{B}_1^T \mathcal{X} \Pi_{\mathcal{Y}} & 0 & -\gamma I & * & * \\ \Pi_H^T R \mathcal{C}_\Delta \Pi_{\mathcal{Y}} & \Pi_H^T R \mathcal{D}_{\Delta\Delta} \Pi_E & \Pi_H^T R \mathcal{D}_{\Delta 1} & -\Pi_H^T R \Pi_H & * \\ \mathcal{C}_1 \Pi_{\mathcal{Y}} & \mathcal{D}_{1\Delta} \Pi_E & \mathcal{D}_{11} & 0 & -\gamma I \end{bmatrix} < 0 \tag{47}$$

to the LFT fuzzy system (9) and (10) can be solved from LMI optimization problem (50). The state-space data of the PDC (11) and (12) are readily obtained from solutions of problem (50) as follows.

- Using (32), compute an SVD factorization of $I - \mathbf{XY}$, $I - \mathbf{RH}$, $I - \mathbf{QE}$ to get $X_{12}, Y_{12}, R_{12}, H_{12}, Q_{12}, E_{12}$.
- Compute matrix data of (11) by sequentially reverting the changes of variable as specified in (37)–(45).
- Deduce Δ_{Ki} of the IF–THEN rule (12) as

$$\Delta_{Ki} := Q_{12}^{-1}(\mathbf{\Delta_{K,i}} - \mathbf{Q}\Delta_i\mathbf{H})H_{12}^{-T}. \tag{51}$$

Remarks: Although only the result with $\mathcal{H}_\infty$ control problem is presented here, it is clear that other problems such as stabilization, $\mathcal{H}_2$ control can be treated in a similar manner. However, for the multiobjective like $\mathcal{H}_\infty/\mathcal{H}_2$ control one, it is possible to use different Lyapunov variables $\mathbf{X}$, $\mathbf{Y}$ and scaling variables $\mathbf{R}$, $\mathbf{H}$, $\mathbf{Q}$, $\mathbf{E}$ for checking each performance to reduce conservatism. However, it may lead to LMIs with additional nonlinear scalar variable which requires additional line search (see [18] for related issues). An alternative approach is to transform the continuous system to a discrete-time one by a bilinear transformation and then applying the result of [4]. We refer the interested reader to [10].

IV. Numerical Examples

In this section, we demonstrate how nonlinear systems can be represented by our proposed fuzzy LFT models and thus can be effectively handled by LFT PDC (11) and (12).

A. RTAC Control

Return back to the RTAC model (15) where it is assumed for simplicity that $(\xi, \theta, \dot{\theta})$ is available for measurement. As in [5] we take the regulated output involving the tracking performance of the translational and angular positions and control:

$$z = [\sqrt{0.1}x_1 \quad \sqrt{0.1}x_3 \quad u]^T.$$

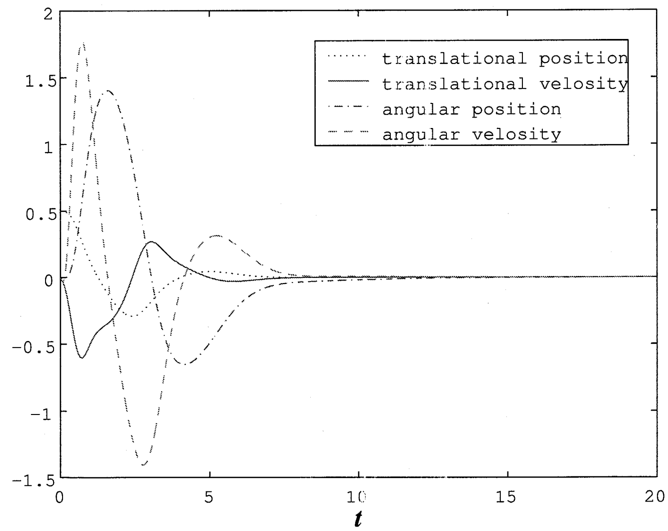

Fig. 2. Tracking performance in the absence of disturbance.

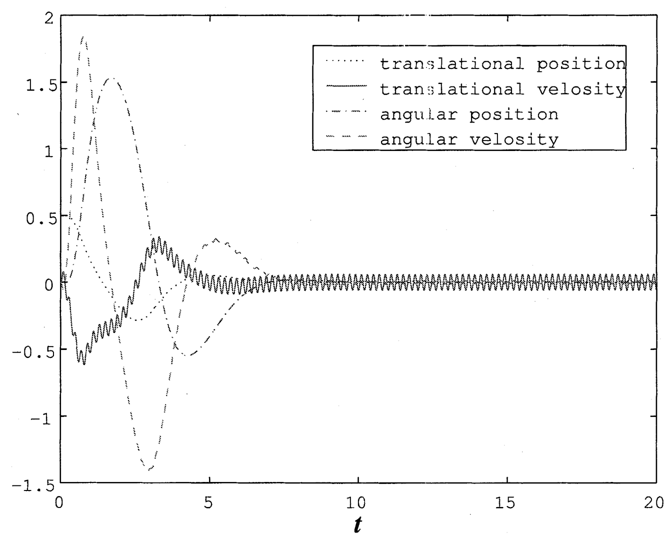

Fig. 3. Tracking performance with the disturbance $w = 1.8 \sin \pi t$.

$$\mathbf{LMI_1} := \begin{bmatrix} (\mathbf{X}A + \mathbf{B_{K1}}C_2) + (*) & * & * & * \\ \mathbf{A_K}^T + A + B_2\mathbf{D_{K11}}C_2 & (A\mathbf{Y} + B_2\mathbf{C_{K1}}) + (*) & * & * \\ B_\Delta^T\mathbf{X} + D_{2\Delta}^T\mathbf{B_{K1}}^T & B_\Delta^T + D_{2\Delta}^T\mathbf{D_{K11}}^T B_2^T & \mathbf{Q} & * \\ \mathbf{B_{K\Delta}^T} & \mathbf{E}B_\Delta^T + \mathbf{D_{K1\Delta}^T}B_2^T & -I & \mathbf{E} \end{bmatrix}$$

$$\mathbf{LMI_2} := \begin{bmatrix} B_1^T\mathbf{X} + D_{21}^T\mathbf{B_{K1}}^T & B_1^T + D_{21}^T\mathbf{D_{K11}}^T B_2^T & 0 & 0 \\ \mathbf{R}C_\Delta + \mathbf{D_{K\Delta1}}C_2 & \mathbf{C_{K\Delta}} & \mathbf{R}D_{\Delta\Delta} + \mathbf{D_{K\Delta1}}D_{2\Delta} & \mathbf{D_{K\Delta\Delta}} \\ C_\Delta + D_{\Delta2}\mathbf{D_{K11}}C_2 & C_\Delta\mathbf{Y} + D_{\Delta2}\mathbf{C_{K1}} & D_{\Delta\Delta} + D_{\Delta2}\mathbf{D_{K11}}D_{2\Delta} & D_{\Delta\Delta}\mathbf{E} + D_{\Delta2}\mathbf{D_{K1\Delta}} \\ C_1 + D_{12}\mathbf{D_{K11}}C_2 & C_1\mathbf{Y} + D_{12}\mathbf{C_{K1}} & D_{1\Delta} + D_{12}\mathbf{D_{K11}}D_{2\Delta} & D_{1\Delta}\mathbf{E} + D_{12}\mathbf{D_{K1\Delta}} \end{bmatrix}$$

$$\mathbf{LMI_3} := \begin{bmatrix} -\gamma I & * & * & * \\ \mathbf{R}D_{\Delta1} + \mathbf{D_{K\Delta1}}D_{21} & -\mathbf{R} & * & * \\ D_{\Delta1} + D_{\Delta2}\mathbf{D_{K11}}D_{21} & -I & -\mathbf{H} & * \\ D_{11} + D_{12}\mathbf{D_{K11}}D_{21} & 0 & 0 & -\gamma I \end{bmatrix}. \tag{49}$$

Under the typical assumption like $(x_3, x_4) \in [-0.5, 0.5]^2$, the data of system (9) for the RTAC model (15) is (see the Appendix)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{1-a_1^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{a_1}{1-a_1^2} & 0 & 0 & 0 \end{bmatrix}$$

$$B_\Delta = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{1-a_1^2} \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{a_1}{1-a_1^2} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ \frac{1}{1-a_1^2} \\ 0 \\ -\frac{a_1}{1-a_1^2} \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -\frac{a_1}{1-a_1^2} \\ 0 \\ \frac{1}{1-a_1^2} \end{bmatrix}$$

$$C_\Delta = \begin{bmatrix} \frac{1}{a_3^2} & 0 & 0 & 0 \\ \frac{1}{a_4^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D_{\Delta\Delta} = \begin{bmatrix} \frac{1}{a_3} & 0 & -\frac{1}{a_3^2} \\ 0 & -\frac{1}{a_4} & -\frac{1}{a_4^2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_{\Delta 1} = \begin{bmatrix} -\frac{1}{a_3^2} \\ -\frac{1}{a_4^2} \\ 0 \end{bmatrix} \quad D_{\Delta 2} = \begin{bmatrix} \frac{1}{a_3^2} \\ -\frac{1}{a_4^2} \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} \sqrt{0.1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{0.1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{52}$$

with the constants a_1, a_2, a_4, a_5 defined by (56).

Then with $\delta_1(t)$ and $\delta_2(t)$ defined by (57), we can rewrite (18) as (21) with Δ_i and the normalized membership functions α_i; see their definition (3) defined by

$$\Delta_1 = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_5 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -a_5 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} -a_2 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_5 \end{bmatrix}$$

$$\Delta_4 = \begin{bmatrix} -a_2 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & a_5 \end{bmatrix}$$

$$\alpha_1(t) = \frac{1}{4a_2a_5}(a_2 - \delta_1(t))(a_5 - \delta_2(t))$$

$$\alpha_2(t) = \frac{1}{4a_2a_5}(a_2 - \delta_1(t))(a_5 + \delta_2(t))$$

$$\alpha_3(t) = \frac{1}{4a_2a_5}(a_2 + \delta_1(t))(a_5 + \delta_2(t))$$

$$\alpha_4(t) = \frac{1}{4a_2a_5}(a_2 + \delta_1(t))(a_5 - \delta_2(t)).$$

and accordingly, the online implementation of the PDC is given by (22).

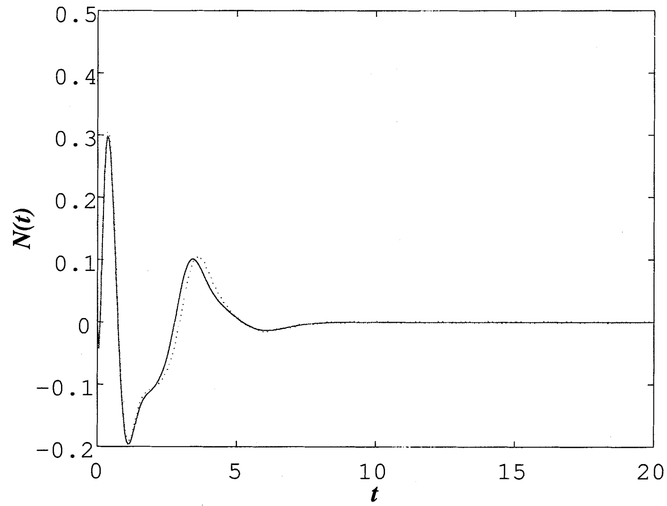

Fig. 4. Control performance in the absence of disturbance (solid) and with disturbance $w = 1.8 \sin \pi t$(dot).

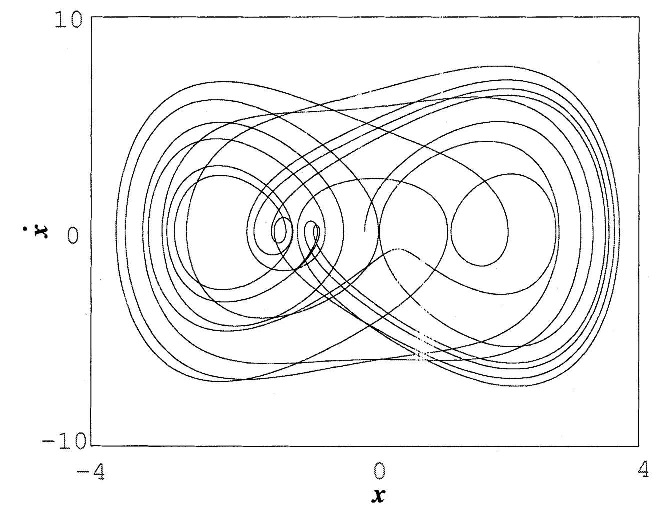

Fig. 5. Chaotic motion without control.

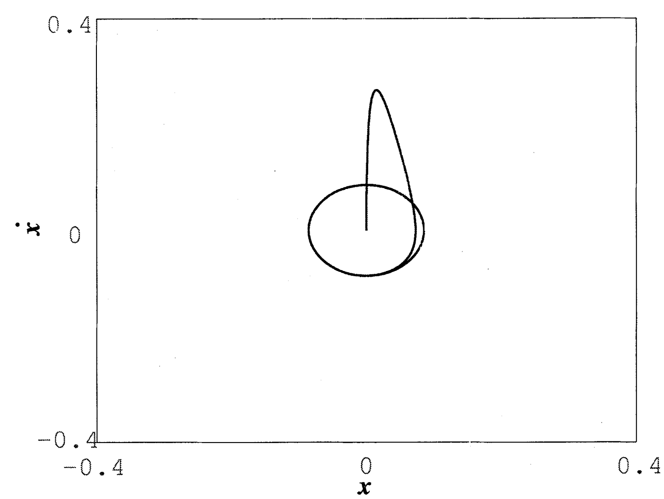

Fig. 6. Stable motion with control.

Under the condition $\epsilon = 0.2$ and $x_0 = [0.5 \quad 0 \quad 0 \quad 0]^T$, the simulation given by Figs. 2–4 clearly show that our PDC stabilizes the system well. Surprisingly, with our PDC, all system states look like "low frequency signals" which is in contrast with the high frequency behavior of the results in in [14], [17]. Perhaps, the high frequency observed in [14], [17] is caused by the nonlinear state transformation. Also, our PDC is much lower gain than that of [14] and [17] because the control performance is taken into account in our problem formulation. Because of complex nonlinear relationship due to nonlinear transformation, it is hard to explicate the control gain in [14], [17].

B. Chaos Control

The Duffing forced-oscillation equation

$$\ddot{\mathbf{x}} + 0.1\dot{\mathbf{x}} + \mathbf{x}^3 - 10\cos t - u(t) = 0 \tag{53}$$

with control input $u(t)$ and measured output $\mathbf{x}$ can be written as (9) with

$$\begin{aligned} x &= \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \quad B_\Delta = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0 \\ 10 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C_\Delta &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad D_{\Delta\Delta} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ D_{\Delta 1} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad D_{\Delta 2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ C_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D_{2\Delta} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad D_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix} \end{aligned} \tag{54}$$

and

$$w_\Delta = x_1(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z_\Delta. \tag{55}$$

Note that without the control input, the system behavior is chaotic (see Fig. 5) and $x_1(t) \in [-4, 4]$, thus we can rewrite (55) as (10) with

$$\begin{aligned} \Delta_1 &= \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \quad \Delta_2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ \alpha_1(t) &= \frac{4 - x_1(t)}{8} \quad \alpha_2(t) = \frac{4 + x_1(t)}{8}. \end{aligned}$$

The simulation with our PDC in the form (11) and (12) is given in Fig. 6, which clearly shows that PDC does a good job to stabilize the oscillation. Similar stability and performances have been also shown in [9]using another form of fuzzy controllers.

APPENDIX
LFT DERIVATION FOR RTAC MODEL

With the definitions

$$\begin{aligned} a_1 &= \epsilon \frac{\cos 0.5 + 1}{2} \quad a_2 = \epsilon \frac{1 - \cos 0.5}{2} \\ a_3 &= 1 - a_1 \quad a_4 = 1 + a_1 \quad a_5 = \epsilon 0.5 \sin 0.5 \end{aligned} \tag{56}$$

and

$$\begin{aligned} \delta_1 &= \epsilon \cos x_3 - a_1 \quad -a_2 \le \delta_1 \le a_2 \\ \delta_2 &= \epsilon x_4 \sin x_3 \quad -a_5 \le \delta_2 \le a_5 \end{aligned} \tag{57}$$

we can write

$$\begin{aligned} \frac{1}{1 - \epsilon^2 \cos^2 x_3} &= \frac{1}{1 - a_1^2} + \frac{1}{2}\left(\frac{\delta_1}{a_3^2 - a_3\delta_1} - \frac{\delta_1}{a_4^2 + a_4\delta_1}\right) \\ \frac{\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} &= \frac{a_1}{1 - a_1^2} + \frac{1}{2}\left(\frac{\delta_1}{a_3^2 - a_3\delta_1} + \frac{\delta_1}{a_4^2 + a_4\delta_1}\right) \\ \frac{\epsilon x_4 \sin x_3}{1 - \epsilon^2 \cos^2 x_3} &= \frac{\delta_2}{1 - a_1^2} + \frac{1}{2}\left(\frac{\delta_1\delta_2}{a_3^2 - a_3\delta_1} - \frac{\delta_1\delta_2}{a_4^2 + a_4\delta_1}\right) \\ \frac{\epsilon^2 \cos x_3 (x_4 \sin x_3)}{1 - \epsilon^2 \cos^2 x_3} &= \frac{\delta_2 a_1}{1 - a_1^2} + \frac{1}{2}\left(\frac{\delta_1\delta_2}{a_3^2 - a_3\delta_1} + \frac{\delta_1\delta_2}{a_4^2 + a_4\delta_1}\right). \end{aligned}$$

Therefore, (16) can be rewritten as

$$\begin{aligned} \dot{x_2} = &- \frac{1}{1 - a_1^2}(x_1 - \delta_2 x_4 - w + a_1 u) \\ &- \frac{1}{2}\frac{\delta_1}{a_3^2 - a_3\delta_1}(x_1 - \delta_2 x_4 - w + u) \\ &+ \frac{1}{2}\frac{\delta_1}{a_4^2 + a_4\delta_1}(x_1 - \delta_2 x_4 - w - u) \\ \dot{x_4} = &\frac{a_1}{1 - a_1^2}\left(x_1 - \delta_2 x_4 - w + \frac{1}{a_1}u\right) \\ &+ \frac{1}{2}\frac{\delta_1}{a_3^2 - a_3\delta_1}(x_1 - \delta_2 x_4 - w + u) \\ &+ \frac{1}{2}\frac{\delta_1}{a_4^2 + a_4\delta_1}(x_1 - \delta_2 x_4 - w - u). \end{aligned}$$

Set

$$\begin{aligned} z_{\Delta 1} &= \frac{1}{a_3^2}(x_1 + a_3 w_{\Delta 1} - w_{\Delta 3} - w + u) \\ z_{\Delta 2} &= \frac{1}{a_4^2}(x_1 - a_4 w_{\Delta 2} - w_{\Delta 3} - w - u) \\ z_{\Delta 3} &= x_4 \\ w_{\Delta 1} &= \delta_1 z_{\Delta 1} \\ w_{\Delta 2} &= \delta_1 z_{\Delta 2} \\ w_{\Delta 3} &= \delta_2 z_{\Delta 3}. \end{aligned} \tag{58}$$

Then

$$\begin{aligned} \dot{x}_2 = &-\frac{1}{1 - a_1^2}(x_1 - w + a_1 u) \\ &- \frac{1}{2}w_{\Delta 1} + \frac{1}{2}w_{\Delta 2} + \frac{1}{1 - a_1^2}w_{\Delta 3} \\ \dot{x}_4 = &\frac{a_1}{1 - a_1^2}\left(x_1 - w + \frac{1}{a_1}u\right) \\ &+ \frac{1}{2}w_{\Delta 1} + \frac{1}{2}w_{\Delta 2} - \frac{a_1}{1 - a_1^2}w_{\Delta 3}. \end{aligned} \tag{59}$$

Using (58) and (59), it is easily seen that that (16) is represented by (17) with data given by (52).

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